Number Theory and RSA Public-Key Encryption

Dr. Natarajan Meghanathan
Professor of Computer Science
Jackson State University
E-mail: natarajan.meghanathan@jsums.edu

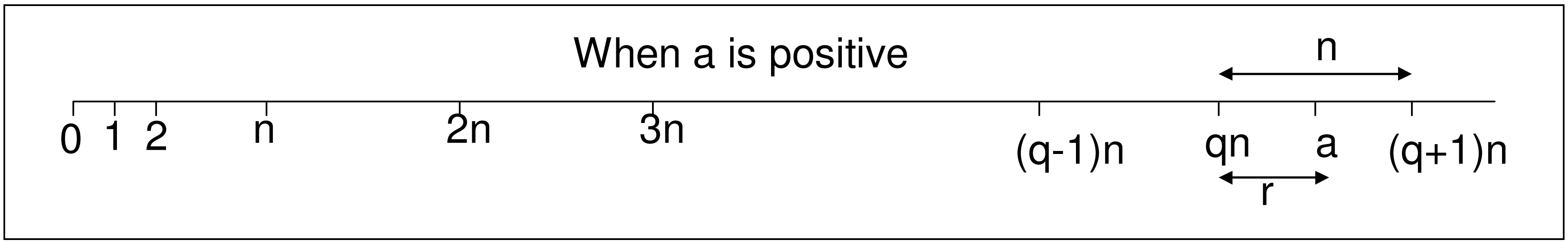
Public Key Encryption

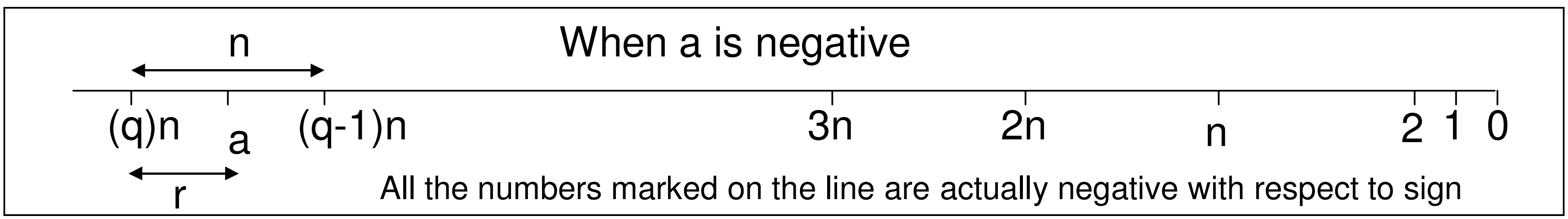
- Motivation: Key distribution problem of symmetric encryption system
- Let K_{PRIV} and K_{PUB} be the private key and public key of a user. Then,
  - $P = D(K_{PRIV}, E(K_{PUB}, P))$
  - With some, public key encryption algorithms like RSA, the following is also true: $P = D(K_{PUB}, E(K_{PRIV}, P))$
- In a system of n users, the number of secret keys for point-to-point communication is $n(n-1)/2 = O(n^2)$. With the public key encryption system, we need 2 keys (one public and one private key) per user. Hence, the total number of keys needed is $2n = O(n)$.

	Secret Key (Symmetric)	Public Key (Asymmetric)
Number of Keys	1	2
Protection of Key	Must be secret	One key must be secret; the key can be publicly exposed
Best uses	Cryptographic workhorse; secrecy and integrity of data	Key exchange, authentication
Key distribution	Must be out-of-band	Public key can be used to distribute other keys
Speed	Fast	Slow

Modular Arithmetic

- Given any positive integer n and any integer a, if we divide a by n, we get a quotient q and a remainder r that obey the following relationship:
  - $a = q * n + r, \quad 0 \le r < n$ and r is the remainder, q is the quotient





  - Example:
    - a = 59; n = 7; 59 = (8)*7 + 3 r = 3; q = 8
    - a = -59; n = 7; -59 = (-9)*7 + 4 r = 4; q = -9
    - 59 mod 7 = 3
    - -59 mod 7 = 4

