Module 1: Analyzing the Efficiency of Algorithms

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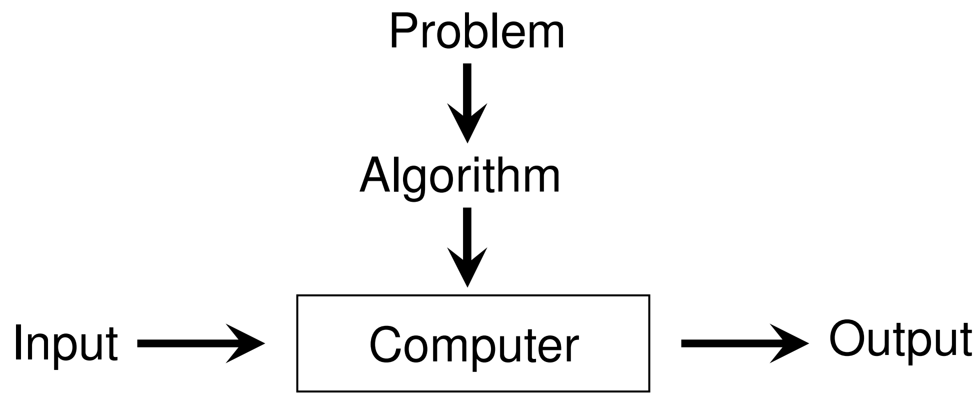
What is an Algorithm?

- An *algorithm* is a sequence of unambiguous instructions for solving a problem, i.e., for obtaining a required output for any legitimate input in a finite amount of time.



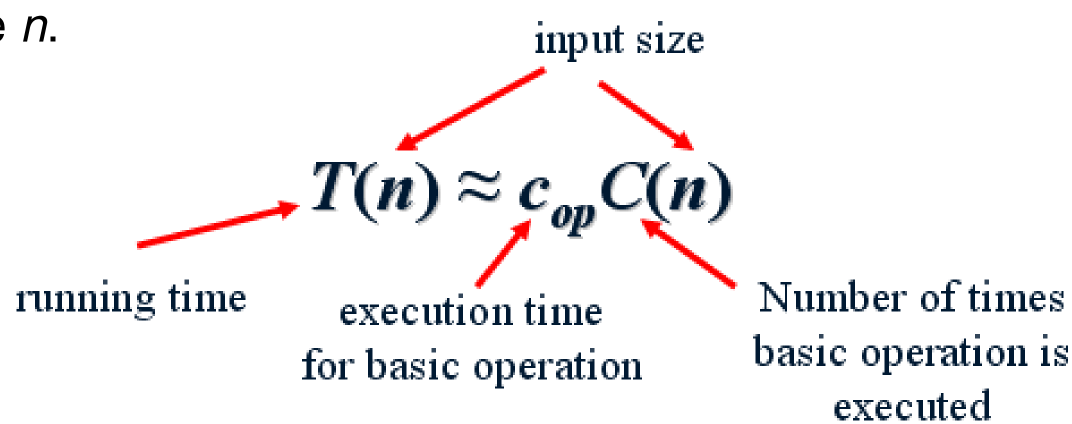
- Important Points about Algorithms
  - The non-ambiguity requirement for each step of an algorithm cannot be compromised
  - The range of inputs for which an algorithm works has to be specified carefully.
  - The same algorithm can be represented in several different ways
  - There may exist several algorithms for solving the same problem.
    - Can be based on very different ideas and can solve the problem with dramatically different speeds

The Analysis Framework

- **Time efficiency (time complexity):** indicates how fast an algorithm runs
- **Space efficiency (space complexity):** refers to the amount of memory units required by the algorithm in addition to the space needed for its input and output
- Algorithms that have non-appreciable space complexity are said to be ***in-place***.
- The time efficiency of an algorithm is typically as a function of the input size (one or more input parameters)
  - Algorithms that input a collection of values:
    - The time efficiency of sorting a list of integers is represented in terms of the number of integers (*n*) in the list
    - For matrix multiplication, the input size is typically referred as n*n.
    - For graphs, the input size is the set of Vertices (V) and edges (E).
  - Algorithms that input only one value:
    - The time efficiency depends on the magnitude of the integer. In such cases, the algorithm efficiency is represented as the number of bits $1+\lfloor \log_2 n \rfloor$ needed to represent the integer n

Units for Measuring Running Time

- The running time of an algorithm is to be measured with a unit that is independent of the extraneous factors like the processor speed, quality of implementation, compiler and etc.
  – At the same time, it is not practical as well as not needed to count the number of times, each operation of an algorithm is performed.
- <u>Basic Operation:</u> The operation contributing the most to the total running time of an algorithm.
  – It is typically the most time consuming operation in the algorithm's innermost loop.
    - **Examples:** Key comparison operation; arithmetic operation (division being the most time-consuming, followed by multiplication)
  – We will count the number of times the algorithm's basic operation is executed on inputs of size *n*.



$$T(n) \approx c_{op} C(n)$$

Examples for Input Size and Basic Operations

Problem	*Input size measure*	*Basic operation*
Searching for key in a list of *n* items	Number of list's items, i.e. *n*	Key comparison
Multiplication of two matrices	Matrix dimensions or total number of elements	Multiplication of two numbers
Checking primality of a given integer *n*	*n*'size = number of digits (in binary representation)	Division
Typical graph problem	#vertices and/or edges	Visiting a vertex or traversing an edge

Orders of Growth

- We are more interested in the order of growth on the number of times the basic operation is executed on the input size of an algorithm.
- Because, for smaller inputs, it is difficult to distinguish efficient algorithms vs. inefficient ones.
- For example, if the number of basic operations of two algorithms to solve a particular problem are n and n^2 respectively, then
  - if $n = 3$, then we may say there is not much difference between requiring 3 basic operations and 9 basic operations and the two algorithms have about the same running time.
  - On the other hand, if $n = 10000$, then it does makes a difference whether the number of times the basic operation is executed is n or n^2.

n	$\log_2 n$	n	$n \log_2 n$	n^2	n^3	2^n	$n!$
10	3.3	10^1	$3.3 \cdot 10^1$	10^2	10^3	10^3	$3.6 \cdot 10^6$
10^2	6.6	10^2	$6.6 \cdot 10^2$	10^4	10^6	$1.3 \cdot 10^{30}$	$9.3 \cdot 10^{157}$
10^3	10	10^3	$1.0 \cdot 10^4$	10^6	10^9		
10^4	13	10^4	$1.3 \cdot 10^5$	10^8	10^{12}		
10^5	17	10^5	$1.7 \cdot 10^6$	10^{10}	10^{15}		
10^6	20	10^6	$2.0 \cdot 10^7$	10^{12}	10^{18}		

Exponential-growth functions

Source: Table 2.1 From Levitin, 3rd ed.

Best-case, Average-case, Worst-case

- For many algorithms, the actual running time may not only depend on the input size; but, also on the specifics of a particular input.
  - For example, sorting algorithms (like insertion sort) may run faster on an input sequence that is *almost-sorted* rather than on a randomly generated input sequence.

