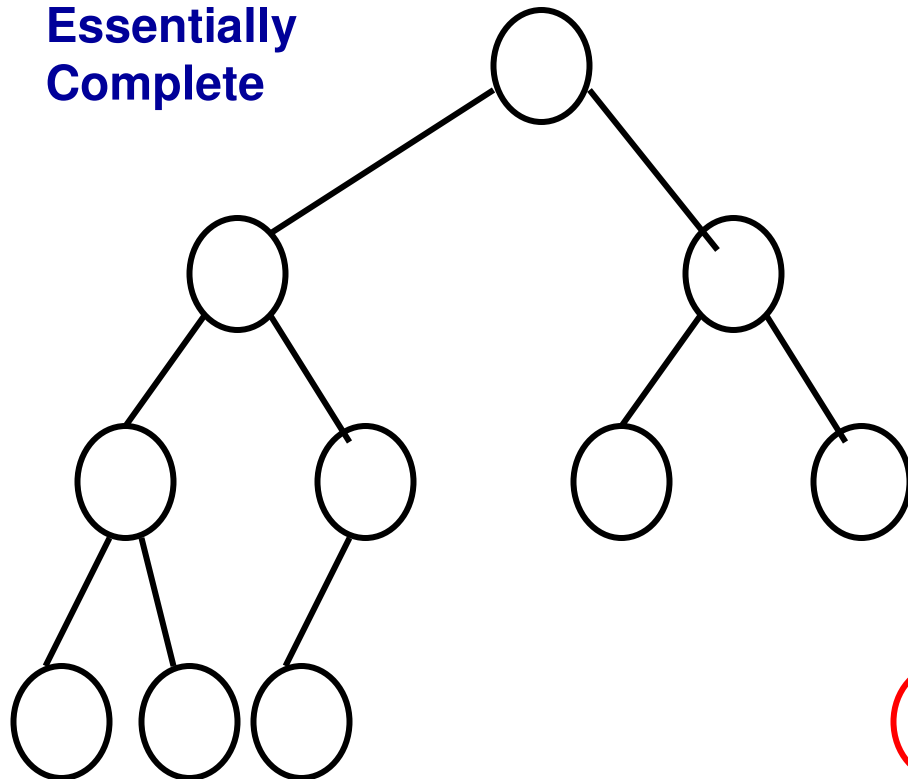
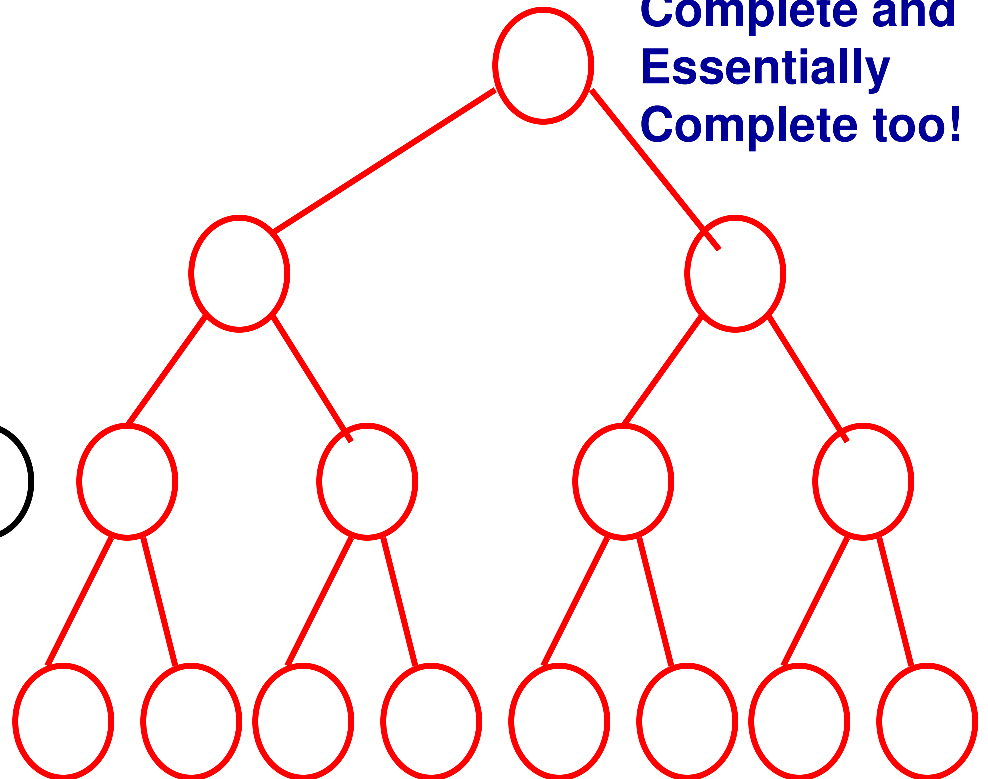
Module 8: Heap

Dr. Natarajan Meghanathan
Professor of Computer Science
Jackson State University
Jackson, MS 39217
E-mail: natarajan.meghanathan@jsums.edu