Modular Arithmetic

- Two integers <u>a and b are said to be congruent modulo n</u>, <u>if a mod n = b mod n</u>. This is written as <u>a ≡ b mod n</u>.
  - We say "<u>a and b are equivalent to each other in class modulo n</u>"

- Example:
  - 73 ≡ 4 mod 23, because 73 mod 23 = 4 = 4 mod 23
  - 21 ≡ -9 mod 10, because 21 mod 10 = 1 = -9 mod 10

- Properties of the Modulo Operator
  - If a ≡ b mod n, then (a – b) mod n = 0
  - If a ≡ b mod n, then b ≡ a mod n
  - If a ≡ b mod n and b ≡ c mod n, then a ≡ c mod n

- Example:
  - 73 ≡ 4 mod 23, then (73 – 4) mod 23 = 0
  - 73 ≡ 4 mod 23, then 4 ≡ 73 mod 23, because 4 mod 23 = 73 mod 23
  - 73 ≡ 4 mod 23 and 4 ≡ 96 mod 23, then 73 ≡ 96 mod 23.

Modular Arithmetic

- Now, that we have studied the meaning of "equivalency" or "congruent modulo n", it is see that the "mod n" operator maps "all integers" (negative and positive) into the set of integers [0, 1, …., n-1].
- Example: Class of modulo 15

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
-60	-59	-58	-57	-56	-55	-54	-53	-52	-51	-50	-49	-48	-47	-46
-45	-44	-43	-42	-41	-40	-39	-38	-37	-36	-35	-34	-33	-32	-31
-30	-29	-28	-27	-26	-25	-24	-23	-22	-21	-20	-19	-18	-17	-16
-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39	40	41	42	43	44
45	46	47	48	49	50	51	52	53	54	55	56	57	58	59
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...

- From the above table, we could say things like
  - $-38 \equiv 22 \bmod 15$ $\quad$ $24 \equiv 54 \bmod 15$
  - $-38 \bmod 15 = 7$ $[-38 = (-3)*15 + 7]$ $\quad$ $24 \bmod 15 = 9$ $[24 = (1)*15 + 9]$
  - $22 \bmod 15 = 7$ $[22 = (1)*15 + 7]$ $\quad$ $54 \bmod 15 = 9$ $[54 = (3)*15 + 9]$

Modular Arithmetic

- Properties:
  - $(x + y) \bmod n = (x \bmod n + y \bmod n) \bmod n$
  - Example:
    - Compute: $(54 + 49) \bmod 15$
      - $(54 + 49) \bmod 15 = 103 \bmod 15 = \underline{13}$
      - $54 \bmod 15 = 9$
      - $49 \bmod 15 = 4$
      - $(54 \bmod 15 + 49 \bmod 15) = 9 + 4 = 13$
      - $(54 \bmod 15 + 49 \bmod 15) \bmod 15 = 13 \bmod 15 = \underline{13}$
  - Example:
    - Compute $(42 + 52) \bmod 15$
      - $(42 + 52) \bmod 15 = 94 \bmod 15 = \underline{4}$
      - $42 \bmod 15 = 12$
      - $52 \bmod 15 = 7$
      - $(42 \bmod 15 + 52 \bmod 15) = 12 + 7 = 19$
      - $(42 \bmod 15 + 52 \bmod 15) \bmod 15 = 19 \bmod 15 = \underline{4}$

Modular Arithmetic

- Properties:
  - (x * y) mod n = (x mod n * y mod n) mod n
  - Example:
    - Compute: (54 * 49) mod 15
      - (54 * 49) mod 15 = 2646 mod 15 = <u>6</u>
      - 54 mod 15 = 9
      - 49 mod 15 = 4
      - (54 mod 15 * 49 mod 15) = 9 * 4 = 36
      - (54 mod 15 * 49 mod 15) mod 15 = 36 mod 15 = <u>6</u>

