Module 1: Analyzing the Efficiency of Algorithms

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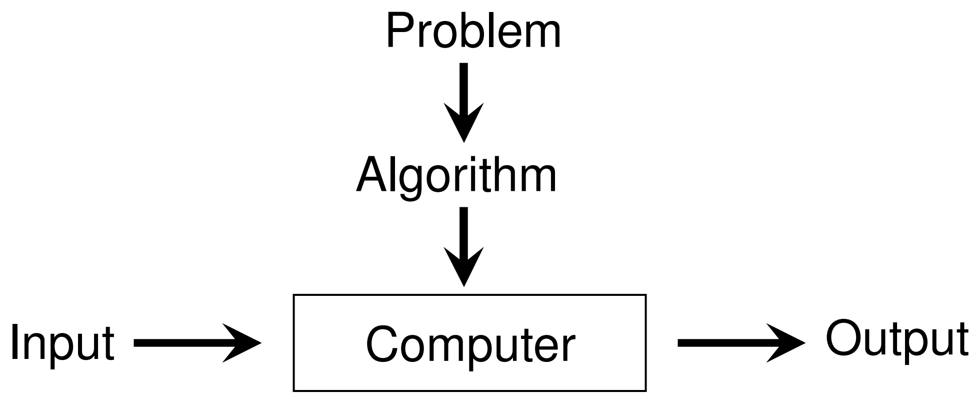
What is an Algorithm?

- An *algorithm* is a sequence of unambiguous instructions for solving a problem, i.e., for obtaining a required output for any legitimate input in a finite amount of time.



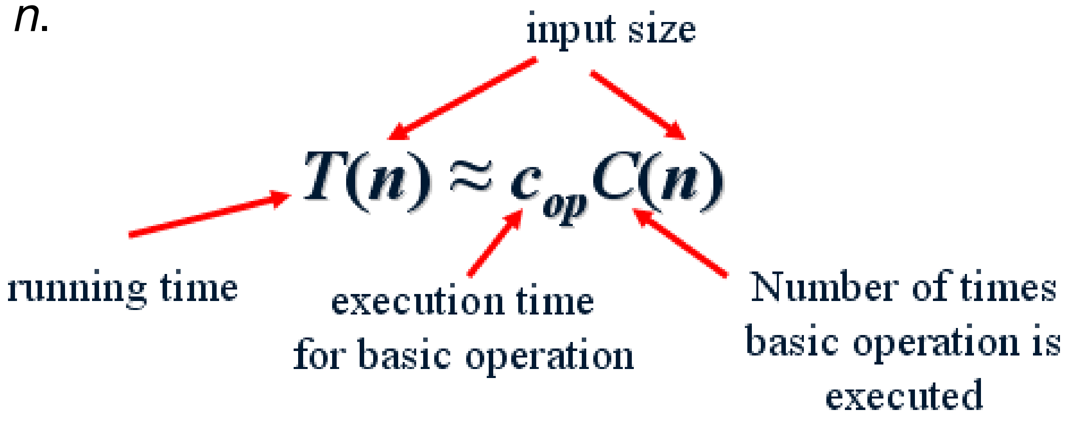
- Important Points about Algorithms
  - The non-ambiguity requirement for each step of an algorithm cannot be compromised
  - The range of inputs for which an algorithm works has to be specified carefully.
  - The same algorithm can be represented in several different ways
  - There may exist several algorithms for solving the same problem.
    - Can be based on very different ideas and can solve the problem with dramatically different speeds

The Analysis Framework

- **Time efficiency (time complexity):** indicates how fast an algorithm runs
- **Space efficiency (space complexity):** refers to the amount of memory units required by the algorithm in addition to the space needed for its input and output
- Algorithms that have non-appreciable space complexity are said to be ***in-place***.
- The time efficiency of an algorithm is typically as a function of the input size (one or more input parameters)
  - Algorithms that input a collection of values:
    - The time efficiency of sorting a list of integers is represented in terms of the number of integers (*n*) in the list
    - For matrix multiplication, the input size is typically referred as n*n.
    - For graphs, the input size is the set of Vertices (V) and edges (E).
  - Algorithms that input only one value:
    - The time efficiency depends on the magnitude of the integer. In such cases, the algorithm efficiency is represented as the number of bits $1+\lfloor \log_2 n \rfloor$ needed to represent the integer n

Units for Measuring Running Time

- The running time of an algorithm is to be measured with a unit that is independent of the extraneous factors like the processor speed, quality of implementation, compiler and etc.
  - At the same time, it is not practical as well as not needed to count the number of times, each operation of an algorithm is performed.
- Basic Operation: The operation contributing the most to the total running time of an algorithm.
  - It is typically the most time consuming operation in the algorithm's innermost loop.
    - **Examples:** Key comparison operation; arithmetic operation (division being the most time-consuming, followed by multiplication)
  - We will count the number of times the algorithm's basic operation is executed on inputs of size *n*.


input size
$T(n) \approx c_{op} C(n)$
running time
execution time for basic operation
Number of times basic operation is executed

Examples for Input Size and Basic Operations

Problem	*Input size measure*	*Basic operation*
Searching for key in a list of *n* items	Number of list's items, i.e. *n*	Key comparison
Multiplication of two matrices	Matrix dimensions or total number of elements	Multiplication of two numbers
Checking primality of a given integer *n*	*n*'size = number of digits (in binary representation)	Division
Typical graph problem	#vertices and/or edges	Visiting a vertex or traversing an edge

Orders of Growth

- We are more interested in the order of growth on the number of times the basic operation is executed on the input size of an algorithm.
- Because, for smaller inputs, it is difficult to distinguish efficient algorithms vs. inefficient ones.
- For example, if the number of basic operations of two algorithms to solve a particular problem are *n* and n^2 respectively, then
  - if $n = 3$, then we may say there is not much difference between requiring 3 basic operations and 9 basic operations and the two algorithms have about the same running time.
  - On the other hand, if $n = 10000$, then it does makes a difference whether the number of times the basic operation is executed is *n* or n^2.

n	$\log_2 n$	n	$n \log_2 n$	n^2	n^3	2^n	$n!$
10	3.3	10^1	$3.3 \cdot 10^1$	10^2	10^3	10^3	$3.6 \cdot 10^6$
10^2	6.6	10^2	$6.6 \cdot 10^2$	10^4	10^6	$1.3 \cdot 10^{30}$	$9.3 \cdot 10^{157}$
10^3	10	10^3	$1.0 \cdot 10^4$	10^6	10^9		
10^4	13	10^4	$1.3 \cdot 10^5$	10^8	10^{12}		
10^5	17	10^5	$1.7 \cdot 10^6$	10^{10}	10^{15}		
10^6	20	10^6	$2.0 \cdot 10^7$	10^{12}	10^{18}		

Exponential-growth functions

Source: Table 2.1 From Levitin, 3rd ed.

Best-case, Average-case, Worst-case

- For many algorithms, the actual running time may not only depend on the input size; but, also on the specifics of a particular input.
  - For example, sorting algorithms (like insertion sort) may run faster on an input sequence that is *almost-sorted* rather than on a randomly generated input sequence.

- **Worst case:** $C_{worst}(n)$ – maximum number of times the basic operation is executed over inputs of size n
- **Best case:** $C_{best}(n)$ – minimum # times over inputs of size n
- **Average case:** $C_{avg}(n)$ – “average” over inputs of size n
  - Number of times the basic operation will be executed on typical input
  - NOT the average of worst and best case
  - Expected number of basic operations considered as a random variable under some assumption about the probability distribution of all possible inputs

Example for Worst and Best-Case Analysis: Sequential Search

```
ALGORITHM  SequentialSearch(A[0..n − 1], K)
    //Searches for a given value in a given array by sequential search
    //Input: An array A[0..n − 1] and a search key K
    //Output: The index of the first element of A that matches K
    //          or −1 if there are no matching elements
    i ← 0
    while i < n and A[i] ≠ K do
        i ← i + 1
    if i < n return i
    else return −1
```

/* Assume the second condition will not be executed if the first condition evaluates to false */

- <u>Worst-Case:</u> $C_{worst}(n) = n$
- <u>Best-Case:</u> $C_{best}(n) = 1$

Probability-based Average-Case Analysis of Sequential Search

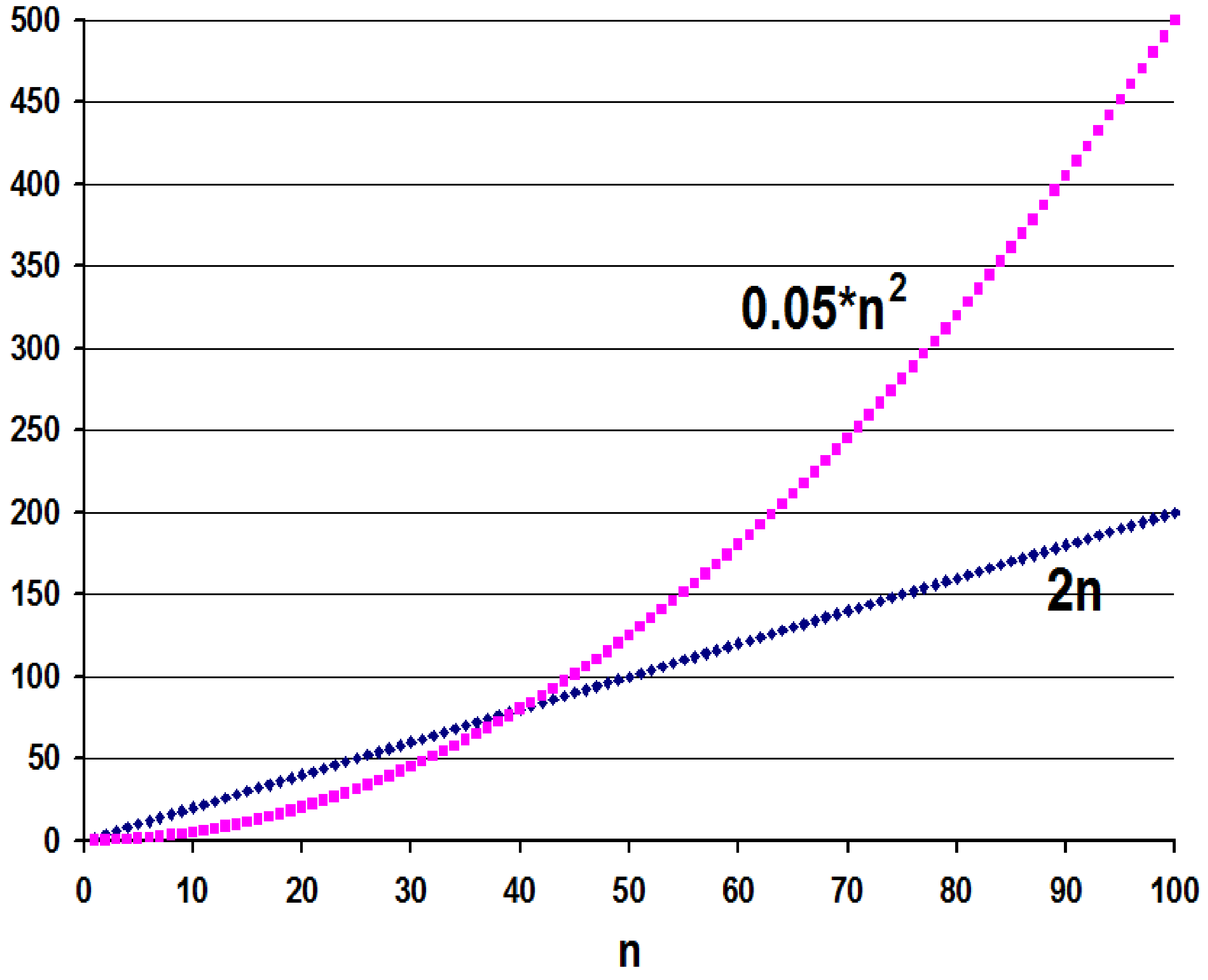
- If *p* is the probability of finding an element in the list, then (1-*p*) is the probability of not finding an element in the list.
- Also, on an *n*-element list, the probability of finding the search key as the i^{th} element in the list is *p/n* for all values of $1 \le i \le n$