Essentially Complete Binary Tree

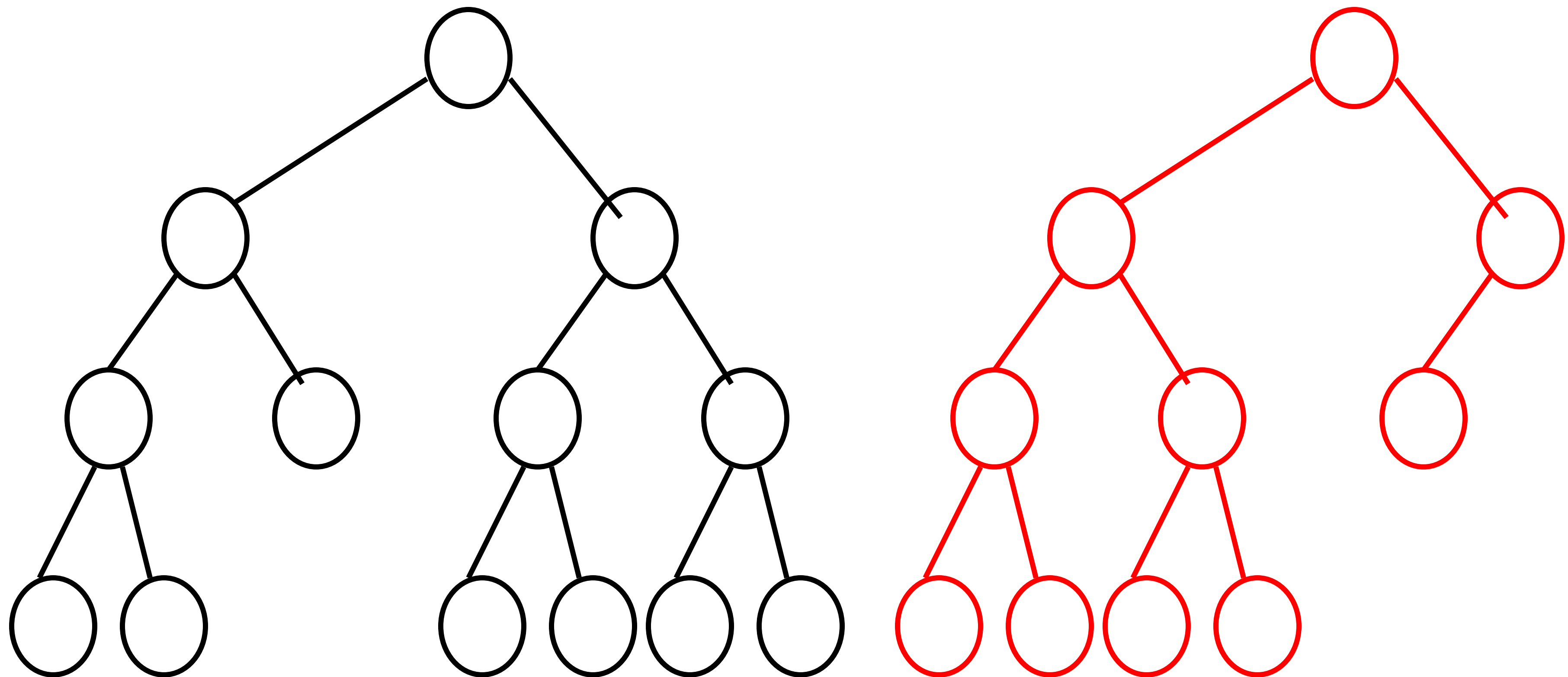
- A binary tree of height ‘h’ is essentially complete if it is a complete binary tree up to level h-1 and the nodes at level h are as far to the left as possible.
- Note: A complete binary tree is also essentially complete.





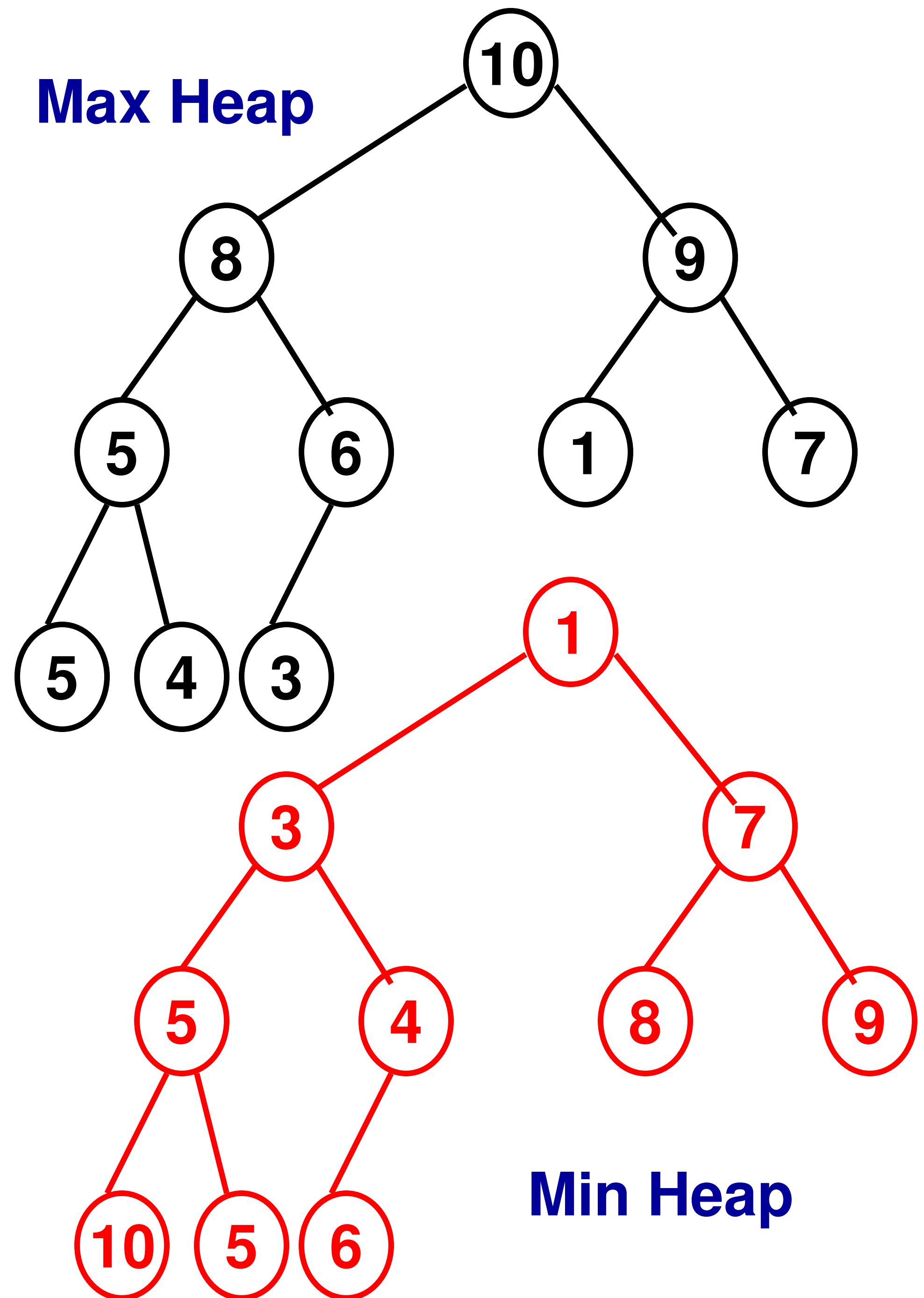
Essentially Complete Binary Tree

- The trees shown below are not essentially complete.