  - Example:
    - Compute (42 * 52) mod 15
      - (42 * 52) mod 15 = 2184 mod 15 = <u>9</u>
      - 42 mod 15 = 12
      - 52 mod 15 = 7
      - (42 mod 15 * 52 mod 15) = 12 * 7 = 84
      - (42 mod 15 * 52 mod 15) mod 15 = 84 mod 15 = <u>9</u>

Modular Arithmetic

- Properties:
  - (a * b * c) mod n = ((a mod n) * (b mod n) * (c mod n)) mod n
  - (a * b * c) mod n = ((((a mod n) * (b mod n)) mod n) * (c mod n)) mod n
  - (a * b * c * d) mod n = ((a mod n) * (b mod n) * (c mod n) * (d mod n)) mod n
  - Similarly, (a * b * c * d * e) mod n….
  - Example:
    - Compute (42 * 56 * 98 * 108) mod 15
    - Straightforward approach: (42 * 56 * 98 * 108) mod 15 = (24893568) mod 15 = 3
    - Optimum approach 1 / Optimum approach 2

Optimum approach 1

- 42 mod 15 = 12
- 56 mod 15 = 11
- 98 mod 15 = 8
- 108 mod 15 = 3
- (42 * 56 * 98 * 108) mod 15
  = (12 * 11 * 8 * 3) mod 15
  = (3168) mod 15 = 3

Optimum approach 2

- First Compute (42 * 56) mod 15
- (42 * 56) mod 15 = (12 * 11) mod 15 = 12
- Then, compute (42 * 56 * 98) mod 15
- (42 * 56 * 98) mod 15 = (12 * 98) mod 15 = (12 * 8) mod 15 = 6
- Now, compute (42 * 56 * 98 * 108) mod 15
- (42 * 56 * 98 * 108) mod 15 = (6 * 108) mod 15 = (6 * 3) mod 15 = 3

Modular Arithmetic

- Modular Exponentiation
  - The Right-to-Left Binary Algorithm

To compute b^e mod n

First, write the exponent e in binary notation.

$$e = \sum_{i=0}^{m-1} a_i 2^i$$

In this notation, the length of e is m bits. For any i, such that $0 \le i < m-1$, the a_i take the value of 0 or 1. By definition, $a_{m-1} = 1$.

$$b^e = b^{\left(\sum_{i=0}^{m-1} a_i 2^i\right)} = \prod_{i=0}^{m-1} \left(b^{2^i}\right)^{a_i}$$

Solution for b^e mod n = $\prod_{i=0}^{m-1} \left(b^{2^i}\right)^{a_i} \bmod n$

Example for Modular Exponentiation

- To compute $5^{41} \bmod 9$
  - Straightforward approach:
    - $5^{41} \bmod 9 = (45474735088646411895751953125) \bmod 9 = 2$
    - Number of multiplications - 40
  - Using the Right-to-Left Binary Algorithm
    - Write 41 in binary: 101001
    - $5^{41} = 5^{32} * 5^{8} * 5^{1}$

32	16	8	4	2	1
1	0	1	0	0	1

$5^1 \bmod 9 = 5 \bmod 9 = 5$

$5^2 \bmod 9 = (5^1 * 5^1) \bmod 9 = (5 \bmod 9 * 5 \bmod 9) \bmod 9 = (5 * 5) \bmod 9 = 25 \bmod 9 = 7$

$5^4 \bmod 9 = (5^2 * 5^2) \bmod 9 = (5^2 \bmod 9 * 5^2 \bmod 9) \bmod 9 = (7 * 7) \bmod 9 = 49 \bmod 9 = 4$

$5^8 \bmod 9 = (5^4 * 5^4) \bmod 9 = (5^4 \bmod 9 * 5^4 \bmod 9) \bmod 9 = (4 * 4) \bmod 9 = 16 \bmod 9 = 7$

$5^{16} \bmod 9 = (5^8 * 5^8) \bmod 9 = (5^8 \bmod 9 * 5^8 \bmod 9) \bmod 9 = (7 * 7) \bmod 9 = 49 \bmod 9 = 4$