- **Worst case:** $C_{worst}(n)$ – maximum number of times the basic operation is executed over inputs of size n
- **Best case:** $C_{best}(n)$ – minimum # times over inputs of size n
- **Average case:** $C_{avg}(n)$ – "average" over inputs of size n
  - Number of times the basic operation will be executed on typical input
  - NOT the average of worst and best case
  - Expected number of basic operations considered as a random variable under some assumption about the probability distribution of all possible inputs

Example for Worst and Best-Case Analysis: Sequential Search

```
ALGORITHM SequentialSearch(A[0..n − 1], K)
    //Searches for a given value in a given array by sequential search
    //Input: An array A[0..n − 1] and a search key K
    //Output: The index of the first element of A that matches K
    //          or −1 if there are no matching elements
    i ← 0
    while i < n and A[i] ≠ K do
        i ← i + 1
    if i < n return i
    else return −1
```

/* Assume the second condition will not be executed if the first condition evaluates to false */

- <u>Worst-Case:</u> $C_{worst}(n) = n$
- <u>Best-Case:</u> $C_{best}(n) = 1$

Probability-based Average-Case Analysis of Sequential Search

- If *p* is the probability of finding an element in the list, then (1-*p*) is the probability of not finding an element in the list.
- Also, on an *n*-element list, the probability of finding the search key as the i^{th} element in the list is *p/n* for all values of $1 \leq i \leq n$

$$C_{avg}(n) = [1 \cdot \frac{p}{n} + 2 \cdot \frac{p}{n} + \cdots + i \cdot \frac{p}{n} + \cdots + n \cdot \frac{p}{n}] + n \cdot (1 - p)$$

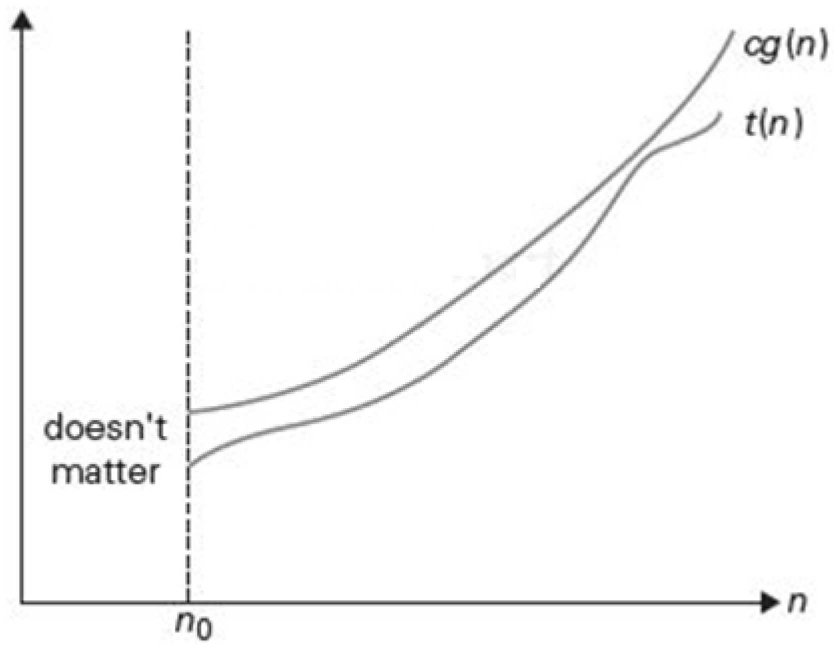
$$= \frac{p}{n}[1 + 2 + \cdots + i + \cdots + n] + n(1 - p)$$

$$= \frac{p}{n}\frac{n(n+1)}{2} + n(1 - p) = \frac{p(n+1)}{2} + n(1 - p).$$

- If p = 1 (the element that we will search for always exists in the list), then $C_{avg}(n) = (n+1)/2$. That is, on average, we visit half of the entries in the list to search for any element in the list.
- If p = 0 (all the time, the element that we will search never exists), then $C_{avg}(n) = n$. That is, we visit all the elements in the list.