$$
\begin{aligned}
C_{avg}(n) &= [1 \cdot \frac{p}{n} + 2 \cdot \frac{p}{n} + \cdots + i \cdot \frac{p}{n} + \cdots + n \cdot \frac{p}{n}] + n \cdot (1 - p) \\
&= \frac{p}{n}[1 + 2 + \cdots + i + \cdots + n] + n(1 - p) \\
&= \frac{p}{n} \frac{n(n+1)}{2} + n(1 - p) = \frac{p(n+1)}{2} + n(1 - p).
\end{aligned}
$$

- If p = 1 (the element that we will search for always exists in the list), then $C_{avg}(n) = (n+1)/2$. That is, on average, we visit half of the entries in the list to search for any element in the list.
- If p = 0 (all the time, the element that we will search never exists), then $C_{avg}(n) = n$. That is, we visit all the elements in the list.

YouTube Link: https://www.youtube.com/watch?v=8V-bHrPykrE

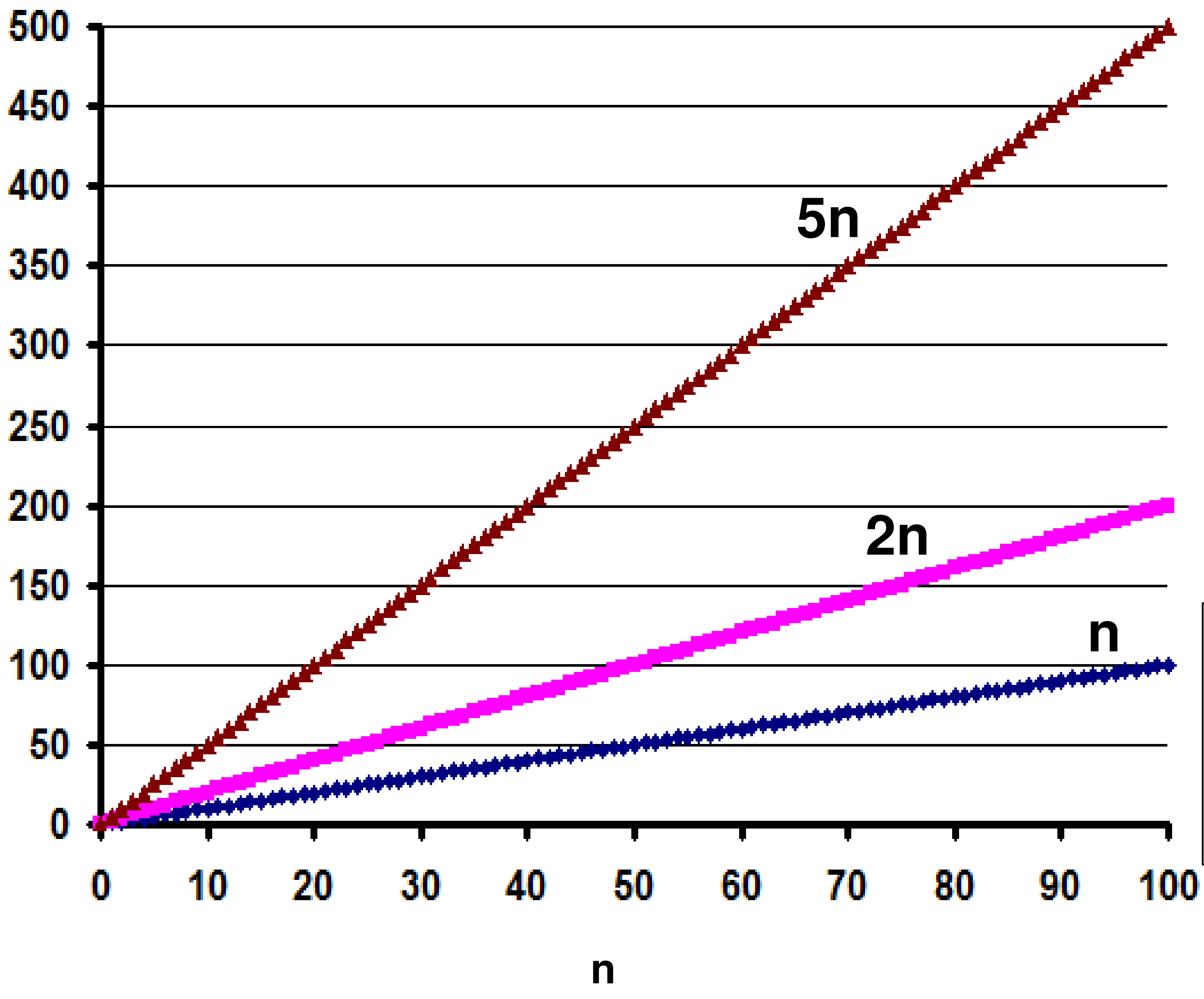
Asymptotic Notations: Intro



$2n \leq 0.05\ n^2$
for $n \geq 40$
$2n = O(n^2)$

$0.05n^2 \geq 2n$
for $n \geq 40$
$0.05n^2 = \Omega(n)$

Asymptotic Notations: Intro

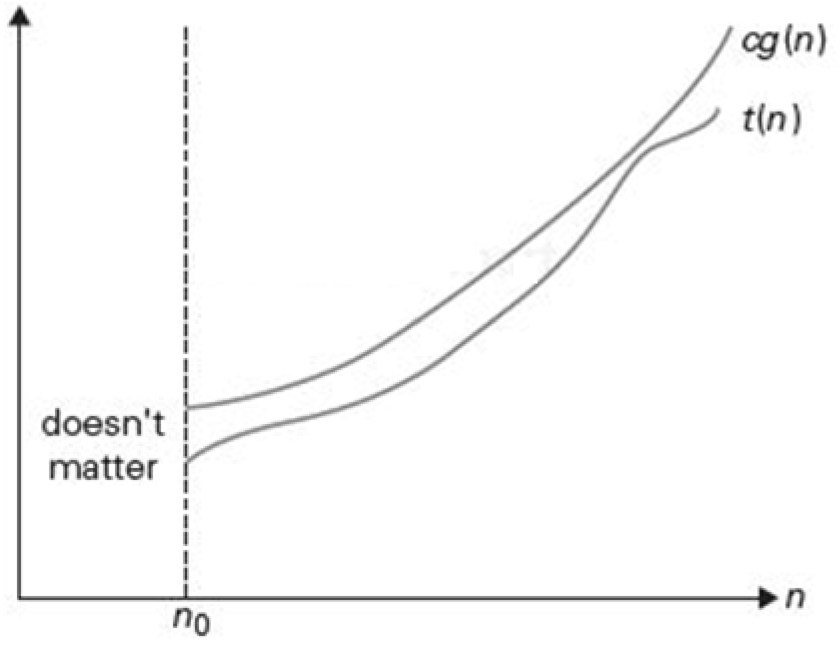


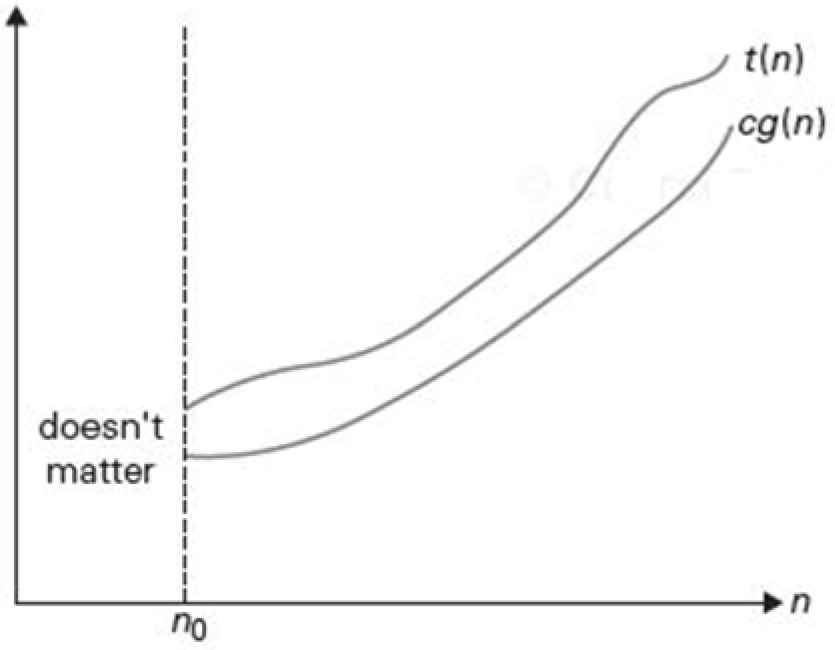
$2n \leq 5n$
for $n \geq 1$
$2n = O(n)$

$2n \geq n$
for $n \geq 1$
$2n = \Omega(n)$

As $2n = O(n)$
and $2n = \Omega(n)$,
we say
$2n = \Theta(n)$

Asymptotic Notations: Formal Intro





$t(n) = O(g(n))$

$t(n) \leq c^*g(n)$ for all $n \geq n_0$

c is a positive constant (> 0)
and n_0 is a non-negative integer

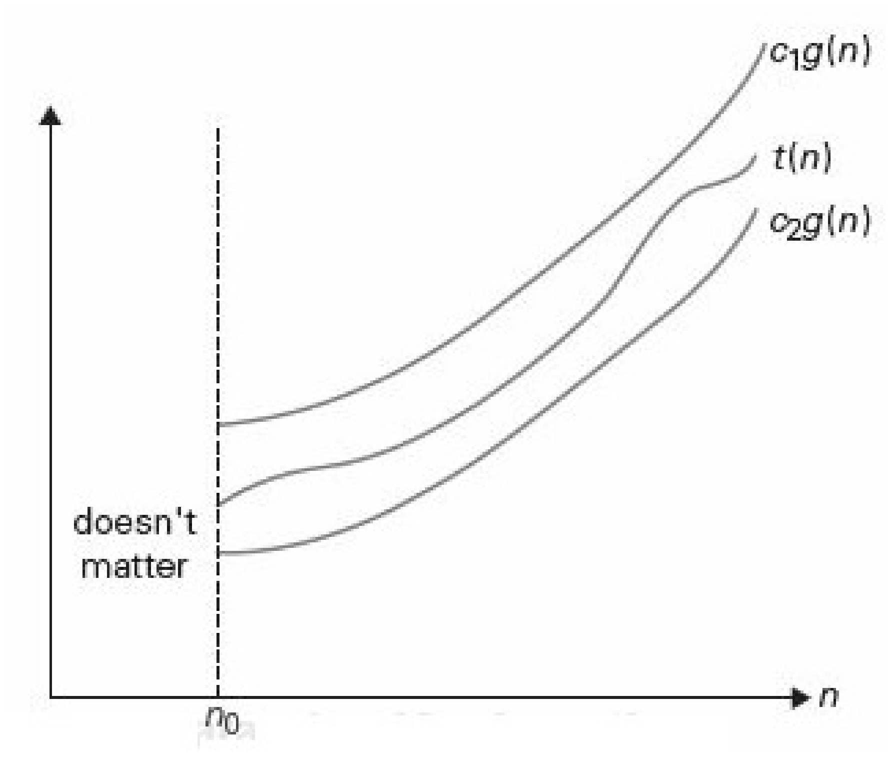
$t(n) = \Omega(g(n))$

$t(n) \geq c^*g(n)$ for all $n \geq n_0$

c is a positive constant (> 0)
and n_0 is a non-negative integer

Note: If $t(n) = O(g(n)) \Rightarrow g(n) = \Omega(t(n))$; also, if $t(n) = \Omega(g(n)) \Rightarrow g(n) = O(t(n))$

Asymptotic Notations: Formal Intro



$t(n) = \Theta(g(n))$

$c2 * g(n) \leq t(n) \leq c1 * g(n)$ for all $n \geq n_0$

c1 and c2 are positive constants (> 0)
and n_0 is a non-negative integer

Asymptotic Notations: Examples

- Let t(n) and g(n) be any non-negative functions defined on a set of all real numbers.
- We say t(n) = O(g(n)) for all functions t(n) that have a lower or the same order of growth as g(n), within a constant multiple as n → ∞.
    - **Examples:** $n \in O(n)$, $n \in O(n^2)$, $100n + 5 \in O(n^2)$, $\frac{1}{2}n(n-1) \in O(n^2)$
      $n^3 \notin O(n^2)$, $0.00001n^3 \notin O(n^2)$, $n^4 + n + 1 \notin O(n^2)$
- We say t(n) = Ω(g(n)) for all functions t(n) that have a higher or the same order of growth as g(n), within a constant multiple as n → ∞.
    - **Examples:** $n \in \Omega(n)$, $n^3 \in \Omega(n^2)$, $\frac{1}{2}n(n-1) \in \Omega(n^2)$, $100n + 5 \notin \Omega(n^2)$
- We say t(n) = Θ(g(n)) for all functions t(n) that have the same order of growth as g(n), within a constant multiple as n → ∞.
    - **Examples:** $an^2 + bn + c = \Theta(n^2)$;
      $n^2 + \log n = \Theta(n^2)$

Useful Property of Asymptotic Notations

- If $t_1(n) \in O(g_1(n))$ and $t_2(n) \in O(g_2(n))$, then

$$t_1(n) + t_2(n) \in O(\max\{g_1(n), g_2(n)\})$$

- If $t_1(n) \in \Theta(g_1(n))$ and $t_2(n) \in \Theta(g_2(n))$, then

$$t_1(n) + t_2(n) \in \Theta(\max\{g_1(n), g_2(n)\})$$

- The property can be applied for the Ω notation with a slight change: Replace the Max with the Min.