$5^{32} \bmod 9 = (5^{16} * 5^{16}) \bmod 9 = (5^{16} \bmod 9 * 5^{16} \bmod 9) \bmod 9 = (4 * 4) \bmod 9 = 16 \bmod 9 = 7$

$$
\begin{aligned}
5^{41} \bmod 9 &= (5^{32} * 5^8 * 5^1) \bmod 9 \\
&= (7 * 7 * 5) \bmod 9 \\
&= ((49 \bmod 9) * (5 \bmod 9)) \bmod 9 \\
&= (4 * 5) \bmod 9 \\
&= 20 \bmod 9 \\
&= 2
\end{aligned}
$$

Number of multiplications: $5 + 2 = 7$

Example for Modular Exponentiation

- To compute 3^{61} mod 8
  - Straightforward approach:
    - 3^{61} mod 8 = (127173474825648619542883299603) mod 8 = 3
    - Number of multiplications - 60
  - Using the Right-to-Left Binary Algorithm
    - <u>Write 61 in binary:</u> 111101
    - $3^{41} = 3^{32} * 3^{16} * 3^{8} * 3^{4} * 3^{1}$

32	16	8	4	2	1
1	1	1	1	0	1

$3^1 \bmod 8 = 3 \bmod 8 = 3$
$3^2 \bmod 8 = (3^1 * 3^1) \bmod 8 = (3 \bmod 8 * 3 \bmod 8) \bmod 8 = (3 * 3) \bmod 8 = 9 \bmod 8 = 1$
$3^4 \bmod 8 = (3^2 * 3^2) \bmod 8 = (3^2 \bmod 8 * 3^2 \bmod 8) \bmod 8 = (1 * 1) \bmod 8 = 1 \bmod 8 = 1$
$3^8 \bmod 8 = (3^4 * 3^4) \bmod 8 = (3^4 \bmod 8 * 3^4 \bmod 8) \bmod 8 = (1 * 1) \bmod 8 = 1 \bmod 8 = 1$
$3^{16} \bmod 8 = (3^8 * 3^8) \bmod 8 = (3^8 \bmod 8 * 3^8 \bmod 8) \bmod 8 = (1 * 1) \bmod 8 = 1 \bmod 8 = 1$
$3^{32} \bmod 8 = (3^{16} * 3^{16}) \bmod 8 = (3^{16} \bmod 8 * 3^{16} \bmod 8) \bmod 8 = (1 * 1) \bmod 8 = 1 \bmod 8 = 1$

$3^{61} \bmod 8 = (3^{32} * 3^{16} * 3^8 * 3^4 * 3^1) \bmod 8$
$= (1 * 1 * 1 * 1 * 3) \bmod 8$
$= ((1 \bmod 8) * (1 * 1 * 3 \bmod 9)) \bmod 8$
$= ((1 * 1) \bmod 8 * (1 * 3)) \bmod 8$
$= ((1 * 1) \bmod 8 * (3)) \bmod 8$
$= (1 * 3) \bmod 8$
$= 3 \bmod 8 = 3$

Number of multiplications: 5 + 4 = 9

Multiplicative Inverse Modulo n

- If (a * b) modulo n = 1, then
  - a is said to be the multiplicative inverse of b in class modulo n
  - b is said to be the multiplicative inverse of a in class modulo n
- Example:
  - Find the multiplicative inverse of 7 in class modulo 15
  - Straightforward approach:
    - Multiply 7 with all the integers [0, 1, …, 14] in class modulo 15
    - There will be only one integer x for which (7*x) modulo 15 = 1

X	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
(7 * X) modulo 15	0	7	14	6	13	5	12	4	11	3	10	2	9	1	8

  - Find the multiplicative inverse of 9 in class modulo 13
    - Multiply 9 with all the integers [0, 1, …, 12] in class modulo 13
    - There will be only one integer x for which (9*x) modulo 13 = 1

X	0	1	2	3	4	5	6	7	8	9	10	11	12
(9 * X) modulo 13	0	9	5	1	10	6	2	11	7	3	12	8	4

- A more efficient approach to find multiplicative inverse in class modulo n is to use the Extended Euclid Algorithm