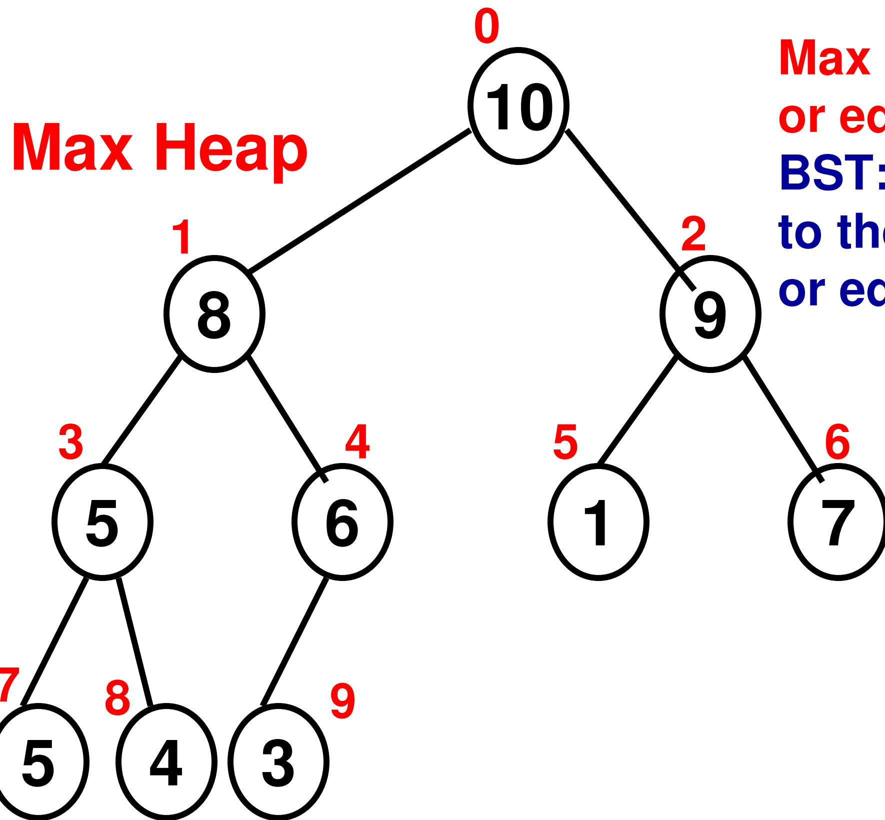


Heap

- A heap is a binary tree that satisfies the following two properties:
  - Essentially complete or complete
  - Max/Min heap
    - Max heap: The data at each internal node is greater than or equal to the data of its immediate child nodes
    - Min heap: The data at each internal node is lower than or equal to the data of its immediate child nodes

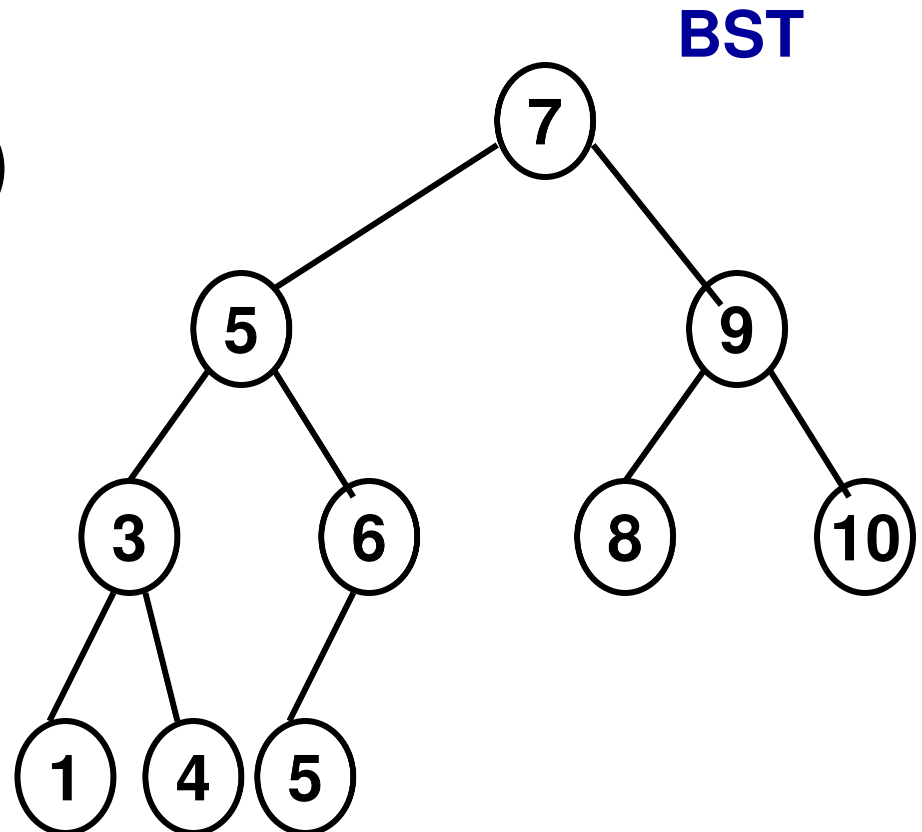


Difference between BST and Heap



Max Heap: Each internal node is greater than or equal to its child nodes.