YouTube Link: https://www.youtube.com/watch?v=8V-bHrPykrE

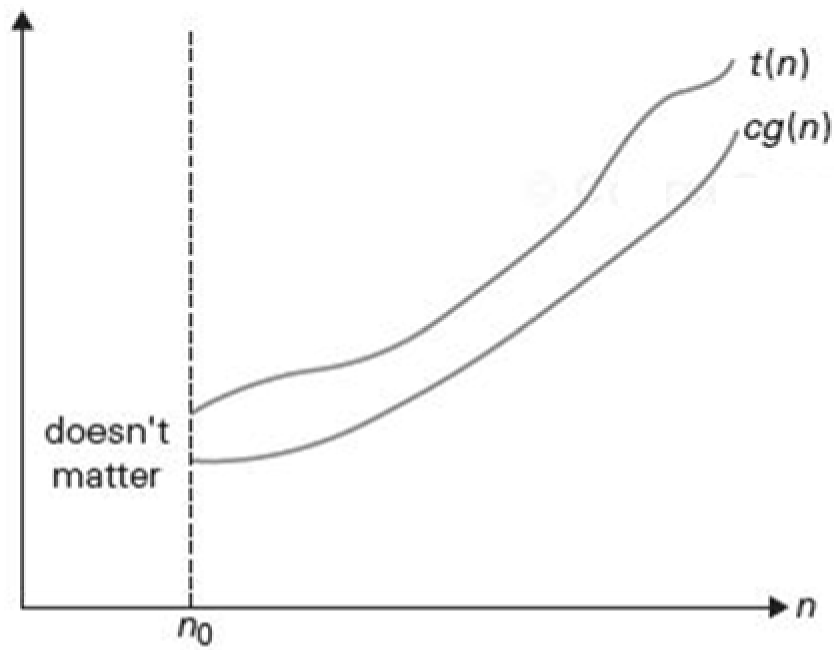
Asymptotic Notations: Formal Intro



$$t(n) = O(g(n))$$

$t(n) \leq c^*g(n)$ for all $n \geq n_0$

c is a positive constant (> 0)
and n_0 is a non-negative integer



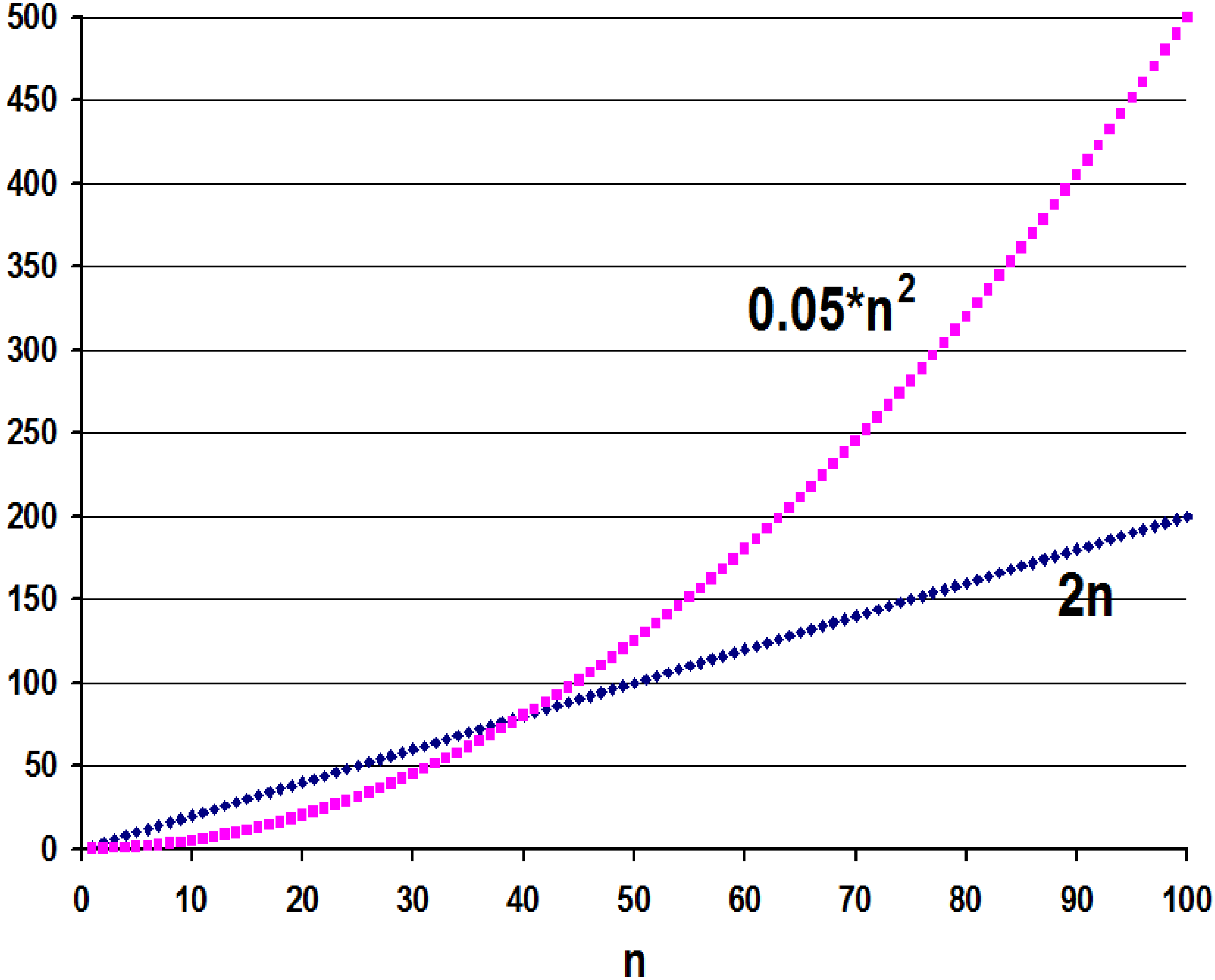
$$t(n) = \Omega(g(n))$$

$t(n) \geq c^*g(n)$ for all $n \geq n_0$

c is a positive constant (> 0)
and n_0 is a non-negative integer

Note: If $t(n) = O(g(n)) \rightarrow g(n) = \Omega(t(n))$; also, if $t(n) = \Omega(g(n)) \rightarrow g(n) = O(t(n))$

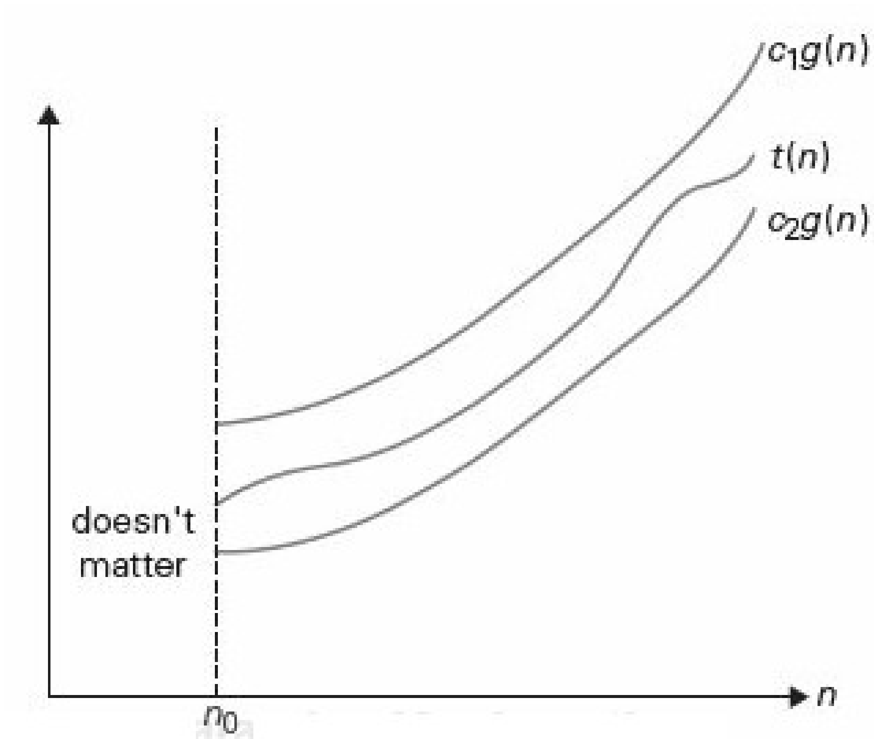
Asymptotic Notations: Intro



$2n \leq 0.05\ n^2$
for $n \geq 40$
$c = 0.05,\ n_0 = 40$
$2n = O(n^2)$
More generally,
$n = O(n^2)$.

$0.05n^2 \geq 2n$
for $n \geq 40$
$c = 2,\ n_0 = 40$
$0.05n^2 = \Omega(n)$
More generally,
$n^2 = \Omega(n)$