Euclid's Algorithm to find the GCD

- Given two integers m and n (say m > n), then
  - GCD (m, n) = GCD (n, m mod n)
  - One can continue using the above recursion until the second term becomes 0. The GCD (m, n) will be then the value of the first term, because GCD (k, 0) = k

- Example: GCD (120, 45)
  - GCD (120, 45) = GCD (45, 30) = GCD (30, 15) = GCD (15, 0) = 15
- Example: GCD (45, 12)
  - GCD (45, 12) = GCD (12, 9) = GCD (9, 3) = GCD (3, 0) = 3
- Example: GCD (53, 30)
  - GCD (53, 30) = GCD (30, 23) = GCD (23, 7) = GCD (7, 2) = GCD (2, 1) = GCD (1, 0) = 1

- Note: Two numbers m and n are said to be relatively prime if
  - GCD (m, n) = 1.

Property of GCD

- For any two integers m and n,
  - We can write m * x + n * y = GCD (m, n)
    - x and y are also integers
    - We find x and y through the <u>Extended Euclid algorithm</u>

- If m and n are relatively prime, then
  - there exists two integers x and y such that m * x + n * y = 1
    - x is the multiplicative inverse of m modulo n
    - y is the multiplicative inverse of n modulo m
    - We could find x and y through the <u>Extended Euclid algorithm</u>

Extended Euclid Algorithm

- Theorem Statement
  - Let m and n be positive integers. Define
    - a[0] = m, a[1] = n
    - x[0] = 1, x[1] = 0, y[0] = 0, y[1] = 1,
    - q[k] = Floor(a[k-1]/ a[k]) for k > 0
    - a[k] = a[k-2] – (a[k-1]*q[k-1]) for k > 1
    - x[k] = x[k-2] – (q[k-1] * x[k-1]) for k > 1
    - y[k] = y[k-2] – (q[k-1] * y[k-1]) for k > 1
  - If a[p] is the last non-zero a[k], then
    - a[p] = GCD (m, n) = x[p] * m + y[p] * n
    - x[p] is the multiplicative inverse of m modulo n
    - y[p] is the multiplicative inverse of n modulo m

Example for Extended Euclid Algorithm

- Find the multiplicative inverse of 30 modulo 53
  - The larger of the two numbers is our m and the smaller is n
  - <u>Initial Setup</u> of the computation table