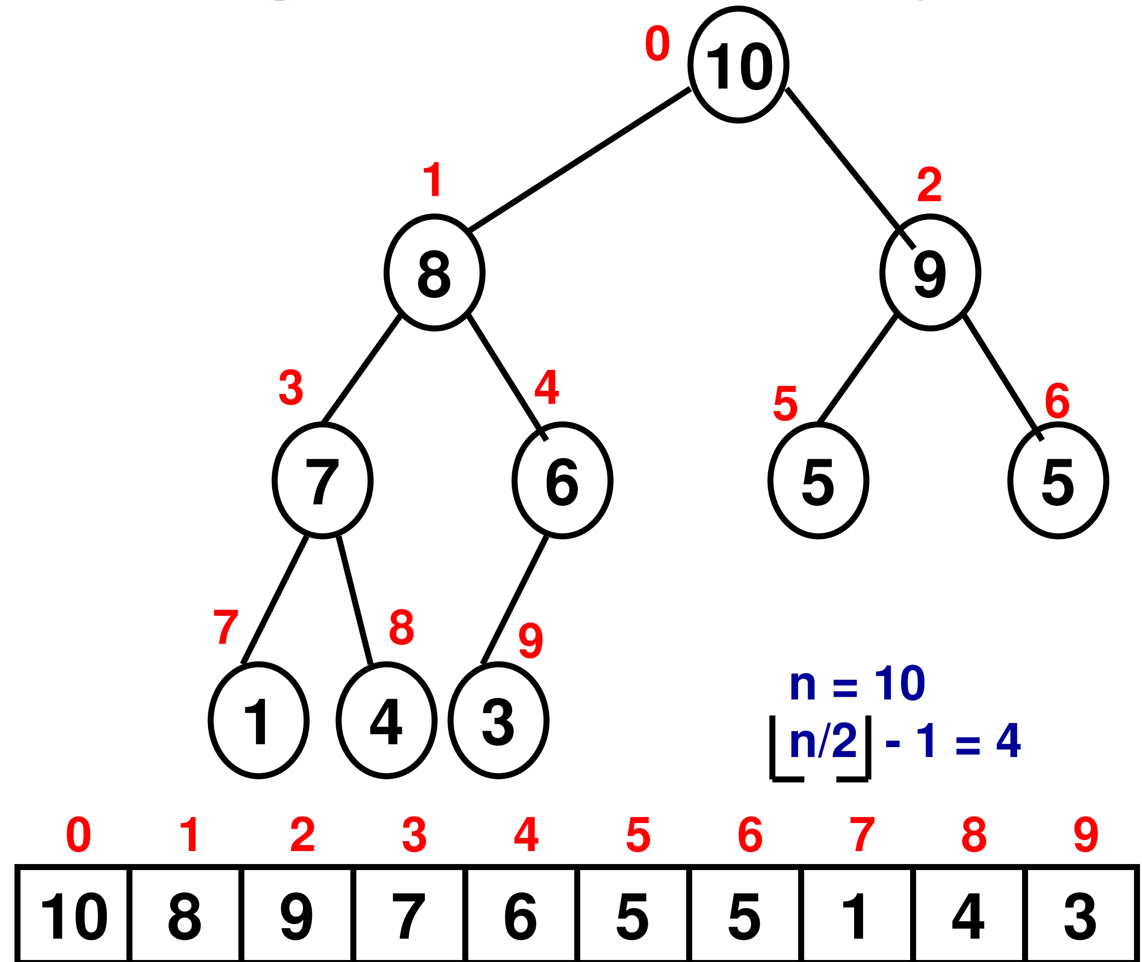
BST: An internal node is greater than or equal to the nodes in its left sub tree and lower than or equal to the nodes in its right sub tree.



7	3	8	1	9	4	0	5	2	6
5	5	4	8	3	6	10	1	9	7
1	3	4	5	5	6	7	8	9	10

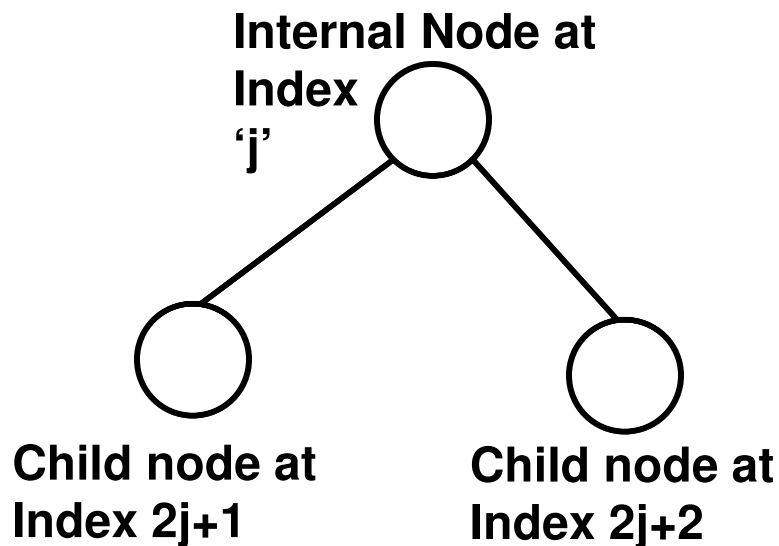
Storing the Heap as an Array

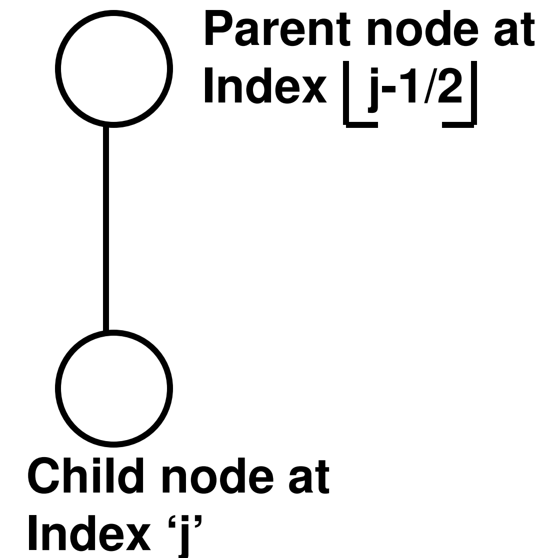
- A heap of ‘n’ elements can be stored in an array (index starting from 0) such that the internal nodes (in the top-down, left-right order) are represented as elements from index 0 to $\lfloor n/2 \rfloor - 1$ and the leaf nodes (again, top-down, left-right order) are represented as elements from index $\lfloor n/2 \rfloor$ to n-1.
- The child nodes of an internal node at index ‘j’ are at indexes 2j+1 and 2j+2.
- The parent node for a node at index j is at index $\lfloor (j-1)/2 \rfloor$



The child nodes of internal node ‘8’ at index 1 are at indexes 2*1+1 = 3 and 2*1 + 2 = 4. The parent node for node ‘7’ at index 3 is at index $\lfloor (3-1)/2 \rfloor$ = 1

Storing the Heap as an Array





For the rest of this module, we will construct and employ a 'max' heap unless otherwise specified.