- If $t_1(n) \in \Omega(g_1(n))$ and $t_2(n) \in \Omega(g_2(n))$, then

$$t_1(n) + t_2(n) \in \Omega(\min\{g_1(n), g_2(n)\})$$

Using Limits to Compare Order of Growth

$$\lim_{n\to\infty}\frac{t(n)}{g(n)}=\begin{cases}0 & \text{implies that } t(n) \text{ has a smaller order of growth than } g(n),\\ c & \text{implies that } t(n) \text{ has the same order of growth as } g(n),\\ \infty & \text{implies that } t(n) \text{ has a larger order of growth than } g(n).\end{cases}$$

The first case means t(n) = O(g(n)
if the second case is true, then t(n) = Θ(g(n))
The last case means t(n) = Ω(g(n))

L'Hopital's Rule $$\lim_{n\to\infty}\frac{t(n)}{g(n)}=\lim_{n\to\infty}\frac{t'(n)}{g'(n)}$$

Note: t'(n) and g'(n) are first-order derivatives of t(n) and g(n)

Stirling's Formula $$n!\approx\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{for large values of } n$$

Example 1: To Determine the Order of Growth

Find the class $O(g(n))$, $\Theta(g(n))$ and $\Omega(g(n))$ for the following function:

(a) $(n^2+1)^{10}$.

$O(g(n))$: Let $g(n) = n^{21}$

$$\lim_{n\to\infty} \frac{(n^2+1)^{10}}{n^{21}} = \lim_{n\to\infty} \frac{(n^2+1)^{10}}{(n^2)^{10}\cdot n} = \lim_{n\to\infty} \frac{1}{n}\left[\frac{n^2+1}{n^2}\right]^{10} = \lim_{n\to\infty} \frac{1}{n}\left[1+\frac{1}{n^2}\right]^{10}$$

$= 0 * 1 = 0 \Rightarrow (n^2+1)^{10} = O(n^{21})$

Example 1: To Determine the Order of Growth (continued…)

(b) $\Theta(g(n))$: Let $g(n) = n^{20}$

$$\lim_{n\to\infty} \frac{(n^2+1)^{10}}{n^{20}} = \lim_{n\to\infty} \frac{(n^2+1)^{10}}{(n^2)^{10}}$$

$$= \lim_{n\to\infty} \left[\frac{n^2+1}{n^2}\right]^{10} = \lim_{n\to\infty} \left[1+\frac{1}{n^2}\right] = 1$$

$$\Rightarrow \underline{\underline{(n^2+1)^{10} = \Theta(n^{20})}}$$

(c) $\Omega(g(n))$: Let $g(n) = n^{10}$.

$$\lim_{n\to\infty} \frac{(n^2+1)^{10}}{n^{10}} = \lim_{n\to\infty} \left[\frac{n^2+1}{n}\right]^{10} = \lim_{n\to\infty} \left[n+\frac{1}{n}\right]^{10} = \lim_{n\to\infty} n^{10} = \infty$$

$$(n^2+1)^{10} = \Omega(n^{10})$$

Example 2: To Determine the Order of Growth

Find the class $O(g(n))$, $\Theta(g(n))$ and $\Omega(g(n))$ for the following functions:

(b) $\sqrt{3n^2+7n+4}$

$\xrightarrow{} t(n)$

<u>$O(g(n))$:</u> $\boxed{\sqrt{3n^2+7n+4}} \simeq \sqrt{n^2} = n.$

Pick $g(n) = n^2 = \sqrt{n^4}$ [degree of $g(n)$ greater than $t(n)$]

$$\lim_{n\to\infty} \frac{\sqrt{3n^2+7n+4}}{\sqrt{n^4}} = \lim_{n\to\infty} \sqrt{\frac{3n^2+7n+4}{n^4}} = \lim_{n\to\infty} \sqrt{\frac{3}{n^2}+\frac{7}{n^3}+\frac{4}{n^4}}$$

$$= \underline{0}$$

$$\Rightarrow \sqrt{3n^2+7n+4} = \underline{O(n^2)}$$

Example 2: To Determine the Order of Growth (continued…)

$\Theta(g(n))$:

Pick $g(n)$ to be of the same degree as $t(n) = \sqrt{3n^2+7n+4}$
$\approx \sqrt{n^2} = \underline{n}$.

$g(n) = \sqrt{n^2} = n$.

$$\lim_{n\to\infty} \frac{\sqrt{3n^2+7n+4}}{\sqrt{n^2}} = \lim_{n\to\infty} \sqrt{\frac{3n^2+7n+4}{n^2}} = \lim_{n\to\infty} \sqrt{3+\frac{7}{n}+\frac{4}{n^2}}$$

$$= \underline{\sqrt{3}}$$

$\underline{\sqrt{3n^2+7n+4} = \Theta(n)}$

$\Omega(g(n))$:

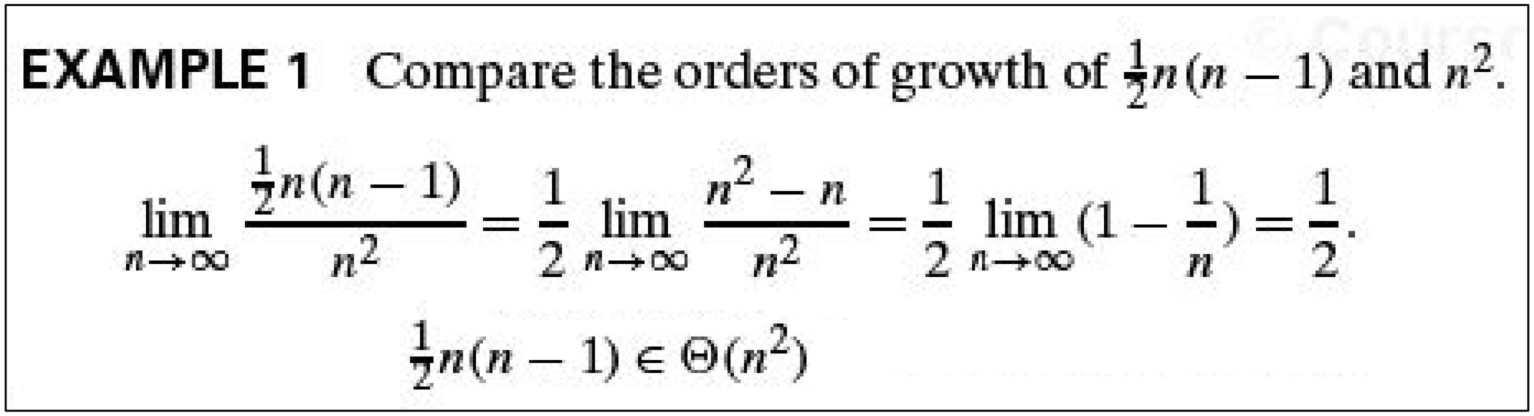
Pick $g(n)$ to be of a lower degree than $t(n) = \sqrt{3n^2+7n+4}$
$\approx n$.