	a	q	x	y
m →	53	-	1	0
n →	30		0	1

We want to find the x and y such that 53x + 30y = 1

Iteration 1

a	q	x	y
53	-	1	0
30	1	0	1

a	q	x	y
53	-	1	0
30	1	0	1
23			

a	q	x	y
53	-	1	0
30	1	0	1
23		1	

a	q	x	y
53	-	1	0
30	1	0	1
23		1	-1

Iteration 2

a	q	x	y
53	-	1	0
30	1	0	1
23	1	1	-1

a	q	x	y
53	-	1	0
30	1	0	1
23	1	1	-1
7			

a	q	x	y
53	-	1	0
30	1	0	1
23	1	1	-1
7		-1	

a	q	x	y
53	-	1	0
30	1	0	1
23	1	1	-1
7		-1	2

Example for Extended Euclid Algorithm

Iteration 3

a	q	x	y
53	-	1	0
30	1	0	1
23	1	1	-1
7	3	-1	2

a	q	x	y
53	-	1	0
30	1	0	1
23	1	1	-1
7	3	-1	2
2			

a	q	x	y
53	-	1	0
30	1	0	1
23	1	1	-1
7	3	-1	2
2		4	

a	q	x	y
53	-	1	0
30	1	0	1
23	1	1	-1
7	3	-1	2
2		4	-7

Iteration 4

a	q	x	y
53	-	1	0
30	1	0	1
23	1	1	-1
7	3	-1	2
2	3	4	-7

a	q	x	y
53	-	1	0
30	1	0	1
23	1	1	-1
7	3	-1	2
2	3	4	-7
1			

a	q	x	y
53	-	1	0
30	1	0	1
23	1	1	-1
7	3	-1	2
2	3	4	-7
1		-13	

a	q	x	y
53	-	1	0
30	1	0	1
23	1	1	-1
7	3	-1	2
2	3	4	-7
1		-13	23

Iteration 5

a	q	x	y
53	-	1	0
30	1	0	1
23	1	1	-1
7	3	-1	2
2	3	4	-7
1	2	-13	23

a	q	x	y
53	-	1	0
30	1	0	1
23	1	1	-1
7	3	-1	2
2	3	4	-7
1	2	-13	23
0			

a	q	x	y
53	-	1	0
30	1	0	1
23	1	1	-1
7	3	-1	2
2	3	4	-7
1	2	-13	23

STOP!

-13*53+30*23 = 1 = GCD

23 is the multiplicative inverse of 30 modulo 53

-13 ≡ 17 is the Multiplicative inverse of 53 modulo 30

Example for Extended Euclid Algorithm

- Find the multiplicative inverse of 17 modulo 89
  - The larger of the two numbers is our m and the smaller is n
  - <u>Initial Setup</u> of the computation table

We want to find the x and y such that 89x + 17y = 1

	a	q	x	y
m →	89	-	1	0
n →	17		0	1

Iteration 1

a	q	x	y
89	-	1	0
17	5	0	1

a	q	x	y
89	-	1	0
17	5	0	1
4			

a	q	x	y
89	-	1	0
17	5	0	1
4		1	

a	q	x	y
89	-	1	0
17	5	0	1
4		1	-5

Iteration 2

a	q	x	y
89	-	1	0
17	5	0	1
4	4	1	-5

a	q	x	y
89	-	1	0
17	5	0	1
4	4	1	-5
1			

a	q	x	y
89	-	1	0
17	5	0	1
4	4	1	-5
1			

a	q	x	y
89	-	1	0
17	5	0	1
4	4	1	-5
1		-4	21

Example for Extended Euclid Algorithm

Iteration 3

a	q	x	y
89	-	1	0
17	5	0	1
4	4	1	-5
1	4	-4	21

a	q	x	y
89	-	1	0
17	5	0	1
4	4	1	-5
1	4	-4	21
0			

a	q	x	y
89	-	1	0
17	5	0	1
4	4	1	-5
1	4	-4	21

STOP!

$-4*89 + 21*17 = 1 = GCD$

21 is the multiplicative inverse of 17 modulo 89

- 4 ≡ 13 is the multiplicative inverse of 89 modulo 17

RSA Algorithm

- The RSA algorithm uses two keys, *d* and *e*, which work in pairs, for decryption and encryption, respectively.
- A plaintext message P is encrypted to ciphertext by:
  - $C = P^e \bmod n$
- The plaintext is recovered by:
  - $P = C^d \bmod n$
- Because of symmetry in modular arithmetic, encryption and decryption are mutual inverses and commutative. Therefore,
  - $P = C^d \bmod n = (P^e)^d \bmod n = (P^d)^e \bmod n$
- Thus, one can apply the encrypting transformation first and then the decrypting one, or the decrypting transformation first followed by the encrypting one.

- <u>On the complexity of RSA:</u> It is very difficult to factorize a large integer into two prime factors. The number of prime numbers between 2 and *n* is $(n/(\ln n))$.
- <u>Euler's Phi Function for Positive Prime Integers:</u> For any positive prime integer p, (p-1) is the number of positive integers less than p and relatively prime to p.