The data for an internal node must be greater than or equal to that of its child nodes.

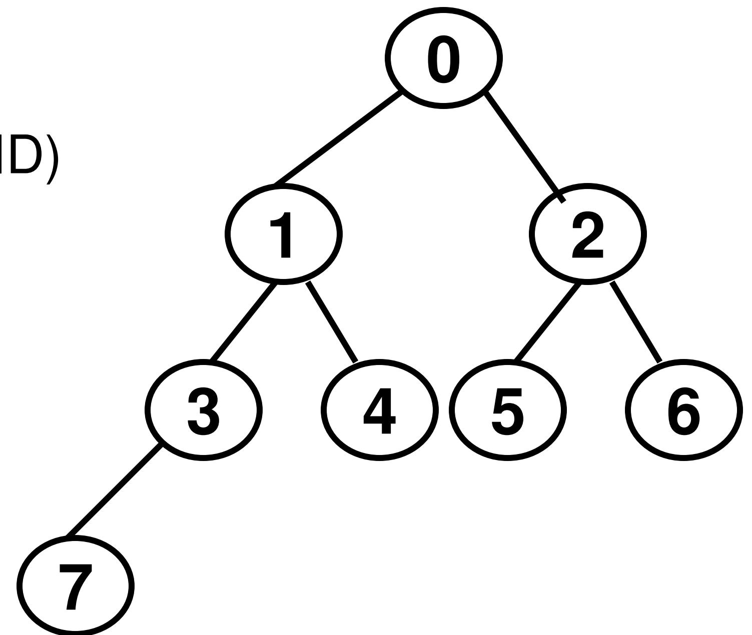
Using BFS to check whether a Binary Tree is Essentially Complete

We keep track of whether we come across a state wherein an internal node does not have a child node. The moment we come across an internal node with a missing Child node (left node or right node), we set the boolean 'noChildZoneStarts' to true If we across a child node when the noChildZoneStarts boolean is true, we declare the tree is not essentially complete!

```
Queue queue
queue.enqueue(root node id 0)
noChildZoneStarts = false

Begin BFS_BinaryTree
while (!queue.isEmpty()) do
  FirstNodeID = queue.dequeue();
  if (noChildZoneStarts == false AND FirstNode.leftChildNodeID == -1)
          noChildZoneStarts = true
  else if (noChildZoneStarts == true AND FirstNode.leftChildNodeID != -1)
          return "the binary tree is not essentially complete"

  if (FirstNode.leftChildNodeID != -1) then
     queue.enqueue(FirstNode.leftChildNodeID)
  end if
```

Breadth First Search (BFS) Algorithm continued…

```
if (noChildZoneStarts == false AND FirstNode.rightChildNodeID == -1)
        noChildZoneStarts = true
else if (noChildZoneStarts == true AND FirstNode.rightChildNodeID != -1)
        return "the binary tree is not essentially complete"

if (FirstNode.rightChildNodeID != -1) then
   queue.enqueue(FirstNode.rightChildNodeID)
end if

end while

return "the binary tree is essentially complete"

End BFS_BinaryTree
```



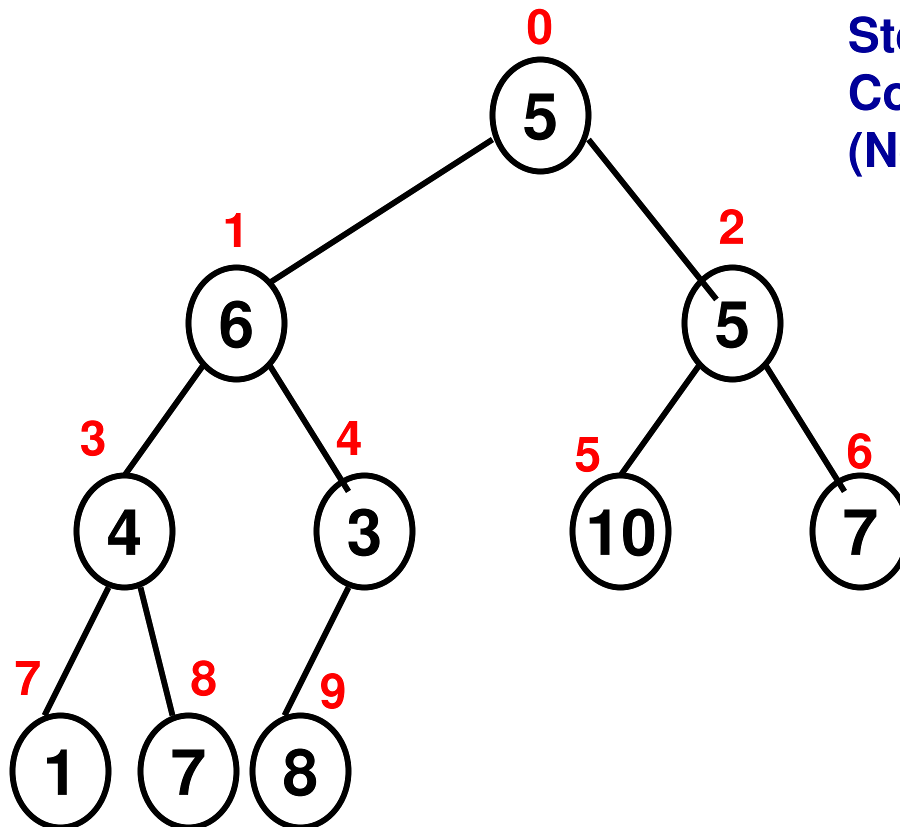
Once we find out that node '3' does not have a right child, all the nodes explored further in BFS should not have any child node. Otherwise, the binary tree is not essentially complete.

Heap Construction

- Given an array of ‘n’ elements,
- Step 1: Construct an essentially complete binary tree and then reheapify the internal nodes of the tree to make sure the max or min heap property is satisfied for each internal node.
- Step 2: Reheapify an internal node for ‘max’ heap: If the data at an internal node is lower than that of one or both of its child nodes, then swap the data for the internal node with the larger of the data of its two child nodes.
  - If any internal node further down is affected because of this swap, the reheapify operation is recursively continued all the way until a leaf node is reached.
- The reheapify operation is started from the node at index $\lfloor n/2 \rfloor - 1$ and continued all the way to the node at index 0.