Asymptotic Notations: Formal Intro

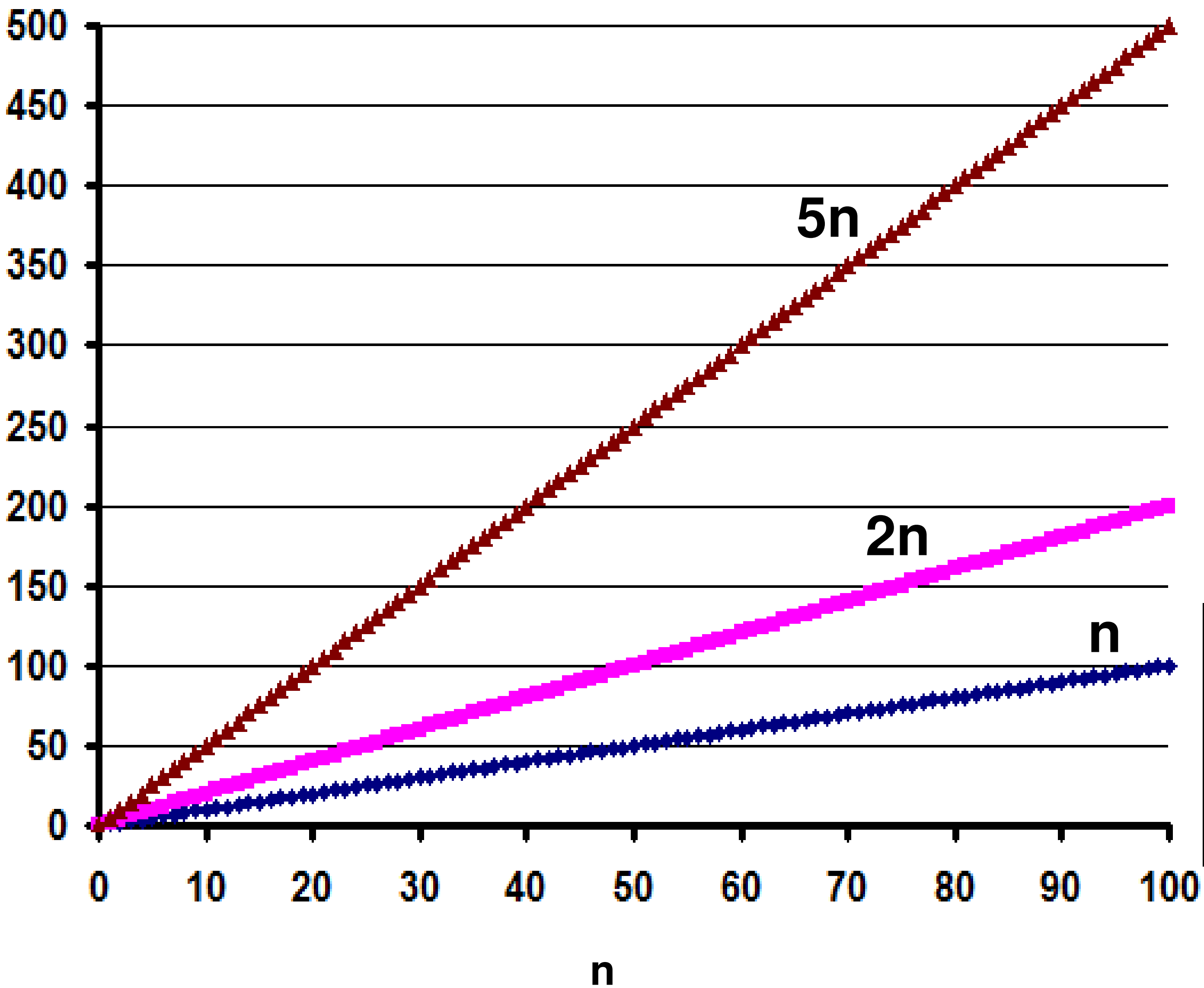


$$t(n) = \Theta(g(n))$$

$$c2*g(n) \le t(n) \le c1*g(n) \text{ for all } n \ge n_0$$

c1 and c2 are positive constants (> 0)
and n_0 is a non-negative integer

Asymptotic Notations: Intro



$2n \leq 5n$
for $n \geq 1$
$2n = O(n)$

$2n \geq n$
for $n \geq 1$
$2n = \Omega(n)$

As $2n = O(n)$
and $2n = \Omega(n)$,
we say
$2n = \Theta(n)$

Relationship and Difference between Big-O and Big-Θ

- If f(n) = Θ(g(n)), then f(n) = O(g(n)).
- If f(n) = O(g(n)), then f(n) need not be Θ(g(n)).
- Note: To come up with the Big-O/Θ term, we exclude the lower order terms of the expression for the time complexity and consider only the most dominating term. Even for the most dominating term, we omit any constant coefficient and only include the variable part inside the asymptotic notation.
- Big-Θ provides a tight bound (useful for precise analysis); whereas, Big-O provides an upper bound (useful for worst-case analysis).
- Examples:

  (1) $5n^2 + 7n + 2 = \Theta(n^2)$
  - Also, $5n^2 + 7n + 2 = O(n^2)$

  (2) $5n^2 + 7n + 2 = O(n^3)$,

  Also, $5n^2 + 7n + 2 = O(n^4)$, But, $5n^2 + 7n + 2 \neq \Theta(n^3)$ and $\neq \Theta(n^4)$
- The Big-O complexity of an algorithm can be technically more than one value, but the Big-Θ of an algorithm can be only one value and it provides a tight bound. For example, if an algorithm has a complexity of $O(n^3)$, its time complexity can technically be also considered as $O(n^4)$.

When to use Big-O and Big-Θ

- If the best-case and worst-case time complexity of an algorithm is guaranteed to be of a certain polynomial all the time, then we will use Big-Θ.
- If the time complexity of an algorithm could fluctuate from a best-case to worst-case of different rates, we will use Big-O notation as it is not possible to come up with a Big-Θ for such algorithms.

- **<u>Sequential key search</u>**
- **Inputs: Array A[0…n-1], Search Key K**
- **Begin**

```
        for (i = 0 to n-1) do
        if (A[i] == K) then
            return "Key K found at index i"
        end if
    end for
    return "Key K not found!!"
  End
```

O(n) only and not Θ(n)

- **<u>Finding the Maximum Integer in an Array</u>**
- **Input: Array A[0…n-1]**
- **Begin**

```
      Max = A[0]
        for (i = 1 to n-1) do
        if (Max < A[i]) then
            Max = A[i]
        end if
    end for
    return Max
  End
```

Θ(n) ➔It is also O(n)

Another Example to Decide whether Big-O or Big-Θ

Skeleton of a pseudo code

```
Input size: n
Begin Algorithm
If (certain condition) then
     for (i = 1 to n) do
       print a statement in unit time
   end for
else
     for (i = 1 to n) do
             for (j = 1 to n) do
                print a statement in unit time
             end for
     end for
End Algorithm
```

Best Case
The condition in the if block is true
-- Loop run 'n' times

Worst Case
The condition in the if block is false
-- Loop run 'n^2' times

Time Complexity: $O(n^2)$
It is not possible to come up with a Θ-based time complexity for this algorithm.

Asymptotic Notations: Examples

- Let t(n) and g(n) be any non-negative functions defined on a set of all real numbers.
- We say t(n) = O(g(n)) for all functions t(n) that have a lower or the same order of growth as g(n), within a constant multiple as n → ∞.
  - **Examples:** $n \in O(n)$, $n \in O(n^2)$, $100n + 5 \in O(n^2)$, $\frac{1}{2}n(n-1) \in O(n^2)$
    $n^3 \notin O(n^2)$, $0.00001n^3 \notin O(n^2)$, $n^4 + n + 1 \notin O(n^2)$
- We say t(n) = Ω(g(n)) for all functions t(n) that have a higher or the same order of growth as g(n), within a constant multiple as n → ∞.
  - **Examples:** $n \in \Omega(n)$, $n^3 \in \Omega(n^2)$, $\frac{1}{2}n(n-1) \in \Omega(n^2)$, $100n + 5 \notin \Omega(n^2)$
- We say t(n) = Θ(g(n)) for all functions t(n) that have the same order of growth as g(n), within a constant multiple as n → ∞.
  - **Examples:** $an^2 + bn + c = \Theta(n^2)$;
    $n^2 + \log n = \Theta(n^2)$