Key Choice for RSA Algorithm

- The encryption key consists of the pair of integers (e, n) and the decryption key consists of the pair of integers (d, n).
- Finding the value of n:
  - Choose two large prime numbers p and q (approximately at least 100 digits each)
  - The value of n is p * q, and hence n is also very large (approximately at least 200 digits).
  - Trump card of RSA: `A large value of n inhibits us to find the prime factors p and q.`
- Choosing e:
  - Choose e to be a very large integer that is relatively prime to (p-1)*(q-1).
  - To guarantee the above requirement, choose e to be greater than both p-1 and q-1
- Choosing d:
  - Select d such that (e * d) mod ((p-1)*(q-1)) = 1
  - In other words, d is the multiplicative inverse of e in class modulo (p-1)*(q-1)

Example 1 for RSA Algorithm

- Let p = 13 and q = 19. Find the encryption and decryption keys. Choose your encryption key to be at least 10.
- <u>Solution:</u>
- The value of n = p*q = 13*19 = 247
- (p-1)*(q-1) = 12*18 = 216
- Choose the encryption key e = 11, which is relatively prime to 216 = (p-1)*(q-1).

a	q	x	y
216	-	1	0
11	19	0	1
7	1	1	-19
4	1	-1	20
3	1	2	-39
1	3	-3	59
0			

- The decryption key d is the multiplicative inverse of 11 modulo 216.
- Run the Extended Euclid algorithm with m = 216 and n = 11.
  - 216x + 11y = 1
  - We need to find 'y': the multiplicative inverse of 11 modulo 216
  - y = 59
- We find the decryption key d to be 59

- The encryption key is (11, 247)
- The decryption key is (59, 247)

Example 2 for RSA Algorithm

- Let p = 11 and q = 13. Find the encryption and decryption keys. Choose your encryption key to be at least 10. Show the encryption and decryption for Plaintext 7

a	q	x	y
120	-	1	0
11	10	0	1
10	1	1	-10
1	10	-1	11
0			

<u>Solution:</u>

- The value of n = p*q = 11*13 = 143
- (p-1)*(q-1) = 10*12 = 120
- Choose the encryption key e = 11, which is relatively prime to 120 = (p-1)*(q-1).
- The decryption key d is the multiplicative inverse of 11 modulo 120.
- Run the Extended Euclid algorithm with m = 120 and n = 11.
- We find the decryption key d to be also 11 (the multiplicative inverse of 11 in class modulo 120)

- The encryption key is (11, 143)
- The decryption key is (11, 143)

Example 2 for RSA Algorithm

- Encryption for Plaintext P = 7
- Ciphertext C = P^e mod n

  = 7^{11} mod 143

8	4	2	1
1	0	1	1

7^1 mod 143 = 7 mod 143 = 7

7^2 mod 143 = (7^1 * 7^1) mod 143 = (7 mod 143 * 7 mod 143) mod 143 = (7 * 7) mod 143 = 49 mod 143 = 49

7^4 mod 143 = (7^2 * 7^2) mod 143 = (7^2 mod 143 * 7^2 mod 143) mod 143 = (49 * 49) mod 143 = 2401 mod 143 = 113

7^8 mod 143 = (7^4 * 7^4) mod 143 = (7^4 mod 143 * 7^4 mod 143) mod 143 = (113 * 113) mod 143 = 12769 mod 143 = 42

7^{11} mod 143 = (7^8 * 7^2 * 7^1) mod 143
= (42 * 49 * 7) mod 143
= (((42 * 49) mod 143) * (7)) mod 143
= (((2058) mod 143) * (7)) mod 143
= ((56) * (7)) mod 143
= (392) mod 143
= 106

Ciphertext is 106

Example 2 for RSA Algorithm

- Decryption for Ciphertext C = 106
- Plaintext P = C^d mod n

  = 106^{11} mod 143

8	4	2	1
1	0	1	1

106^1 mod 143 = 106 mod 143 = 106

106^2 mod 143 = (106^1 * 106^1) mod 143 = (106 mod 143 * 106 mod 143) mod 143 = (106 * 106) mod 143 = 49 mod 143 = 82

106^4 mod 143 = (106^2 * 106^2) mod 143 = (106^2 mod 143 * 106^2 mod 143) mod 143 = (82 * 82) mod 143 = 6724 mod 143 = 3

106^8 mod 143 = (106^4 * 106^4) mod 143 = (106^4 mod 143 * 106^4 mod 143) mod 143 = (3 * 3) mod 143 = 9 mod 143 = 9