Heap Construction Example 1

0	1	2	3	4	5	6	7	8	9
5	6	5	4	3	10	7	1	7	8

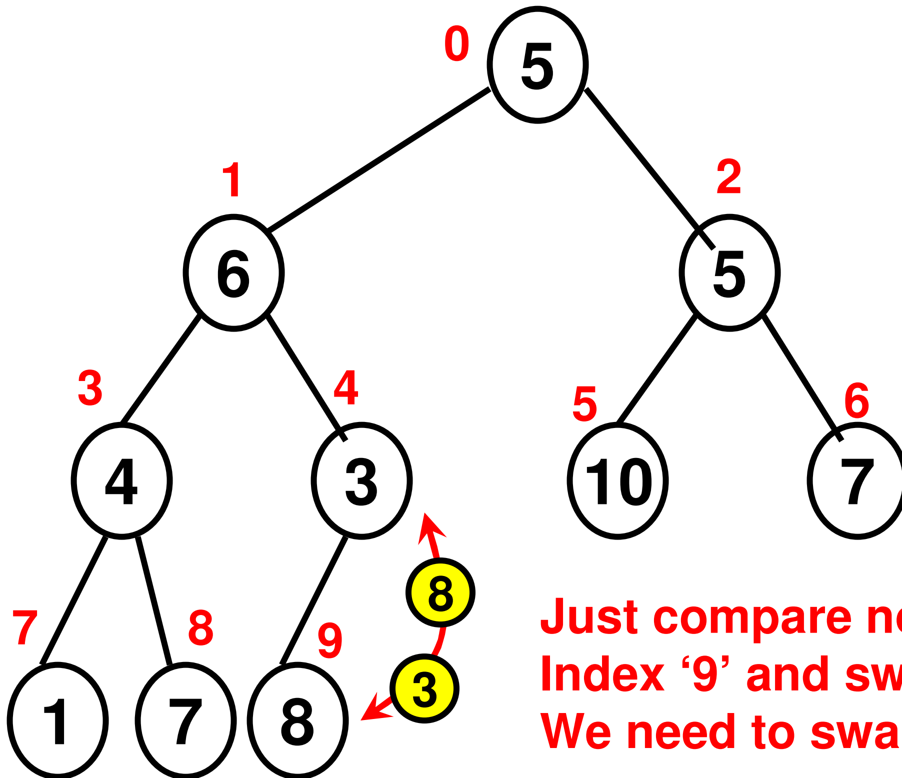


Step 1: Essentially Complete Binary Tree (Not a heap yet!)

Heap Construction Example 1

Before (Reheapify at Index '4'):

0	1	2	3	4	5	6	7	8	9
5	6	5	4	3	10	7	1	7	8



Step 2: Reheapify node at index '4' and down further if needed

Compare the node at index '4' with its child nodes at index 2*4 + 1 = 9 and index 2*4 + 2 = 10. Since index '10' does not exist and index 9 exists, it implies we have reached a leaf node (at index 9) and there is no need to proceed further down.

Just compare node at index '4' with the child node at Index '9' and swap them, if needed. In this case: Yes, We need to swap.

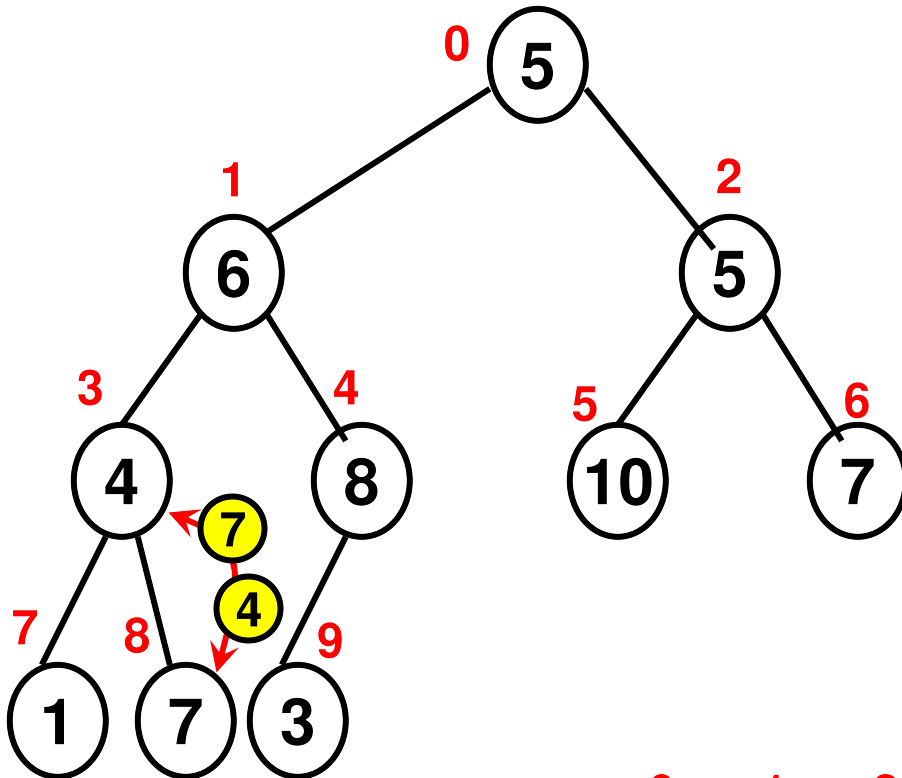
After (Reheapify at Index '4'):

0	1	2	3	4	5	6	7	8	9
5	6	5	4	8	10	7	1	7	3

Heap Construction Example 1

Before (Reheapify at Index '3'):

0	1	2	3	4	5	6	7	8	9
5	6	5	4	8	10	7	1	7	3



Step 2: Reheapify node at index '3' and down further if needed

Compare the node at index '3' with its child nodes at index 2*3 + 1 = 7 and index 2*3 + 2 = 8. In this case, We swap element at index '3' with element at index '8'. Since 8 is already a leaf node, we do not proceed down further.