Useful Property of Asymptotic Notations

- If $t_1(n) \in O(g_1(n))$ and $t_2(n) \in O(g_2(n))$, then

$$t_1(n) + t_2(n) \in O(\max\{g_1(n), g_2(n)\})$$

- If $t_1(n) \in \Theta(g_1(n))$ and $t_2(n) \in \Theta(g_2(n))$, then

$$t_1(n) + t_2(n) \in \Theta(\max\{g_1(n), g_2(n)\})$$

- The property can be applied for the Ω notation with a slight change: Replace the Max with the Min.
- If $t_1(n) \in \Omega(g_1(n))$ and $t_2(n) \in \Omega(g_2(n))$, then

$$t_1(n) + t_2(n) \in \Omega(\min\{g_1(n), g_2(n)\})$$

Using Limits to Compare Order of Growth

$$\lim_{n\to\infty} \frac{t(n)}{g(n)} = \begin{cases} 0 & \text{implies that } t(n) \text{ has a smaller order of growth than } g(n), \\ c & \text{implies that } t(n) \text{ has the same order of growth as } g(n), \\ \infty & \text{implies that } t(n) \text{ has a larger order of growth than } g(n). \end{cases}$$

The first case means t(n) = O(g(n)
if the second case is true, then t(n) = Θ(g(n))
The last case means t(n) = Ω(g(n))

L'Hopital's Rule $$\lim_{n\to\infty} \frac{t(n)}{g(n)} = \lim_{n\to\infty} \frac{t'(n)}{g'(n)}$$

Note: t'(n) and g'(n) are first-order derivatives of t(n) and g(n)

Stirling's Formula $$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \text{ for large values of } n$$

Example 1: To Determine the Order of Growth

Find the class $O(g(n))$, $\Theta(g(n))$ and $\Omega(g(n))$ for the following function:

(a) $(n^2+1)^{10}$.

$O(g(n))$: Let $g(n) = n^{21}$

$$\lim_{n\to\infty} \frac{(n^2+1)^{10}}{n^{21}} = \lim_{n\to\infty} \frac{(n^2+1)^{10}}{(n^2)^{10}\cdot n} = \lim_{n\to\infty} \frac{1}{n}\left[\frac{n^2+1}{n^2}\right]^{10} = \lim_{n\to\infty} \frac{1}{n}\left[1+\frac{1}{n^2}\right]^{10}$$

$$= 0 * 1 = 0 \Rightarrow (n^2+1)^{10} = O(n^{21})$$

Example 1: To Determine the Order of Growth (continued…)

(b) $\Theta(g(n))$: Let $g(n) = n^{20}$

$$\lim_{n\to\infty} \frac{(n^2+1)^{10}}{n^{20}} = \lim_{n\to\infty} \frac{(n^2+1)^{10}}{(n^2)^{10}}$$

$$= \lim_{n\to\infty} \left[\frac{n^2+1}{n^2}\right]^{10} = \lim_{n\to\infty} \left[1+\frac{1}{n^2}\right] = 1$$

$$\Rightarrow (n^2+1)^{10} = \Theta(n^{20})$$

(c) $\Omega(g(n))$: Let $g(n) = n^{10}$.

$$\lim_{n\to\infty} \frac{(n^2+1)^{10}}{n^{10}} = \lim_{n\to\infty} \left[\frac{n^2+1}{n}\right]^{10} = \lim_{n\to\infty} \left[n+\frac{1}{n}\right]^{10} = \lim_{n\to\infty} n^{10} = \infty$$

$$(n^2+1)^{10} = \Omega(n^{10})$$

Example 2: To Determine the Order of Growth

Find the class $O(g(n))$, $\Theta(g(n))$ and $\Omega(g(n))$ for the following functions:

(b) $\sqrt{3n^2+7n+4}$

$O(g(n))$: $\sqrt{3n^2+7n+4}$ (→ $t(n)$) $\approx \sqrt{n^2} = n$.

Pick $g(n) = n^2 = \sqrt{n^4}$ [degree of $g(n)$ greater than $t(n)$]

$$\lim_{n\to\infty} \frac{\sqrt{3n^2+7n+4}}{\sqrt{n^4}} = \lim_{n\to\infty} \sqrt{\frac{3n^2+7n+4}{n^4}} = \lim_{n\to\infty} \sqrt{\frac{3}{n^2}+\frac{7}{n^3}+\frac{4}{n^4}}$$

$$= \underline{0}$$

$$\Rightarrow \sqrt{3n^2+7n+4} = \underline{O(n^2)}$$

Example 2: To Determine the Order of Growth (continued…)

$\Theta(g(n))$:

Pick $g(n)$ to be of the same degree as $t(n) = \sqrt{3n^2+7n+4}$

$\approx \sqrt{n^2} = n$.

$g(n) = \sqrt{n^2} = n$.

$$\lim_{n\to\infty} \frac{\sqrt{3n^2+7n+4}}{\sqrt{n^2}} = \lim_{n\to\infty} \sqrt{\frac{3n^2+7n+4}{n^2}} = \lim_{n\to\infty} \sqrt{3+\frac{7}{n}+\frac{4}{n^2}}$$

$$= \sqrt{3}$$

$$\sqrt{3n^2+7n+4} = \Theta(n)$$

$\Omega(g(n))$:

Pick $g(n)$ to be of a lower degree than $t(n) = \sqrt{3n^2+7n+4}$

$\approx n$.

So, pick $g(n) = n^{1/2} = \sqrt{n}$

$$\lim_{n\to\infty} \frac{\sqrt{3n^2+7n+4}}{\sqrt{n}} = \lim_{n\to\infty} \sqrt{\frac{3n^2+7n+4}{n}} = \lim_{n\to\infty} \sqrt{3n+7+\frac{4}{n}} = \infty.$$

So, $\sqrt{3n^2+7n+4} = \Omega(\sqrt{n})$

Examples to Compare the Order of Growth

EXAMPLE 1 Compare the orders of growth of $\frac{1}{2}n(n-1)$ and n^2.

$$\lim_{n\to\infty}\frac{\frac{1}{2}n(n-1)}{n^2}=\frac{1}{2}\lim_{n\to\infty}\frac{n^2-n}{n^2}=\frac{1}{2}\lim_{n\to\infty}(1-\frac{1}{n})=\frac{1}{2}.$$

$$\frac{1}{2}n(n-1)\in\Theta(n^2)$$

EXAMPLE 2 Compare the orders of growth of $\log_2 n$ and $\sqrt{n}$.

$$\lim_{n\to\infty}\frac{\log_2 n}{\sqrt{n}}=\lim_{n\to\infty}\frac{(\log_2 n)'}{(\sqrt{n})'}=\lim_{n\to\infty}\frac{(\log_2 e)\frac{1}{n}}{\frac{1}{2\sqrt{n}}}=2\log_2 e\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0.$$

$$\log_2 n\in O(\sqrt{n}).$$

Example 3: Compare the order of growth of $\log^2 n$ and $\log n^2$.

$$\lim_{n->\infty}\frac{\log^2 n}{\log n^2}=\lim_{n->\infty}\frac{\log n*\log n}{\log(n*n)}=\lim_{n->\infty}\frac{\log n*\log n}{\log n+\log n}=\lim_{n->\infty}\frac{\log n*\log n}{2\log n}=\lim_{n->\infty}\frac{\log n}{2}=\infty$$

Hence, $\log^2 n=\Omega(\log n^2)$ That is, $\log n^2=O(\log^2 n)$

Some More Examples: Order of Growth

For each of the following functions, indicate the class $\Theta(g(n))$ the function belongs to. (Use the simplest $g(n)$ possible in your answers.) Prove your assertions.

a. $(n^2+1)^{10}$ **b.** $\sqrt{10n^2+7n+3}$

c. $2n\lg(n+2)^2+(n+2)^2\lg\frac{n}{2}$ **d.** $2^{n+1}+3^{n-1}$

- a) $(n^2+1)^{10}$: Informally, $= (n^2+1)^{10} \approx n^{20}$.