So, pick $g(n) = n^{1/2} = \sqrt{n}$

$$\lim_{n\to\infty} \frac{\sqrt{3n^2+7n+4}}{\sqrt{n}} = \lim_{n\to\infty} \sqrt{\frac{3n^2+7n+4}{n}} = \lim_{n\to\infty} \sqrt{3n+7+\frac{4}{n}} = \underline{\infty}.$$

So, $\underline{\sqrt{3n^2+7n+4} = \Omega(\sqrt{n})}$

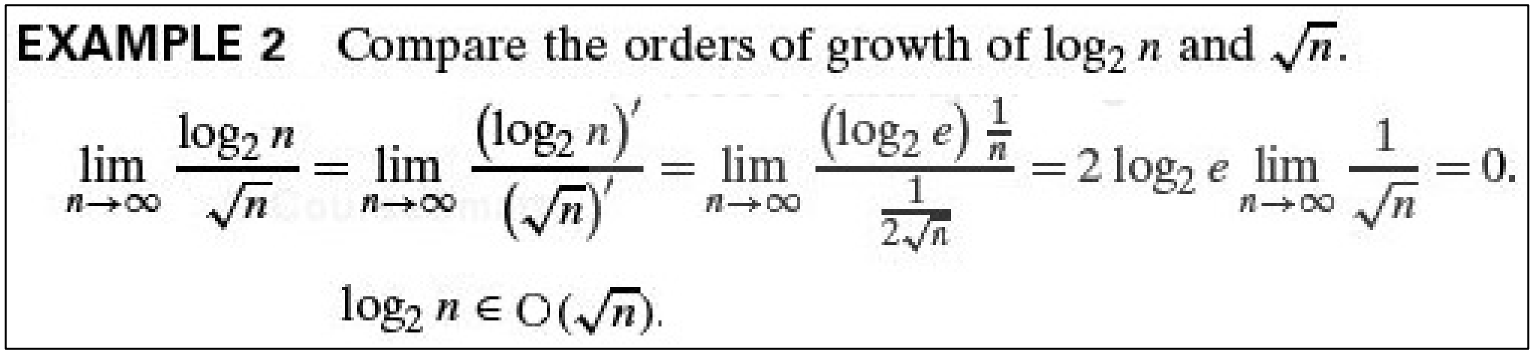
Examples to Compare the Order of Growth



EXAMPLE 1 Compare the orders of growth of $\frac{1}{2}n(n-1)$ and n^2.

$$\lim_{n\to\infty}\frac{\frac{1}{2}n(n-1)}{n^2}=\frac{1}{2}\lim_{n\to\infty}\frac{n^2-n}{n^2}=\frac{1}{2}\lim_{n\to\infty}(1-\frac{1}{n})=\frac{1}{2}.$$

$$\frac{1}{2}n(n-1)\in\Theta(n^2)$$



EXAMPLE 2 Compare the orders of growth of $\log_2 n$ and $\sqrt{n}$.

$$\lim_{n\to\infty}\frac{\log_2 n}{\sqrt{n}}=\lim_{n\to\infty}\frac{(\log_2 n)'}{(\sqrt{n})'}=\lim_{n\to\infty}\frac{(\log_2 e)\frac{1}{n}}{\frac{1}{2\sqrt{n}}}=2\log_2 e\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0.$$

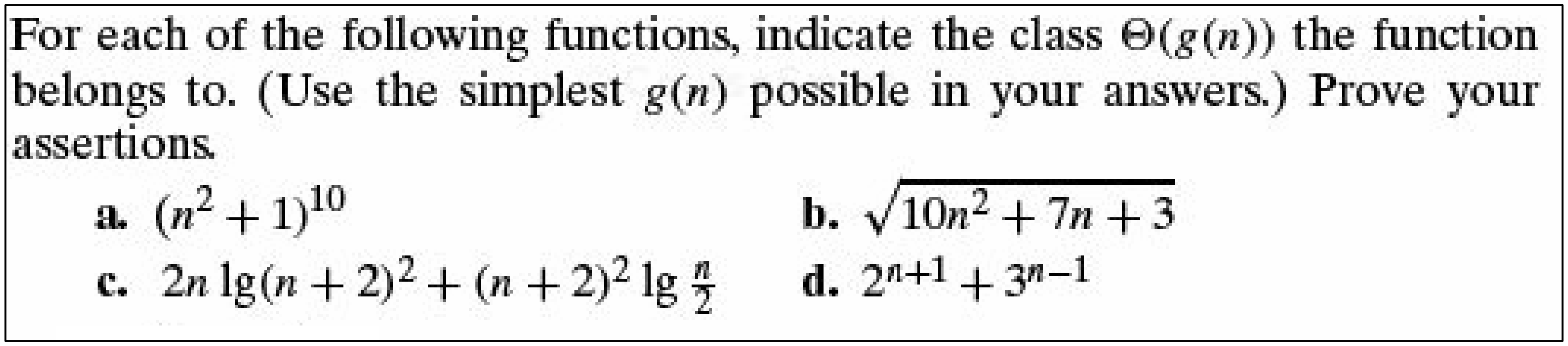
$$\log_2 n\in O(\sqrt{n}).$$

Example 3: Compare the order of growth of $\log^2 n$ and $\log n^2$.

$$\lim_{n->\infty}\frac{\log^2 n}{\log n^2}=\lim_{n->\infty}\frac{\log n*\log n}{\log(n*n)}=\lim_{n->\infty}\frac{\log n*\log n}{\log n+\log n}=\lim_{n->\infty}\frac{\log n*\log n}{2\log n}=\lim_{n->\infty}\frac{\log n}{2}=\infty$$

Hence, $\log^2 n = \Omega(\log n^2)$ That is, $\log n^2 = O(\log^2 n)$

Some More Examples: Order of Growth



For each of the following functions, indicate the class $\Theta(g(n))$ the function belongs to. (Use the simplest $g(n)$ possible in your answers.) Prove your assertions.

a. $(n^2+1)^{10}$ **b.** $\sqrt{10n^2+7n+3}$

c. $2n\lg(n+2)^2+(n+2)^2\lg\frac{n}{2}$ **d.** $2^{n+1}+3^{n-1}$

- a) $(n^2+1)^{10}$: Informally, = $(n^2+1)^{10} \approx n^{20}$.

Formally,
$$\lim_{n\to\infty}\frac{(n^2+1)^{10}}{n^{20}} = \lim_{n\to\infty}\frac{(n^2+1)^{10}}{(n^2)^{10}} = \lim_{n\to\infty}\left(\frac{n^2+1}{n^2}\right)^{10} == \lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{10} = 1.$$

Hence $(n^2+1)^{10} \in \Theta(n^{20})$.

b) Informally, $\sqrt{10n^2+7n+3} \approx \sqrt{10n^2} = \sqrt{10}n \in \Theta(n)$. Formally,

$$\lim_{n\to\infty}\frac{\sqrt{10n^2+7n+3}}{n} = \lim_{n\to\infty}\sqrt{\frac{10n^2+7n+3}{n^2}} = \lim_{n\to\infty}\sqrt{10+\frac{7}{n}+\frac{3}{n^2}} = \sqrt{10}.$$

Hence $\sqrt{10n^2+7n+3} \in \Theta(n)$.

c) $2n\lg(n+2)^2 + (n+2)^2\lg\frac{n}{2} = 2n2\lg(n+2) + (n+2)^2(\lg n - 1)$
$\in \Theta(n\lg n) + \Theta(n^2\lg n) = \Theta(n^2\lg n)$.

d) $\boxed{2^{n+1}+3^{n-1} = 2^n 2 + 3^n\frac{1}{3} \in \Theta(2^n)+\Theta(3^n) = \Theta(3^n)}$

Some More Examples: Order of Growth

List the following functions according to their order of growth from the lowest to the highest:

$$(n-2)!,\quad 5\lg(n+100)^{10},\quad 2^{2n},\quad 0.001n^4+3n^3+1,\quad \ln^2 n,\quad \sqrt[3]{n},\quad 3^n$$

$$\boxed{(n-2)! \in \Theta((n-2)!)} \quad \boxed{5\lg(n+100)^{10} = 50\lg(n+100) \in \Theta(\log n)}$$

$$\boxed{2^{2n} = (2^2)^n \in \Theta(4^n)} \quad \boxed{0.001n^4+3n^3+1 \in \Theta(n^4)} \quad \boxed{\ln^2 n \in \Theta(\log^2 n)}$$

$$\boxed{\sqrt[3]{n} \in \Theta(n^{\frac{1}{3}})} \quad \boxed{3^n \in \Theta(3^n)}$$

The listing of the functions in the increasing Order of growth is as follows:

$$5\lg(n+100)^{10},\quad \ln^2 n,\quad \sqrt[3]{n},\quad 0.001n^4+3n^3+1,\quad 3^n,\quad 2^{2n},\quad (n-2)!$$

$$\lim_{n\to\infty}\frac{\log^2 n}{n^{1/3}} = \lim_{n\to\infty}\frac{2\log n}{n*(1/3)n^{(-2/3)}} = \frac{6\log n}{n^{(1/3)}} = \frac{6}{n*(1/3)n^{(-2/3)}} = \lim_{n\to\infty}\frac{18}{n^{(1/3)}} = 0$$

Hence, $\log^2 n = O(n^{1/3})$

Time Efficiency of Non-recursive Algorithms: *General Plan for Analysis*

- Decide on parameter *n* indicating *input size*
- Identify algorithm's *basic operation*
- Determine *worst*, *average*, and *best* cases for input of size *n*, if the number of times the basic operation gets executed varies with specific instances (inputs)
- Set up a sum for the number of times the basic operation is executed
- Simplify the sum using standard formulas and rules

Useful Summation Formulas and Rules

$\Sigma_{l\le i\le u} 1 = 1+1+\ldots+1 = u - l + 1$

In particular, $\Sigma_{1\le i\le n} 1 = n - 1 + 1 = n \in \Theta(n)$

$\Sigma_{1\le i\le n}\, i = 1+2+\ldots+n = n(n+1)/2 \approx n^2/2 \in \Theta(n^2)$