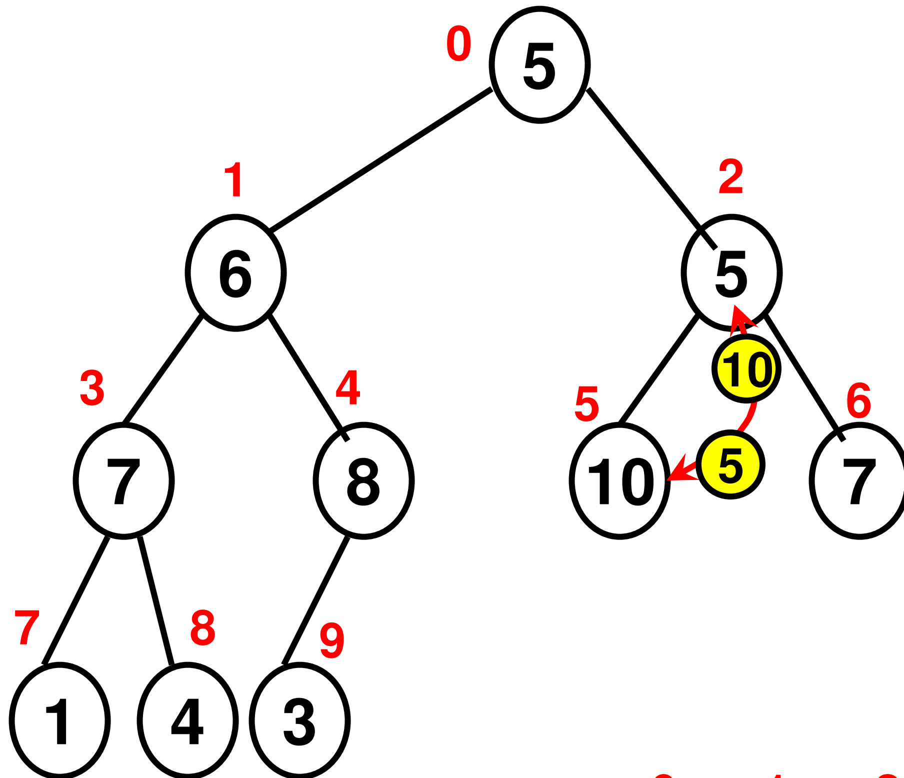
After (Reheapify at Index '3'):

0	1	2	3	4	5	6	7	8	9
5	6	5	7	8	10	7	1	4	3

Heap Construction Example 1

Before (Reheapify at Index '2'):

0	1	2	3	4	5	6	7	8	9
5	6	5	7	8	10	7	1	4	3



Step 2: Reheapify node at index '2' and down further if needed

Compare the node at index '2' with its child nodes at index 2*2 + 1 = 5 and index 2*2 + 2 = 6. In this case, We swap element at index '2' with element at index '5'. Since 5 is already a leaf node, we do not proceed down further.

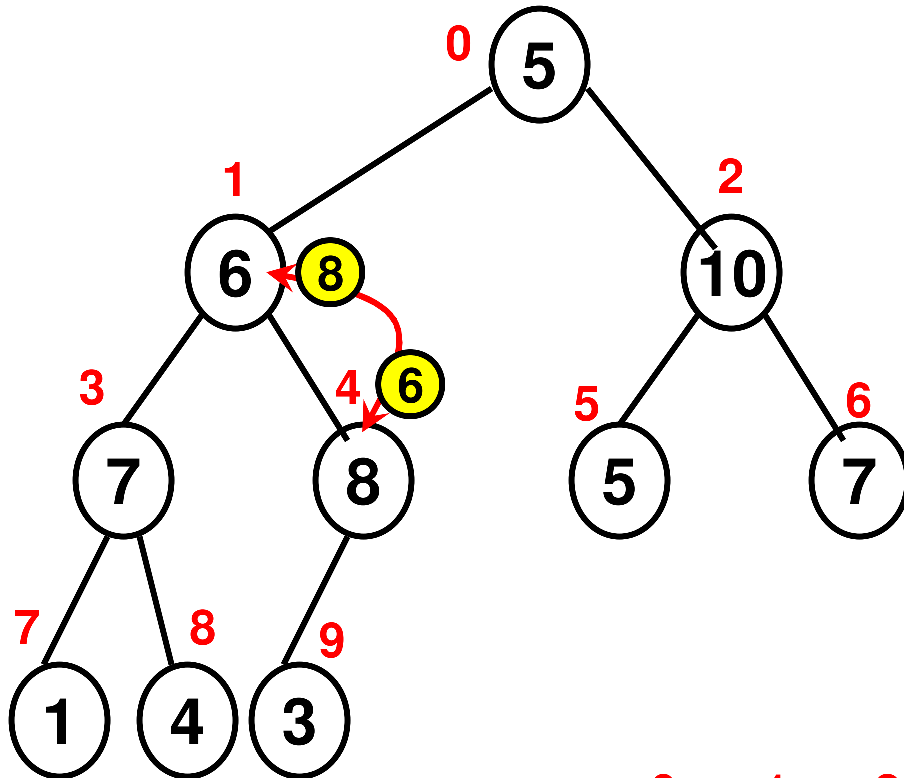
After (Reheapify at Index '2'):

0	1	2	3	4	5	6	7	8	9
5	6	10	7	8	5	7	1	4	3

Heap Construction Example 1

Before (Reheapify at Index ‘1’):

0	1	2	3	4	5	6	7	8	9
5	6	10	7	8	5	7	1	4	3



Step 2: Reheapify node at index ‘1’ and down further if needed

Compare the node at index ‘1’ with its child nodes at index 2*1 + 1 = 3 and index 2*1 + 2 = 4. In this case, We swap element at index ‘1’ with element at index ‘4’.
Again do a reheapify at index ‘4’, if needed and continue in a recursive fashion until it is no longer needed.