106^{11} mod 143 = (106^8 * 106^2 * 106^1) mod 143
= (9 * 82 * 106) mod 143
= (((9 * 82) mod 143) * (106)) mod 143
= (((738) mod 143) * (106)) mod 143
= ((23) * (106)) mod 143
= (2438) mod 143
= 7

Plaintext is 7

Another Example for RSA Algorithm

- Let p = 17 and q = 23. Find the encryption and decryption keys. Choose your encryption key to be at least 10. Show the encryption and decryption for Plaintext 127

a	q	x	y
352	-	1	0
13	27	0	1
1	13	1	-27
0			

<u>Solution:</u>

- The value of n = p*q = 17*23 = 391
- (p-1)*(q-1) = 16*22 = 352
- Choose the encryption key e = 13, which is relatively prime to 352 = (p-1)*(q-1).
- The decryption key d is the multiplicative inverse of 13 modulo 352.
- Run the Extended Euclid algorithm with m = 352 and n = 13.
- The multiplicative inverse is -27 ≡ (-27 + 352) = 325
- We find the decryption key d to be 325 (the multiplicative inverse of 13 in class modulo 352)

- The encryption key is (13, 391)
- The decryption key is (325, 391)

Another Example for RSA Algorithm

- Encryption for Plaintext P = 127
- Ciphertext C = P^e mod n

  = 127^{13} mod 391

8	4	2	1
1	1	0	1

127^1 mod 391 = 127 mod 391 = 127

127^2 mod 391 = (127^1 * 127^1) mod 391 = (127 mod 391 * 127 mod 391) mod 391 = (127 * 127) mod 391 = 16129 mod 391 = 98

127^4 mod 391 = (127^2 * 127^2) mod 391 = (127^2 mod 391 * 127^2 mod 391) mod 391 = (98 * 98) mod 391 = 9604 mod 391 = 220

127^8 mod 391 = (127^4 * 127^4) mod 391 = (127^4 mod 391 * 127^4 mod 391) mod 391 = (220 * 220) mod 391 = 48400 mod 391 = 307

127^{13} mod 391 = (127^8 * 127^4 * 127^1) mod 391
= (307 * 220 * 127) mod 391
= (((307 * 220) mod 391) * (127)) mod 391
= (((67540) mod 391) * (127)) mod 391
= ((288) * (127)) mod 391
= (36576) mod 391
= 213

Ciphertext is 213

Another Example for RSA Algorithm

- Decryption for Ciphertext C = 213
- Plaintext P = C^d mod n

  = 213^{325} mod 391

256	128	64	32	16	8	4	2	1
1	0	1	0	0	0	1	0	1

$213^1 \bmod 391 = 213 \bmod 391 = 213$

$213^2 \bmod 391 = (213 * 213) \bmod 391 = 45369 \bmod 391 = 13$

$213^4 \bmod 391 = (13 * 13) \bmod 391 = 169 \bmod 391 = 169$

$213^8 \bmod 391 = (169 * 169) \bmod 391 = 28561 \bmod 391 = 18$

$213^{16} \bmod 391 = (18 * 18) \bmod 391 = 324 \bmod 391 = 324$

$213^{32} \bmod 391 = (324 * 324) \bmod 391 = 104976 \bmod 391 = 188$

$213^{64} \bmod 391 = (188 * 188) \bmod 391 = 35344 \bmod 391 = 154$

$213^{128} \bmod 391 = (154 * 154) \bmod 391 = 23716 \bmod 391 = 256$

$213^{256} \bmod 391 = (256 * 256) \bmod 391 = 65536 \bmod 391 = 239$

$$\begin{aligned}
213^{325} \bmod 391 &= (213^{256} * 213^{64} * 213^{4} * 213^{1}) \bmod 391 \\
&= (239 * 154 * 169 * 213) \bmod 391 \\
&= (52 * 169 * 213) \bmod 391 \\
&= (186 * 213) \bmod 391 \\
&= 127
\end{aligned}$$

Plaintext is 127