$\Sigma_{1\le i\le n}\, i^2 = 1^2+2^2+\ldots+n^2 = n(n+1)(2n+1)/6 \approx n^3/3 \in \Theta(n^3)$

$\Sigma_{0\le i\le n}\, a^i = 1 + a + \ldots + a^n = (a^{n+1} - 1)/(a - 1)$ for any $a \neq 1$

In particular, $\Sigma_{0\le i\le n}\, 2^i = 2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1 \in \Theta(2^n)$

$\Sigma(a_i \pm b_i) = \Sigma a_i \pm \Sigma b_i \qquad \Sigma c a_i = c\Sigma a_i \qquad \Sigma_{l\le i\le u} a_i = \Sigma_{l\le i\le m} a_i + \Sigma_{m+1\le i\le u} a_i$

$$\sum_{i=l}^{u} 1 = (u - l + 1)$$

Examples on Summation

- 1 + 3 + 5 + 7 + …. + 999

$$= [1 + 2 + 3 + 4 + 5 + \ldots .\ 999] - [2 + 4 + 6 + 8 + \ldots + 998]$$

$$= \frac{999 * 1000}{2} - 2[1 + 2 + 3 + \ldots . + 499]$$

$$= 999 * 500 - 2\left[\frac{499 * 500}{2}\right] = 999 * 500 - 499 * 500$$

$$= 500*(999\text{-}499) = 500*500 = 250{,}000$$

- 2 + 4 + 8 + 16 + … + 1024

$$= 2^1 + 2^2 + 2^3 + 2^4 + \ldots + 2^{10}$$

$$= [2^0 + 2^1 + 2^2 + 2^3 + 2^4 + \ldots + 2^{10}] - 1$$

$$= \left[\sum_{i=0}^{10} 2^i\right] - 1 = \left[2^{11} - 1\right] - 1 = \mathbf{2046}$$

$$\sum_{i=3}^{n+1} 1 = [(n+1) - 3 + 1] = n+1 - 2 = n-1 = \Theta(n)$$

$$\sum_{i=3}^{n+1} i = 3 + 4 + ... + (n+1) = [1 + 2 + 3 + 4 + ... + (n+1)] - [1 + 2]$$

$$= \frac{(n+1)(n+2)}{2} - 3 = \Theta(n^2) - \Theta(1) = \Theta(n^2)$$

$$\sum_{i=0}^{n-1} i(i+1) = \sum_{i=0}^{n-1} i^2 + i = \sum_{i=0}^{n-1} i^2 + \sum_{i=0}^{n-1} i$$

$$= \left[\frac{[n-1][(n-1)+1][2(n-1)+1]}{6}\right] + \left[\frac{[n-1][(n-1)+1]}{2}\right]$$

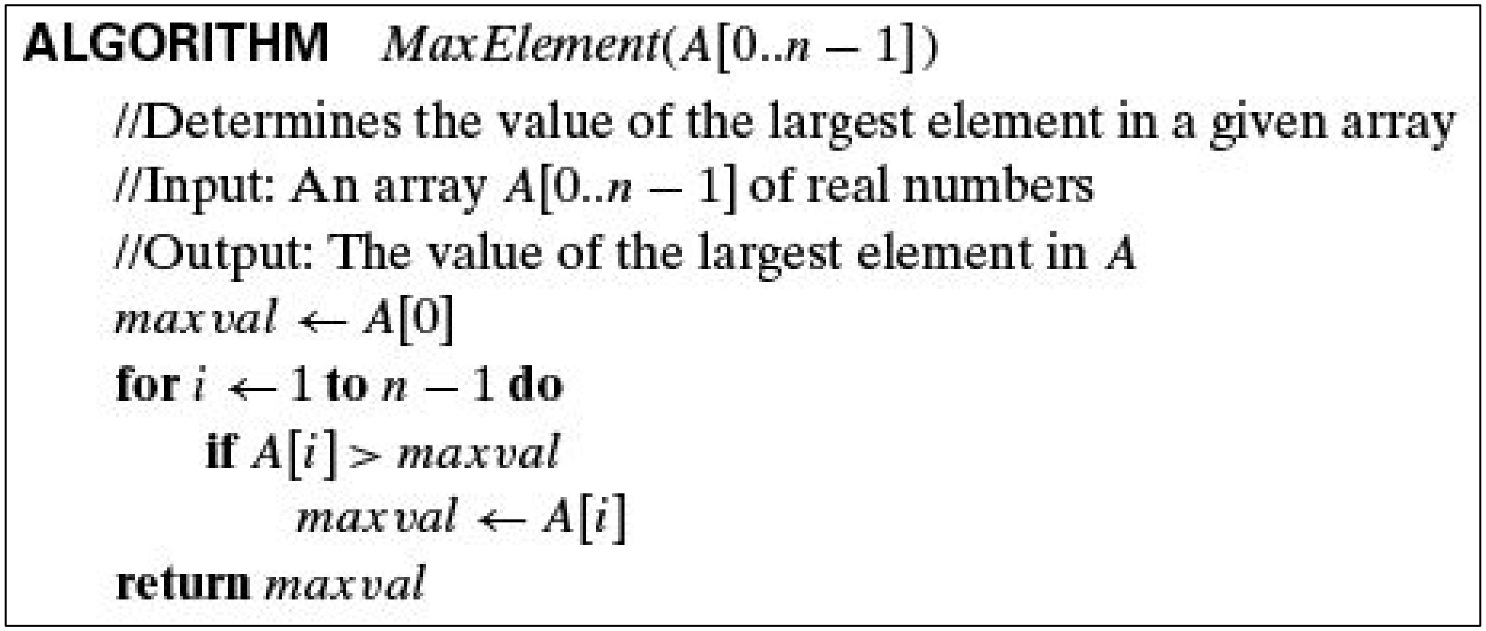
$$= \left[\frac{[n-1][n][2n-1]}{6}\right] + \left[\frac{[n-1][n]}{2}\right]$$

$$= \Theta(n^3) + \Theta(n^2) = \Theta(n^3)$$

$$\sum_{i=0}^{n-1}(i^2+1)^2 = \sum_{i=0}^{n-1}(i^4+2i^2+1) = \sum_{i=0}^{n-1} i^4 + 2\sum_{i=0}^{n-1} i^2 + \sum_{i=0}^{n-1} 1$$

$$\in \Theta(n^5) + \Theta(n^3) + \Theta(n) = \Theta(n^5)$$

Example 1: Finding Max. Element



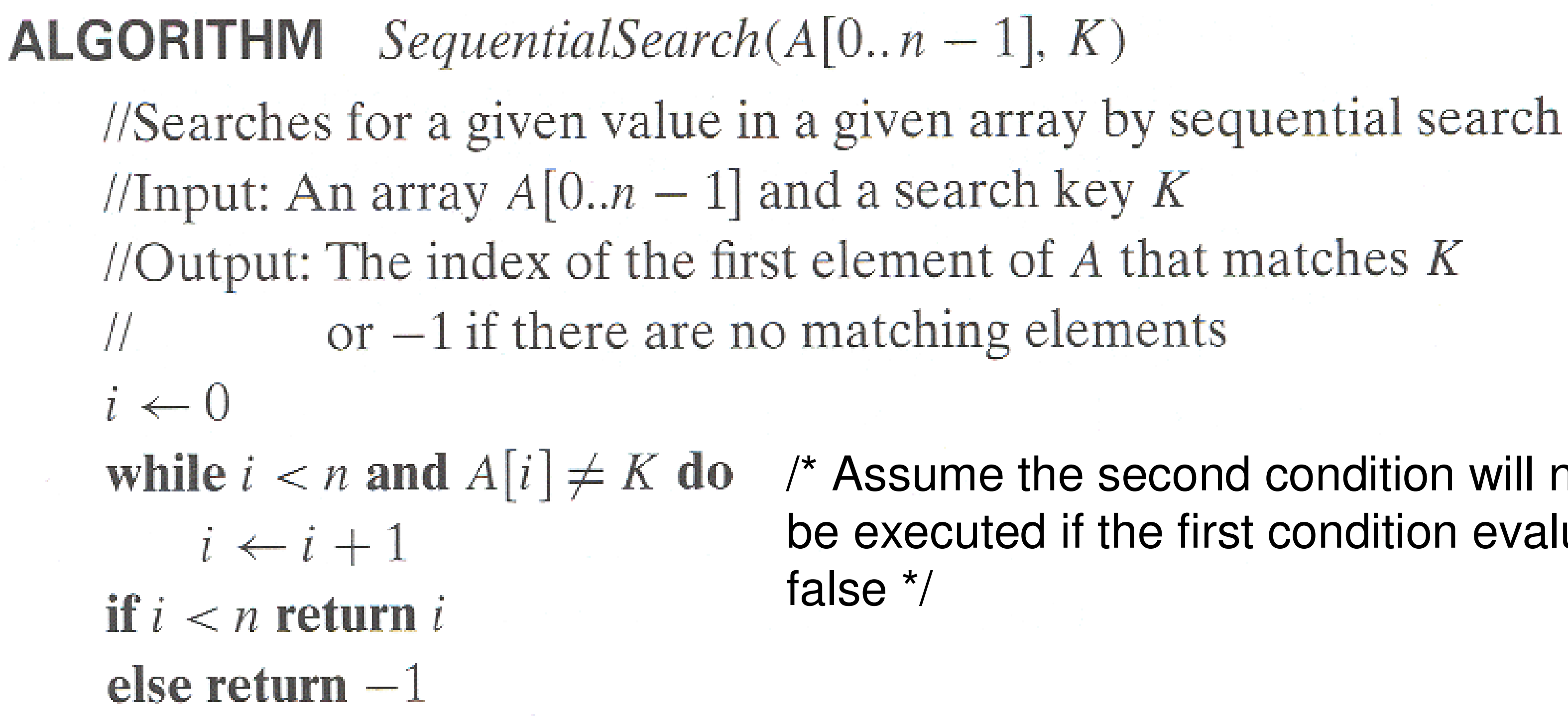
```
ALGORITHM  $MaxElement(A[0..n-1])$
    //Determines the value of the largest element in a given array
    //Input: An array $A[0..n-1]$ of real numbers
    //Output: The value of the largest element in $A$
    $maxval \leftarrow A[0]$
    for $i \leftarrow 1$ to $n-1$ do
        if $A[i] > maxval$
            $maxval \leftarrow A[i]$
    return $maxval$
```