Formally,

$$\lim_{n\to\infty}\frac{(n^2+1)^{10}}{n^{20}} = \lim_{n\to\infty}\frac{(n^2+1)^{10}}{(n^2)^{10}} = \lim_{n\to\infty}\left(\frac{n^2+1}{n^2}\right)^{10} == \lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{10} = 1.$$

Hence $(n^2+1)^{10} \in \Theta(n^{20})$.

b) Informally, $\sqrt{10n^2+7n+3} \approx \sqrt{10n^2} = \sqrt{10}n \in \Theta(n)$. Formally,

$$\lim_{n\to\infty}\frac{\sqrt{10n^2+7n+3}}{n} = \lim_{n\to\infty}\sqrt{\frac{10n^2+7n+3}{n^2}} = \lim_{n\to\infty}\sqrt{10+\frac{7}{n}+\frac{3}{n^2}} = \sqrt{10}.$$

Hence $\sqrt{10n^2+7n+3} \in \Theta(n)$.

c) $2n\lg(n+2)^2 + (n+2)^2\lg\frac{n}{2} = 2n2\lg(n+2) + (n+2)^2(\lg n - 1)$

$\in \Theta(n\lg n) + \Theta(n^2\lg n) = \Theta(n^2\lg n)$.

d) $2^{n+1}+3^{n-1} = 2^n 2 + 3^n\frac{1}{3} \in \Theta(2^n)+\Theta(3^n) = \Theta(3^n)$

Some More Examples: Order of Growth

List the following functions according to their order of growth from the lowest to the highest:

$$(n-2)!,\quad 5\lg(n+100)^{10},\quad 2^{2n},\quad 0.001n^4+3n^3+1,\quad \ln^2 n,\quad \sqrt[3]{n},\quad 3^n$$

$(n-2)! \in \Theta((n-2)!)$

$5\lg(n+100)^{10} = 50\lg(n+100) \in \Theta(\log n)$

$2^{2n} = (2^2)^n \in \Theta(4^n)$

$0.001n^4 + 3n^3 + 1 \in \Theta(n^4)$

$\ln^2 n \in \Theta(\log^2 n)$

$\sqrt[3]{n} \in \Theta(n^{\frac{1}{3}})$

$3^n \in \Theta(3^n)$

The listing of the functions in the increasing Order of growth is as follows:

$$5\lg(n+100)^{10},\quad \ln^2 n,\quad \sqrt[3]{n},\quad 0.001n^4+3n^3+1,\quad 3^n,\quad 2^{2n},\quad (n-2)!$$

$$\lim_{n\to\infty} \frac{\log^2 n}{n^{1/3}} = \lim_{n\to\infty} \frac{2\log n}{n*(1/3)n^{(-2/3)}} = \frac{6\log n}{n^{(1/3)}} = \frac{6}{n*(1/3)n^{(-2/3)}} = \lim_{n\to\infty} \frac{18}{n^{(1/3)}} = 0$$

Hence, $\log^2 n = O(n^{1/3})$

Time Efficiency of Non-recursive Algorithms: *General Plan for Analysis*

- Decide on parameter *n* indicating *input size*
- Identify algorithm's *basic operation*
- Determine *worst*, *average*, and *best* cases for input of size *n*, if the number of times the basic operation gets executed varies with specific instances (inputs)
- Set up a sum for the number of times the basic operation is executed
- Simplify the sum using standard formulas and rules

Useful Summation Formulas and Rules

$\Sigma_{l \le i \le u} 1 = 1+1+\ldots+1 = u - l + 1$

In particular, $\Sigma_{1 \le i \le n} 1 = n - 1 + 1 = n \in \Theta(n)$

$\Sigma_{1 \le i \le n}\, i = 1+2+\ldots+n = n(n+1)/2 \approx n^2/2 \in \Theta(n^2)$

$\Sigma_{1 \le i \le n}\, i^2 = 1^2+2^2+\ldots+n^2 = n(n+1)(2n+1)/6 \approx n^3/3 \in \Theta(n^3)$

$\Sigma_{0 \le i \le n}\, a^i = 1 + a + \ldots + a^n = (a^{n+1} - 1)/(a - 1)$ for any $a \neq 1$

In particular, $\Sigma_{0 \le i \le n}\, 2^i = 2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1 \in \Theta(2^n)$

$\Sigma(a_i \pm b_i) = \Sigma a_i \pm \Sigma b_i$ $\quad \Sigma c a_i = c \Sigma a_i$ $\quad \Sigma_{l \le i \le u} a_i = \Sigma_{l \le i \le m} a_i + \Sigma_{m+1 \le i \le u} a_i$

$$\sum_{i=l}^{u} 1 = (u - l + 1)$$

Examples on Summation

- 1 + 3 + 5 + 7 + …. + 999

$$= [1 + 2 + 3 + 4 + 5 + \ldots. 999] - [2 + 4 + 6 + 8 + \ldots + 998]$$

$$= \frac{999 * 1000}{2} - 2[1 + 2 + 3 + \ldots. + 499]$$

$$= 999 * 500 - 2\left[\frac{499 * 500}{2}\right] = 999 * 500 - 499 * 500$$

$$= 500*(999\text{-}499) = 500*500 = 250{,}000$$

- 2 + 4 + 8 + 16 + … + 1024

$$= 2^1 + 2^2 + 2^3 + 2^4 + \ldots + 2^{10}$$

$$= [2^0 + 2^1 + 2^2 + 2^3 + 2^4 + \ldots + 2^{10}] - 1$$

$$= \left[\sum_{i=0}^{10} 2^i\right] - 1 = \left[2^{11} - 1\right] - 1 = \mathbf{2046}$$

$$\sum_{i=3}^{n+1} 1 = [(n+1) - 3 + 1] = n+1 - 2 = n-1 = \Theta(n)$$

$$\sum_{i=3}^{n+1} i = 3 + 4 + ... + (n+1) = [1 + 2 + 3 + 4 + ... + (n+1)] - [1 + 2]$$

$$= \frac{(n+1)(n+2)}{2} - 3 = \Theta(n^2) - \Theta(1) = \Theta(n^2)$$

$$\sum_{i=0}^{n-1} i(i+1) = \sum_{i=0}^{n-1} i^2 + i = \sum_{i=0}^{n-1} i^2 + \sum_{i=0}^{n-1} i$$

$$= \left[\frac{[n-1][(n-1)+1][2(n-1)+1]}{6}\right] + \left[\frac{[n-1][(n-1)+1]}{2}\right]$$

$$= \left[\frac{[n-1][n][2n-1]}{6}\right] + \left[\frac{[n-1][n]}{2}\right]$$

$$= \Theta(n^3) + \Theta(n^2) = \Theta(n^3)$$

$$\sum_{i=0}^{n-1}(i^2+1)^2 = \sum_{i=0}^{n-1}(i^4+2i^2+1) = \sum_{i=0}^{n-1} i^4 + 2\sum_{i=0}^{n-1} i^2 + \sum_{i=0}^{n-1} 1$$

$$\in \Theta(n^5) + \Theta(n^3) + \Theta(n) = \Theta(n^5)$$

Example 1: Finding Max. Element

```
ALGORITHM  MaxElement(A[0..n − 1])
    //Determines the value of the largest element in a given array
    //Input: An array A[0..n − 1] of real numbers
    //Output: The value of the largest element in A
    maxval ← A[0]
    for i ← 1 to n − 1 do
        if A[i] > maxval
            maxval ← A[i]
    return maxval
```