- The basic operation is the comparison executed on each repetition of the loop.
- In this algorithm, the number of comparisons is the same for all arrays of size n.
- The algorithm makes one comparison on each execution of the loop, which is repeated for each value of the loop's variable i within the bounds 1 and n-1 (inclusively). Hence,

$$C(n) = \sum_{i=1}^{n-1} 1 = n - 1 \in \Theta(n)$$

Note: Best case = Worst case for this problem

Example 2: Sequential Key Search

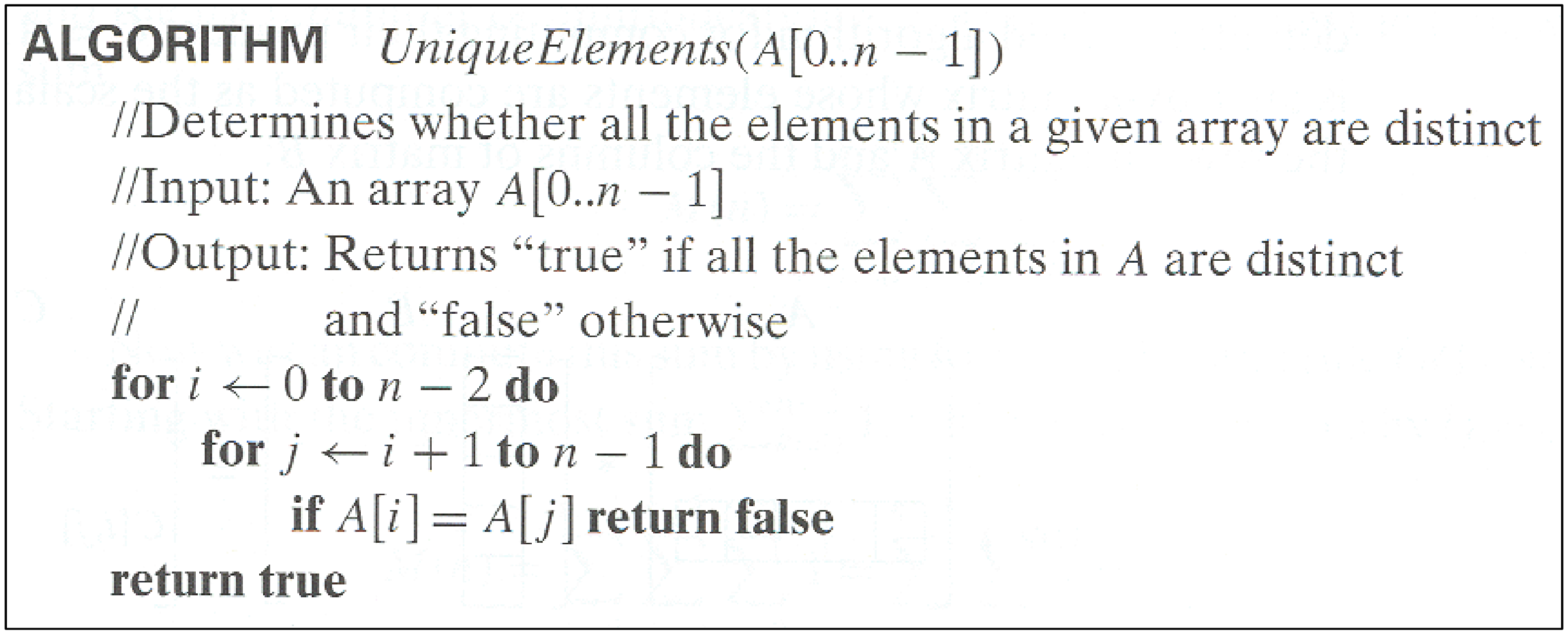


```
ALGORITHM SequentialSearch($A[0..n-1]$, $K$)
    //Searches for a given value in a given array by sequential search
    //Input: An array $A[0..n-1]$ and a search key $K$
    //Output: The index of the first element of $A$ that matches $K$
    //        or $-1$ if there are no matching elements
    $i \leftarrow 0$
    while $i < n$ and $A[i] \neq K$ do
        $i \leftarrow i + 1$
    if $i < n$ return $i$
    else return $-1$
```

/* Assume the second condition will not be executed if the first condition evaluates to false */

- Worst-Case: $C_{worst}(n) = n = \Theta(n)$
- Best-Case: $C_{best}(n) = 1$

Example 3: Element Uniqueness Problem



```
ALGORITHM UniqueElements(A[0..n − 1])
    //Determines whether all the elements in a given array are distinct
    //Input: An array A[0..n − 1]
    //Output: Returns "true" if all the elements in A are distinct
    //          and "false" otherwise
    for $i \leftarrow 0$ to $n - 2$ do
        for $j \leftarrow i + 1$ to $n - 1$ do
            if $A[i] = A[j]$ return false
    return true
```

<u>Best-case situation:</u>

If the two first elements of the array are the same, then we can exit after one comparison. Best case = 1 comparison.

<u>Worst-case situation:</u>

- The basic operation is the comparison in the inner loop. The worst-case happens for two-kinds of inputs:
  – Arrays with no equal elements
  – Arrays in which only the last two elements are the pair of equal elements

Example 3: Element Uniqueness Problem

- For these kinds of inputs, one comparison is made for each repetition of the innermost loop, i.e., for each value of the loop's variable j between its limits i+1 and n-1; and this is repeated for each value of the outer loop i.e., for each value of the loop's variable i between its limits 0 and n-2. Accordingly, we get,

$$C_{worst}(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} [(n-1) - (i+1) + 1] = \sum_{i=0}^{n-2} (n-1-i)$$

$$= \sum_{i=0}^{n-2} (n-1) - \sum_{i=0}^{n-2} i = (n-1) \sum_{i=0}^{n-2} 1 - \frac{(n-2)(n-1)}{2}$$

$$= (n-1)^2 - \frac{(n-2)(n-1)}{2} = \frac{(n-1)n}{2} \approx \frac{1}{2} n^2 = \Theta(n^2)$$

Example 4: Insertion Sort

- Given an array A[0…n-1], at any time, we have the array divided into two parts: A[0,…,i-1] and A[i…n-1].
  - The A[0…i-1] is the sorted part and A[i…n-1] is the unsorted part.
  - In any iteration, we pick an element *v* = A[i] and scan through the sorted sequence A[0…i-1] to insert *v* at the appropriate position.
    - The scanning is proceeded from right to left (i.e., for index j running from i-1 to 0) until we find the right position for *v*.
    - During this scanning process, v = A[i] is compared with A[j].
    - If A[j] > v, then we v has to be placed somewhere before A[j] in the final sorted sequence. So, A[j] cannot be at its current position (in the final sorted sequence) and has to move at least one position to the right. So, we copy A[j] to A[j+1] and decrement the index j, so that we now compare v with the next element to the left.

$$\underbrace{A[0] \le \cdots \le A[j]}_{\text{smaller than or equal to } A[i]} < \underbrace{A[j+1] \le \cdots \le A[i-1]}_{\text{greater than } A[i]} \;\Big|\; A[i] \cdots A[n-1]$$

    - If A[j] ≤ v, we have found the right position for v; we copy v to A[j+1]. This also provides the stable property, in case v = A[j].

Insertion Sort Pseudo Code and Analysis

```
ALGORITHM  InsertionSort($A[0..n-1]$)
    //Sorts a given array by insertion sort
    //Input: An array $A[0..n-1]$ of $n$ orderable elements
    //Output: Array $A[0..n-1]$ sorted in nondecreasing order
    for $i \leftarrow 1$ to $n-1$ do
        $v \leftarrow A[i]$
        $j \leftarrow i-1$
        while $j \geq 0$ and $A[j] > v$ do
            $A[j+1] \leftarrow A[j]$
            $j \leftarrow j-1$
        $A[j+1] \leftarrow v$
```

Best Case (if the array is already sorted): the element v at A[i] will be just compared with A[i-1] and since A[i-1] ≤ A[i] = v, we retain v at A[i] itself and do not scan the rest of the sequence A[0…i-1]. There is only one comparison for each value of index i.

$$\sum_{i=1}^{n-1} 1 = (n-1) - 1 + 1 = (n-1)$$

= Θ(n)

The comparison A[j] > v is the basic operation.

Worst Case (if the array is reverse-sorted): the element v at A[i] has to be moved all the way to index 0, by scanning through the entire sequence A[0…i-1].

$$\sum_{i=1}^{n-1}\sum_{j=i-1}^{0} 1 = \sum_{i=1}^{n-1}\sum_{j=0}^{i-1} 1 = \sum_{i=1}^{n-1} (i-1) - 0 + 1 = \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$$

= Θ(n^2)

Insertion Sort: Analysis and Example

Average Case: On average for a random input sequence, we would be visiting half of the sorted sequence A[0…i-1] to put A[i] at the proper position.

$$C(n) = \sum_{i=1}^{n-1} \sum_{j=i-1}^{(i-1)/2} 1 = \sum_{i=1}^{n-1} \frac{(i-1)}{2} + 1 = \sum_{i=1}^{n-1} \frac{(i+1)}{2} = \Theta(n^2)$$

Example: Given sequence (also initial): **45** 23 8 12 90 21

Index -1

Iteration 1 (v = 23):

45 **45** 8 12 90 21

23 **45** 8 12 90 21

Iteration 2 (v = 8):

23 **45** **45** 12 90 21

23 **23** **45** 12 90 21

8 **23** **45** 12 90 21

Iteration 3 (v = 12):

8 **23** **45** **45** 90 21

8 **23** **23** **45** 90 21

8 **12** **23** **45** 90 21

Iteration 4 (v = 90):

8 **12** **23** **45** 90 21

9 **12** **23** **45** **90** 21

Iteration 5 (v = 21):

9 **12** **23** **45** **90** **90**

9 **12** **23** **45** **45** **90**

9 **12** **23** **23** **45** **90**

9 **12** **21** **23** **45** **90**

The **colored** elements are in the sorted sequence and the circled element is at index *j* of the algorithm.

Time Efficiency of Recursive Algorithms: *General Plan for Analysis*

- Decide on a parameter indicating an input’s size.
- Identify the algorithm’s basic operation.
- Check whether the number of times the basic op. is executed may vary on different inputs of the same size. (If it may, the worst, average, and best cases must be investigated separately.)
- Set up a recurrence relation with an appropriate initial condition expressing the number of times the basic op. is executed.
- Solve the recurrence (or, at the very least, establish its solution’s order of growth) by backward substitutions or another method.

Recursive Evaluation of n!

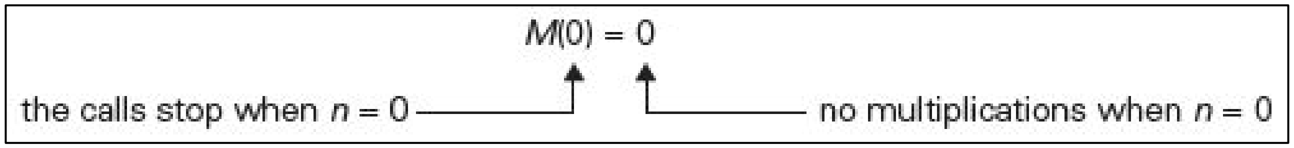
Definition: $n! = 1 * 2 * \ldots * (n-1) * n$ for $n \geq 1$ and $0! = 1$

- Recursive definition of $n!$: $F(n) = F(n-1) * n$ for $n \geq 1$ and
  $F(0) = 1$

```
ALGORITHM  F(n)
    //Computes n! recursively
    //Input: A nonnegative integer n
    //Output: The value of n!
    if n = 0 return 1
    else return F(n − 1) ∗ n
```

$$M(n) = \underbrace{M(n-1)}_{\substack{\text{to compute}\\ F(n-1)}} + \underbrace{1}_{\substack{\text{to multiply}\\ F(n-1)\text{ by } n}} \quad \text{for } n > 0.$$

$$M(n) = M(n-1) + 1 \quad \text{for } n > 0,$$
$$M(0) = 0.$$



M(n-1) = M(n-2) + 1;   **M(n-2) = M(n-3)+1**

M(n) = [M(n-2)+1] + 1 = M(n-2) + 2 = [M(n-3)+1+2] = M(n-3) + 3
= M(n-n) + n = n

Overall time Complexity: Θ(n)

YouTube Link: https://www.youtube.com/watch?v=K25MWuKKYAY

Counting the # Bits of an Integer

```
ALGORITHM  BinRec(n)
    //Input: A positive decimal integer n
    //Output: The number of binary digits in n's binary representation
    if $n = 1$ return 1
    else return BinRec($\lfloor n/2 \rfloor$) + 1
```

# bits (n) = # bits($\lfloor n/2 \rfloor$) + 1; for n > 1
# bits (1) = 1

1	1 bit
2-3	2 bits
4-7	3 bits
8-15	4 bits
16-31	5 bits
32-63	6 bits

Either Division or Addition could be considered the Basic operation, as both are executed once for each recursion. We will treat "addition" as the basic operation.

Let A(n) be the number of additions needed to compute # bits(n)

# Additions $A(n) = A(\lfloor n/2 \rfloor) + 1 \quad \text{for } n > 1.$

Since the recursive calls end when *n* is equal to 1 and there are no additions made, the initial condition is: A(1) = 0.

Counting the # Bits of an Integer

Solution Approach: If we use the backward substitution method (as we did in the previous two examples, we will get stuck for values of n that are not powers of 2).

We proceed by setting n = 2^k for k ≥ 0.

New recurrence relation to solve:

$$\boxed{\begin{aligned} &A(2^k) = A(2^{k-1}) + 1 \quad \text{for } k > 0, \\ &A(2^0) = 0. \end{aligned}}$$

$$\begin{aligned}
A(2^k) &= A(2^{k-1}) + 1 && \text{substitute } A(2^{k-1}) = A(2^{k-2}) + 1 \\
&= [A(2^{k-2}) + 1] + 1 = A(2^{k-2}) + 2 && \text{substitute } A(2^{k-2}) = A(2^{k-3}) + 1 \\
&= [A(2^{k-3}) + 1] + 2 = A(2^{k-3}) + 3 && \dots \\
&\dots \\
&= A(2^{k-i}) + i \\
&\dots \\
&= A(2^{k-k}) + k.
\end{aligned}$$

$$\boxed{A(n) = \log_2 n \in \Theta(\log n).}$$

Examples for Solving Recurrence Relations

X(n) = X(n-1) + 5, for n > 1, X(1) = 0

$$\begin{aligned} x(n) &= x(n-1)+5 \\ &= [x(n-2)+5]+5 = x(n-2)+5\cdot 2 \\ &= [x(n-3)+5]+5\cdot 2 = x(n-3)+5\cdot 3 \\ &= \ldots \\ &= x(n-i)+5\cdot i \\ &= \ldots \\ &= x(1)+5\cdot(n-1) = 5(n-1). \end{aligned}$$

$$= \Theta(n)$$

X(n) = 3*X(n-1) for n > 1, X(1) = 4

$$\begin{aligned} x(n) &= 3x(n-1) \\ &= 3[3x(n-2)] = 3^2x(n-2) \\ &= 3^2[3x(n-3)] = 3^3x(n-3) \\ &= \ldots \\ &= 3^ix(n-i) \\ &= \ldots \\ &= 3^{n-1}x(1) = 4\cdot 3^{n-1}. \end{aligned}$$

$$= (4/3)3^n = \Theta(3^n)$$

X(n) = X(n/3) + 1 for n > 1, X(1) = 1 [Solve for n = 3^k]

$$\begin{aligned}
x(3^k) &= x(3^{k-1}) + 1 \\
&= [x(3^{k-2}) + 1] + 1 = x(3^{k-2}) + 2 \\
&= [x(3^{k-3}) + 1] + 2 = x(3^{k-3}) + 3 \\
&= \ldots \\
&= x(3^{k-i}) + i \\
&= \ldots \\
&= x(3^{k-k}) + k = x(1) + k = 1 + \log_3 n.
\end{aligned}$$

X(n) = Θ(logn)

Master Theorem to Solve Recurrence Relations

- Assuming that size n is a power of b to simplify analysis, we have the following recurrence for the running time, T(n) = a T(n/b) + f(n)
  - where f(n) is a function that accounts for the time spent on dividing an instance of size n into instances of size n/b and combining their solutions.
- Master Theorem:

If $f(n) \in \Theta(n^d)$ where $d \geq 0$, then

$$T(n) \in \begin{cases} \Theta(n^d) & \text{if } a < b^d, \\ \Theta(n^d \log n) & \text{if } a = b^d, \\ \Theta(n^{\log_b a}) & \text{if } a > b^d. \end{cases}$$

The same results hold good for O and Ω too.

Examples:

1) T(n) = 4T(n/2) + n
a = 4; b = 2; d = 1 ➔ a > b^d

$$T(n) = \Theta\left(n^{\log_2 4}\right) = \Theta(n^2)$$

2) T(n) = 4T(n/2) + n^2
a = 4; b = 2; d = 2 ➔ a = b^d

$$T(n) = \Theta\left(n^2 \log n\right)$$

3) T(n) = 4T(n/2) + n^3
a = 4; b = 2; d = 3 ➔ a < b^d

$$T(n) = \Theta\left(n^3\right)$$

4) T(n) = 2T(n/2) + 1
a = 2; b = 2; d = 0 ➔ a > b^d

$$T(n) = \Theta\left(n^{\log_2 2}\right) = \Theta(n)$$

Master Theorem: More Problems

$T(n) = 3T(n/3) + \sqrt{n}$

$T(n) = 3T(n/3) + n^{(1/2)}$

$a = 3; b = 3; d = 1/2$

$b^d = 3^{1/2} = 1.732$

$a = 3 > b^d = 1.732$

$T(n) = \Theta(n^{\log_3 3}) = \Theta(n)$

$T(n) = 4T(n/2) + \log n$

$a = 4; b = 2; d < 1$, because $\log n < n^1$

$b^d = 2^{<1} < 2$

$a > b^d$

$T(n) = \Theta(n^{\log_2 4}) = \Theta(n^2)$

$T(n) = 6T(n/3) + n^2 \log n$

$a = 6; b = 3; 2 < d < 3$, because $\log n < n$ and hence $n^2 \log n < n^3$

$b^d = 3^{2<d<3} > 9 > a$

$a < b^d$

Hence, $T(n) = \Theta(n^d) = \Theta(n^2 \log n)$

Space-Time Tradeoff

In-place vs. Out-of-place Algorithms

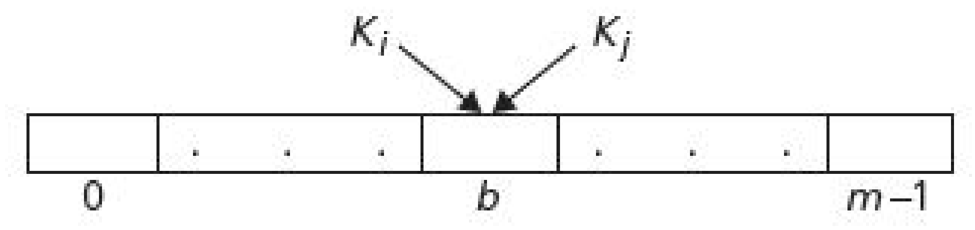
- An algorithm is said to be “in-place” if it uses a minimum and/or constant amount of extra storage space to transform or process an input to obtain the desired output.
  - Depending on the nature of the problem, an in-place algorithm may sometime overwrite an input to the desired output as the algorithm executes (as in the case of in-place sorting algorithms); the output space may sometimes be a constant (for example in the case of string-matching algorithms).
- Algorithms that use significant amount of extra storage space (sometimes, additional space as large as the input – example: merge sort) are said to be out-of-place in nature.
- Time-Space Complexity Tradeoffs of Sorting Algorithms:
  - In-place sorting algorithms like Selection Sort, Bubble Sort, Insertion Sort and Quick Sort have a worst-case time complexity of $\Theta(n^2)$.
  - On the other hand, Merge sort has a space-complexity of $\Theta(n)$, but has a worst-case time complexity of $\Theta(n\log n)$.

Hashing

- A very efficient method for implementing a *dictionary,* i.e., a set with the operations: find, insert and delete
- Based on representation-change and space-for-time tradeoff ideas
- We consider the problem of implementing a dictionary of *n* records with keys K_1, K_2, …, K_n.
- Hashing is based on the idea of distributing keys among a one-dimensional array H[0…m-1] called a hash table.
  - The distribution is done by computing, for each of the keys, the value of some pre-defined function *h* called the ***hash function***.
  - The hash function assigns an integer between 0 and *m*-1, called the hash address to a key.
  - The size of a hash table *m* is typically a prime integer.
- Typical hash functions
  - For non-negative integers as key, a hash function could be h(K)=K mod m;
  - If the keys are letters of some alphabet, the position of the letter in the alphabet (for example, A is at position 1 in alphabet A – Z) could be used as the key for the hash function defined above.
  - If the key is a character string $c_0\, c_1\, \ldots\, c_{s-1}$ of characters from an alphabet, then, the hash function could be: $\left(\sum_{i=0}^{s-1} ord(c_i)\right) \bmod m$

Collisions and Collision Resolution

If $h(K_1) = h(K_2)$, there is a *collision*



- Good hash functions result in fewer collisions but some collisions should be expected
- In this module, we will look at open hashing that works for arrays of any size, irrespective of the hash function.