- The basic operation is the comparison executed on each repetition of the loop.
- In this algorithm, the number of comparisons is the same for all arrays of size n.
- The algorithm makes one comparison on each execution of the loop, which is repeated for each value of the loop's variable i within the bounds 1 and n-1 (inclusively). Hence,

$$C(n) = \sum_{i=1}^{n-1} 1 = n - 1 \in \Theta(n)$$

Note: Best case = Worst case for this problem

Example 2: Sequential Key Search

```
Input: Array A[0...n-1], Search Key K
Begin
        for (index i = 0 to n-1) do
                if (A[i] == K)          <- Basic Operation: Comparison
                        return "index i"
                end if
        end for
        return "index not found"
End
```

Asymptotic time complexity: O(n)

- <u>Worst-Case:</u> $C_{worst}(n) = n$
- <u>Best-Case:</u> $C_{best}(n) = 1$

Example 3: Element Uniqueness Problem

```
ALGORITHM  UniqueElements(A[0..n − 1])
    //Determines whether all the elements in a given array are distinct
    //Input: An array A[0..n − 1]
    //Output: Returns "true" if all the elements in A are distinct
    //           and "false" otherwise
    for i ← 0 to n − 2 do
        for j ← i + 1 to n − 1 do
            if A[i] = A[j] return false
    return true
```

<u>Best-case situation:</u>

If the two first elements of the array are the same, then we can exit after one comparison. Best case = 1 comparison.

<u>Worst-case situation:</u>

- The basic operation is the comparison in the inner loop. The worst-case happens for two-kinds of inputs:
  - Arrays with no equal elements
  - Arrays in which only the last two elements are the pair of equal elements

Example 3: Element Uniqueness Problem

- For these kinds of inputs, one comparison is made for each repetition of the innermost loop, i.e., for each value of the loop's variable j between its limits i+1 and n-1; and this is repeated for each value of the outer loop i.e., for each value of the loop's variable i between its limits 0 and n-2. Accordingly, we get,

$$C_{worst}(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} [(n-1) - (i+1) + 1] = \sum_{i=0}^{n-2} (n-1-i)$$

$$= \sum_{i=0}^{n-2} (n-1) - \sum_{i=0}^{n-2} i = (n-1) \sum_{i=0}^{n-2} 1 - \frac{(n-2)(n-1)}{2}$$

$$= (n-1)^2 - \frac{(n-2)(n-1)}{2} = \frac{(n-1)n}{2} \approx \frac{1}{2} n^2$$

Best-case: 1 comparison
Worst-case: $n^2/2$ comparisons

Asymptotic time complexity = $O(n^2)$

Example 4: Bubble Sort

- A classical sorting algorithm in which (for an array of ‘n’ elements, with indexes 0 to n-1) during the ith iteration, the (n-i-1)th largest element is bubbled all the way to its final position.
- During the ith iteration, starting from index j = 0 to n-i-2, the element at index j is compared with the element at index j+1 and is swapped if the former is larger than the latter.
  - Optimization: If there is no swap during an iteration, the array is sorted and we can stop!

- Example

	45	78	23	12	59	72	
Iteration 0	45	23	12	59	72	**78**	
Iteration 1	23	12	45	59	**72**	**78**	
Iteration 2	12	23	45	**59**	**72**	**78**	
Iteration 3	12	23	**45**	**59**	**72**	**78**	**(no swap: STOP!!)**

Bubble Sort: Pseudo Code and Analysis

```
Input: Array A [0….n-1]
Begin
    for (i = 0 to n-2) do
         boolean didSwap = false
        for (j = 0 to n-i-2) do
             if A[j] > A[j+1] then
                  swap(A[j], A[j+1])
                  didSwap = true
             end if
        end for
        if (didSwap == false) then
            return; // STOP the algorithm
       end if
    end for
End
```

Best Case (array is already sorted):
1 Iteration
(i = 0): j = 0 to n-2
n-1 comparisons ~ n

Worst Case (array is reverse sorted): all iterations

$$\sum_{i=0}^{n-2} \sum_{j=0}^{n-i-2} 1$$

$$\sum_{i=0}^{n-2} [n-i-2]-[0]+1$$

$$= \sum_{i=0}^{n-2} [n-1]-i$$

$$= (n-1)+(n-2)+\ldots+1$$

$$= \frac{n(n-1)}{2} \sim \mathbf{n^2}$$

Asymptotic time complexity = $O(n^2)$

Example 5: Insertion Sort

- Given an array A[0…n-1], at any time, we have the array divided into two parts: A[0,…,i-1] and A[i…n-1].
  - The A[0…i-1] is the sorted part and A[i…n-1] is the unsorted part.
  - In any iteration, we pick an element *v* = A[i] and scan through the sorted sequence A[0…i-1] to insert *v* at the appropriate position.
    - The scanning is proceeded from right to left (i.e., for index j running from i-1 to 0) until we find the right position for *v*.
    - During this scanning process, v = A[i] is compared with A[j].
    - If A[j] > v, then we v has to be placed somewhere before A[j] in the final sorted sequence. So, A[j] cannot be at its current position (in the final sorted sequence) and has to move at least one position to the right. So, we copy A[j] to A[j+1] and decrement the index j, so that we now compare v with the next element to the left.

$$A[0] \leq \cdots \leq A[j] < A[j+1] \leq \cdots \leq A[i-1] \mid A[i] \cdots A[n-1]$$

smaller than or equal to $A[i]$ greater than $A[i]$

    - If A[j] ≤ v, we have found the right position for v; we copy v to A[j+1]. This also provides the stable property, in case v = A[j].

Insertion Sort
Pseudo Code and Analysis

```
Input: Array A[0...n-1]
Begin
for (index i = 1 to n-1) do
      v = A[i]
      index j = i-1
      while (index j ≥ 0) do
            if (v ≥ A[j]) then
                  break 'j' loop
            else        // v < A[j]
                  A[j+1] = A[j]
            end if
      end while
      A[j+1] = v
End
```

Best Case: If the array is already sorted For each value of index i, we just do one comparison (A[i] with A[i-1]), and decide to keep v = A[i] at its current location. Index i varies from 1 to n-1. Hence, there are 'n-1' comparisons.