Open Hashing

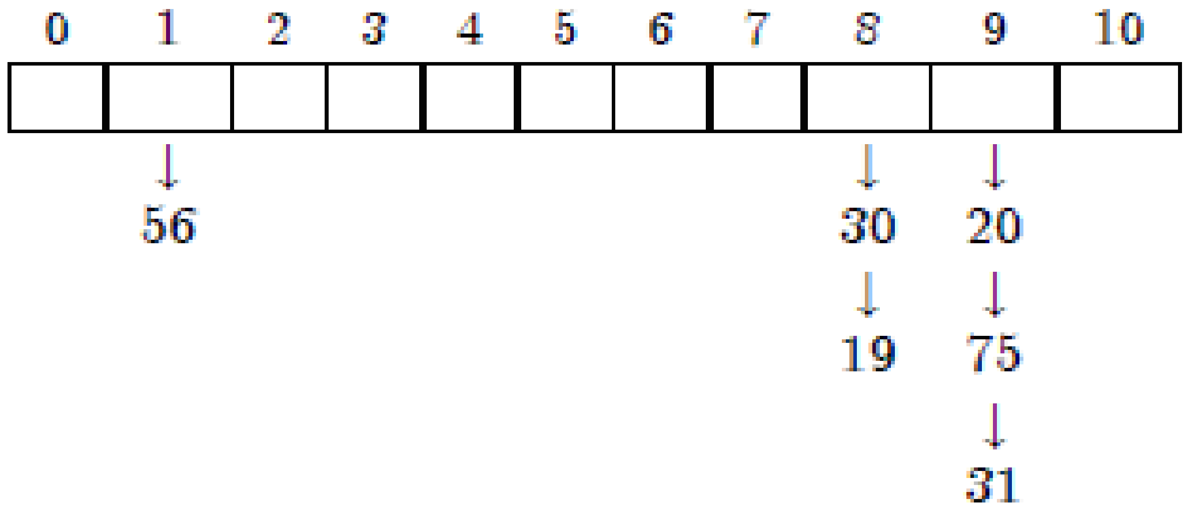
The list of keys: 30, 20, 56, 75, 31, 19

The hash function: $h(K) = K \bmod 11$

The hash addresses:

K	30	20	56	75	31	19
$h(K)$	8	9	1	9	9	8

The open hash table:



The largest number of key comparisons in a successful search in this table is 3 (in searching for $K = 31$).

The average number of key comparisons in a successful search in this table, assuming that a search for each of the six keys is equally likely, is

$$\frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 2 = \frac{10}{6} \approx 1.7.$$

Open Hashing

- Inserting and Deleting from the hash table is of the same complexity as searching.
- If hash function distributes keys uniformly, average length of linked list will be $\alpha = n/m$. This ratio is called *load factor*.
- Average-case number of key comparisons for a successful search is $\alpha/2$; Average-case number of key comparisons for an unsuccessful search is α.
- Worst-case number of key comparisons is $\Theta(n)$ – occurs if we get a linked list containing all the n elements hashing to the same index. To avoid this, we need to be careful in selecting a proper hashing function.
  - Mod-based hashing functions with a prime integer as the divisor are more likely to result in hash values that are evenly distributed across the keys.
- Open hashing still works if the number of keys, $n >$ the size of the hash table, m.

Applications of Hashing (1)

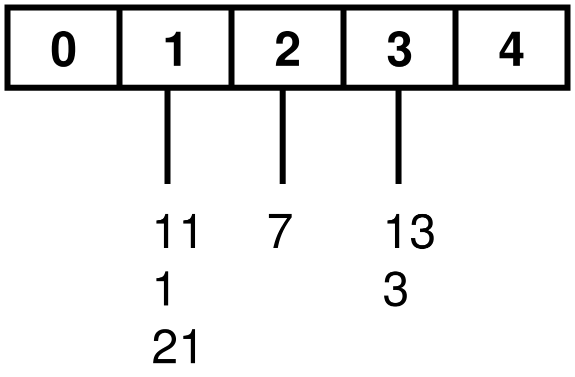
Finding whether an array is a Subset of another array

- Given two arrays A_L (larger array) and A_S (smaller array) of distinct elements, we want to find whether A_S is a subset of A_L.
- Example: $A_L = \{11, 1, 13, 21, 3, 7\}$; $A_S = \{11, 3, 7, 1\}$; A_S is a subset of A_L.
- Solution: Use (open) hashing. Hash the elements of the larger array, and for each element in the smaller array: search if it is in the hash table for the larger array. If even one element in the smaller array is not there in the larger array, we could stop!
- Time-complexity:
  - $\Theta(n)$ to construct the hash table on the larger array of size n, and another $\Theta(n)$ to search the elements of the smaller array.
  - A brute-force approach would have taken $\Theta(n^2)$ time.
- Space-complexity: $\Theta(n)$ with the hash table approach and $\Theta(1)$ with the brute-force approach.
- Note: The above solution could also be used to find whether two sets are disjoint or not. Even if one element in the smaller array is there in the larger array, we could stop!

Applications of Hashing (1)

Finding whether an array is a Subset of another array

- **Example 1**: A_L = {11, 1, 13, 21, 3, 7};
- A_S = {11, 3, 7, 1}; A_S is a subset of A_L.
- Let H(K) = K mod 5.



Hash table approach

# comparisons = 1 (for 11) + 2 (for 3) +
1 (for 7) + 2 (for 1) = 6

Brute-force approach: Pick every element in the smaller array and do a linear search for it in the larger array.

# comparisons = 1 (for 11) + 5 (for 3) +
6 (for 7) + 2 (for 1) = 14

- **Example 2**: A_L = {11, 1, 13, 21, 3, 7};
- A_S = {11, 3, 7, 4}; A_S is NOT a subset of A_L.
- Let H(K) = K mod 5.

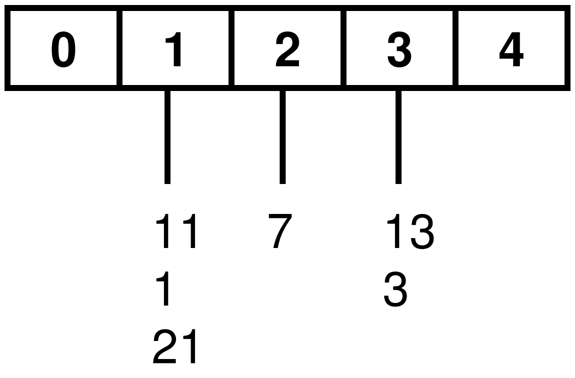
The **hash table approach** would take just 1 (for 11) + 2 (for 3) + 1 (for 7) + 0 (for 4) = 4 comparisons

The **brute-force approach** would take: 1 (for 11) + 5 (for 3) + 6 (for 7) + 6 (for 4) = 18 comparisons.

Applications of Hashing (1)

Finding whether two arrays are disjoint are not

- **Example 1**: A_L = {11, 1, 13, 21, 3, 7};
- A_S = {22, 25, 27, 28}; They are disjoint.
- Let H(K) = K mod 5.



Hash table approach

# comparisons = 1 (for 22) + 0 (for 25) + 1 (for 27) + 3 (for 28) = 5

Brute-force approach: Pick every element in the smaller array and do a linear search for it in the larger array.

# comparisons = 6 comparisons for each element * 4 = 24

- **Example 2**: A_L = {11, 1, 13, 21, 3, 7};
- A_S = {22, 25, 27, 1}; They are NOT disjoint.
- Let H(K) = K mod 5.

The **hash table approach** would take just 1 (for 22) + 0 (for 25) + 1 (for 27) + 2 (for 1) = 4 comparisons

The **brute-force approach** would take: 6 (for 22) + 6 (for 25) + 6 (for 27) + 2 (for 1) = 20 comparisons.

Applications of Hashing (2)

Finding Consecutive Subsequences in an Array

- Given an array A of unique integers, we want to find the contiguous subsequences of length 2 or above as well as the length of the largest subsequence.
- Assume it takes Θ(1) time to insert or search for an element in the hash table.

36 41 56 35 44 33
34 92 43 32 42

H(K) = K mod 7

0	1	2	3	4	5	6
56	36	44		32	33	41
35	92					34
42	43					

36	41	56	35	44	33	34	92	43	32	42
35	40	55	34	43	32	33	91	42	31	41
	42	57					93		33	
	43								34	
	44								35	
	45								36	
									37	

41	**32**
42	**33**
43	**34**
44	**35**
	36

Applications of Hashing (1)

Finding Consecutive Subsequences in an Array

- Algorithm

```
Insert the elements of A in a hash table H
Largest Length = 0
for i = 0 to n-1 do
    if (A[i] – 1 is not in H) then
        j = A[i]   // A[i] is the first element of a possible cont. sub seq.
        j = j + 1
        while ( j  is in H) do
            j = j + 1
        end while
        if ( j – A[i] > 1) then  // we have found a cont. sub seq. of length > 1
            Print all integers from A[i] … (j-1)
            if (Largest Length < j – A[i]) then
                Largest Length = j – A[i]
            end if
        end if
    end if
end for
```

L searches in the Hash table H for sub sequences of length L

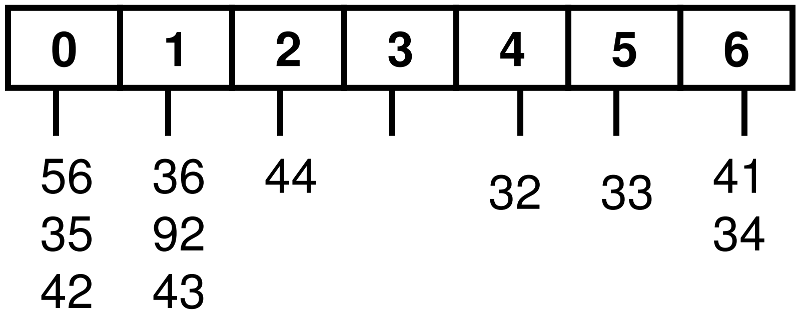
Applications of Hashing (2)

Finding Consecutive Subsequences in an Array

- **Time Complexity Analysis**
- For each element at index i in the array A we do at least one search (for element A[i] – 1) in the hash table.
- For every element that is the first element of a sub seq. of length 1 or above (say length L), we do L searches in the Hash table.
- The sum of all such Ls should be n.
- For an array of size n, we do n + n = 2n = Θ(n) hash searches. The first 'n' corresponds to the sum of all the lengths of the contiguous sub sequences and the second 'n' is the sum of all the 1s (one 1 for each element in the array)

36 41 56 35 44 33
34 92 43 32 42

H(K) = K mod 7



36	**41**	**56**	**35**	**44**	**33**	**34**	**92**	**43**	**32**	**42**
35	**40**	**55**	**34**	**43**	**32**	**33**	**91**	**42**	**31**	**41**
	42	**57**					**93**		**33**	
	43								**34**	
	44								**35**	
	45								**36**	
									37	