Since the sub array from index 0 to i-1 is sorted, there is no way we can move 'v' further to the left, if we come across an A[j] such that v ≥ A[j]

// v < A[j] → The element A[j] is not in its final position Needs to be moved to the right

Worst Case: If the array is reverse sorted. For each value of index i, the element A[i] needs to be compared with all the values to its left (i.e., from j index i-1 to 0).

$$\sum_{i=1}^{n-1}\sum_{j=i-1}^{0}1=\sum_{i=1}^{n-1}\sum_{j=0}^{i-1}1=\sum_{i=1}^{n-1}(i-1)-0+1=\sum_{i=1}^{n-1}i=\frac{n(n-1)}{2}$$

Insertion Sort: Analysis and Example

Average Case: On average for a random input sequence, we would be visiting half of the sorted sequence A[0…i-1] to put A[i] at the proper position.

$$C(n) = \sum_{i=1}^{n-1} \sum_{j=i-1}^{(i-1)/2} 1 = \sum_{i=1}^{n-1} \frac{(i-1)}{2} + 1 = \sum_{i=1}^{n-1} \frac{(i+1)}{2} = \Theta(n^2)$$

Example: Given sequence (also initial): **45** 23 8 12 90 21

Index -1

Iteration 1 (v = 23):

45	**45**	8	12	90	21
23	**45**	8	12	90	21

Iteration 2 (v = 8):

23	**45**	**45**	12	90	21
23	**23**	**45**	12	90	21
8	**23**	**45**	12	90	21

Iteration 3 (v = 12):

8	**23**	**45**	**45**	90	21
8	**23**	**23**	**45**	90	21
8	**12**	**23**	**45**	90	21

Iteration 4 (v = 90):

8	**12**	**23**	**45**	90	21
9	**12**	**23**	**45**	**90**	21

Iteration 5 (v = 21):

9	**12**	**23**	**45**	**90**	**90**
9	**12**	**23**	**45**	**45**	**90**
9	**12**	**23**	**23**	**45**	**90**
9	**12**	**21**	**23**	**45**	**90**

Asymptotic time complexity = O(n^2)

The **colored** elements are in the sorted sequence and the circled element is at index *j* of the algorithm.

Time Efficiency of Recursive Algorithms: *General Plan for Analysis*

- Decide on a parameter indicating an input's size.
- Identify the algorithm's basic operation.
- Check whether the number of times the basic op. is executed may vary on different inputs of the same size. (If it may, the worst, average, and best cases must be investigated separately.)
- Set up a recurrence relation with an appropriate initial condition expressing the number of times the basic op. is executed.
- Solve the recurrence (or, at the very least, establish its solution's order of growth) by backward substitutions or another method.

Recursive Evaluation of n!

Definition: $n! = 1 * 2 * \ldots *(n\text{-}1) * n$ for $n \geq 1$ and $0! = 1$

- Recursive definition of n!: $F(n) = F(n\text{-}1) * n$ for $n \geq 1$ and

  $F(0) = 1$

ALGORITHM $F(n)$
  //Computes n! recursively
  //Input: A nonnegative integer n
  //Output: The value of n!
  if $n = 0$ **return** 1
  else return $F(n-1) * n$

$$M(n) = \underset{\text{to compute } F(n-1)}{M(n-1)} + \underset{\text{to multiply } F(n-1) \text{ by } n}{1} \quad \text{for } n > 0.$$

$$M(n) = M(n-1) + 1 \quad \text{for } n > 0,$$
$$M(0) = 0.$$

$M(0) = 0$

the calls stop when $n = 0$ — no multiplications when $n = 0$

M(n-1) = M(n-2) + 1;   M(n-2) = M(n-3)+1

M(n) = [M(n-2)+1] + 1 = M(n-2) + 2 = [M(n-3)+1+2] = M(n-3) + 3
= M(n-n) + n = n

Overall time Complexity: Θ(n)

YouTube Link: https://www.youtube.com/watch?v=K25MWuKKYAY

Counting the # Bits of an Integer

ALGORITHM *BinRec(n)*

//Input: A positive decimal integer n
//Output: The number of binary digits in n's binary representation
if $n = 1$ **return** 1
else return $BinRec(\lfloor n/2 \rfloor) + 1$

bits (n) = # bits($\lfloor n/2 \rfloor$) + 1; for n > 1
bits (1) = 1

1	1 bit
2-3	2 bits
4-7	3 bits
8-15	4 bits
16-31	5 bits
32-63	6 bits

Either Division or Addition could be considered the Basic operation, as both are executed once for each recursion. We will treat "addition" as the basic operation.

Let A(n) be the number of additions needed to compute # bits(n)

 # Additions $A(n) = A(\lfloor n/2 \rfloor) + 1 \quad \text{for } n > 1.$

Since the recursive calls end when *n* is equal to 1 and there are no additions made, the initial condition is: A(1) = 0.

Counting the # Bits of an Integer

Solution Approach: If we use the backward substitution method (as we did in the previous two examples, we will get stuck for values of n that are not powers of 2).

We proceed by setting $n = 2^k$ for $k \geq 0$.

New recurrence relation to solve:

$$A(2^k) = A(2^{k-1}) + 1 \quad \text{for } k > 0,$$
$$A(2^0) = 0.$$

$$
\begin{aligned}
A(2^k) &= A(2^{k-1}) + 1 && \text{substitute } A(2^{k-1}) = A(2^{k-2}) + 1 \\
&= [A(2^{k-2}) + 1] + 1 = A(2^{k-2}) + 2 && \text{substitute } A(2^{k-2}) = A(2^{k-3}) + 1 \\
&= [A(2^{k-3}) + 1] + 2 = A(2^{k-3}) + 3 && \dots \\
&\dots \\
&= A(2^{k-i}) + i \\
&\dots \\
&= A(2^{k-k}) + k.
\end{aligned}
$$

$$A(n) = \log_2 n \in \Theta(\log n).$$

Examples for Solving Recurrence Relations

X(n) = X(n-1) + 5, for n > 1, X(1) = 0

$$
\begin{aligned}
x(n) &= x(n-1)+5 \\
&= [x(n-2)+5]+5 = x(n-2)+5\cdot 2 \\
&= [x(n-3)+5]+5\cdot 2 = x(n-3)+5\cdot 3 \\
&= \ldots \\
&= x(n-i)+5\cdot i \\
&= \ldots \\
&= x(1)+5\cdot(n-1) = 5(n-1).
\end{aligned}
$$

$= \Theta(n)$

X(n) = 3*X(n-1) for n > 1, X(1) = 4

$$
\begin{aligned}
x(n) &= 3x(n-1) \\
&= 3[3x(n-2)] = 3^2x(n-2) \\
&= 3^2[3x(n-3)] = 3^3x(n-3) \\
&= \ldots \\
&= 3^ix(n-i) \\
&= \ldots \\
&= 3^{n-1}x(1) = 4\cdot 3^{n-1}.
\end{aligned}
$$

$= (4/3)3^n = \Theta(3^n)$

X(n) = X(n/3) + 1 for n > 1, X(1) = 1 [Solve for n = 3^k]

$$
\begin{aligned}
x(3^k) &= x(3^{k-1}) + 1 \\
&= [x(3^{k-2}) + 1] + 1 = x(3^{k-2}) + 2 \\
&= [x(3^{k-3}) + 1] + 2 = x(3^{k-3}) + 3 \\
&= \ldots \\
&= x(3^{k-i}) + i \\
&= \ldots \\
&= x(3^{k-k}) + k = x(1) + k = 1 + \log_3 n.
\end{aligned}
$$

X(n) = Θ(logn)