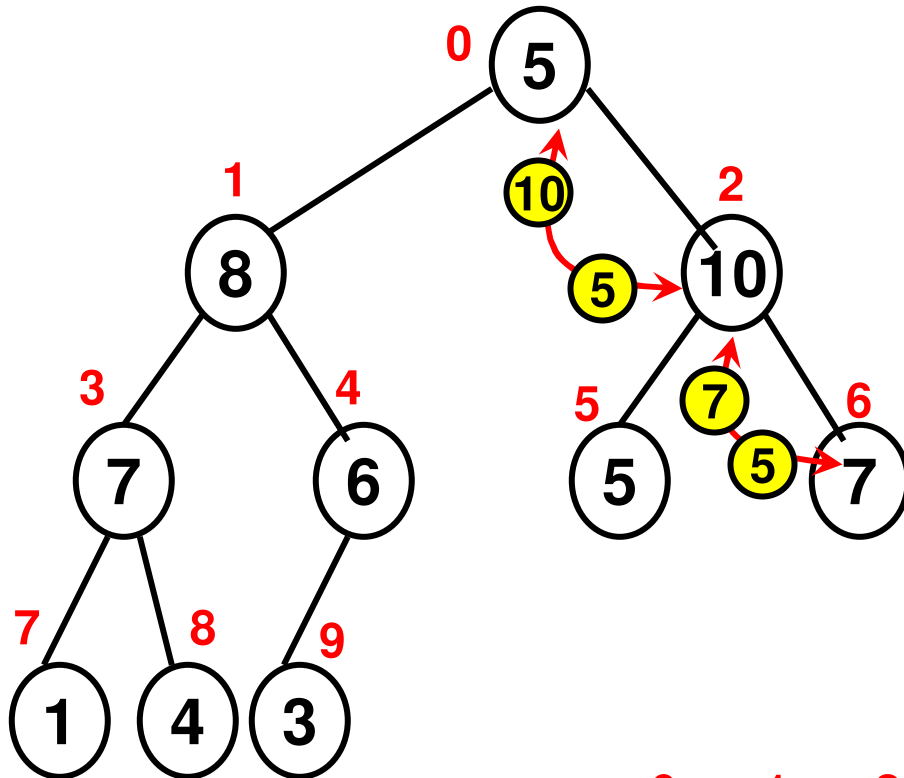
After (Reheapify at Index ‘1’):

0	1	2	3	4	5	6	7	8	9
5	8	10	7	6	5	7	1	4	3

Heap Construction Example 1

Before (Reheapify at Index '0'):

0	1	2	3	4	5	6	7	8	9
5	8	10	7	6	5	7	1	4	3



Step 2: Reheapify node at index '0' and down further if needed

Compare the node at index '0' with its child nodes at index 2*0 + 1 = 1 and index 2*0 + 2 = 2. In this case, We swap element at index '0' with element at index '2'.
Again do a reheapify at index '2' as the element now at index '2' (which is 5) is lower than the maximum of its two child nodes (which is 9 at index '6').

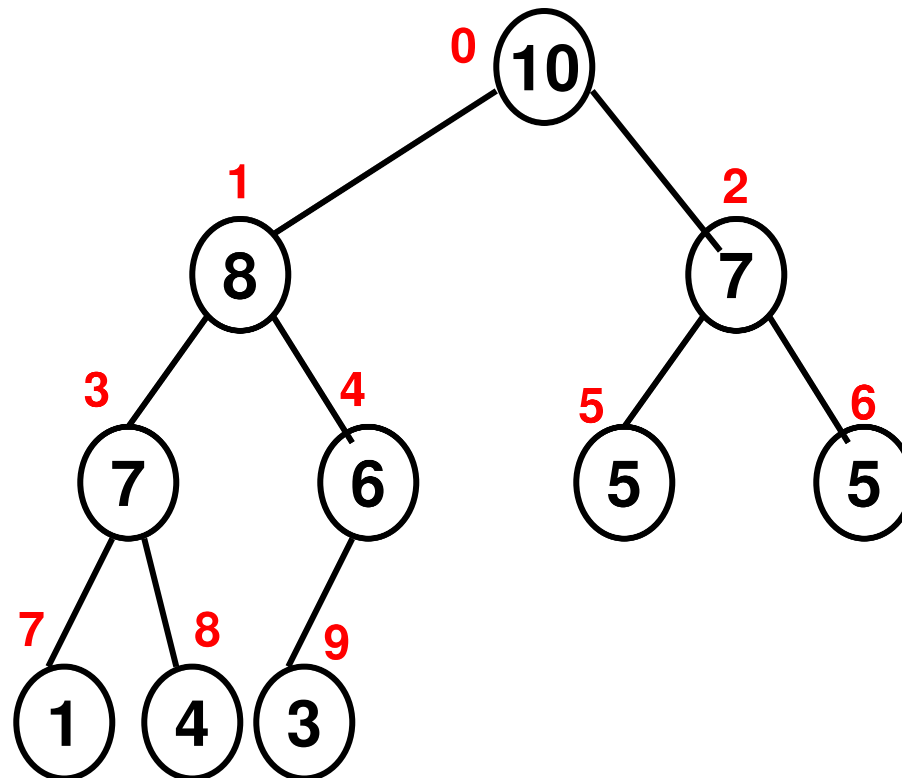
After (Reheapify at Index '0'):

0	1	2	3	4	5	6	7	8	9
10	8	7	7	6	5	5	1	4	3

Heap Construction Example 1

Final Array Representing Max Heap

0	1	2	3	4	5	6	7	8	9
10	8	7	7	6	5	5	1	4	3



Max Heap Construction (Code 8.1: C++)

Main Function

```
int arraySize;
cout << "Enter array size: ";
cin >> arraySize;

int array[arraySize];

int maxValue;
cout << "Enter the max. value for any element: ";
cin >> maxValue;

srand(time(NULL));

cout << "Generated array: ";
for (int i = 0; i < arraySize; i++){
    array[i] = rand() % maxValue;
    cout << array[i] << " ";
}
```

```
//max. heap construction

for (int index = (arraySize/2)-1; index  >= 0; index--)
        rearrangeHeapArray(array, arraySize, index);
}


cout << "After Heap construction..." << endl;
for (int index = 0; index < arraySize; index++)
        cout  << array[index] << " ";

cout << endl;
```

7.1: Reheapify Code (C++)

```
void rearrangeHeapArray(int *array, int arraySize, int index){
        // max heap construction

        int leftChildIndex = 2*index + 1;
        int rightChildIndex = 2*index + 2;
                // If the node at 'index' does not have a left child (implies it does
        if (leftChildIndex >= arraySize)    // not have right child too), then there
                return;                     // is no need to reheapify at that index

        // If the node at 'index' does not have a right child (if the control reaches
        if (rightChildIndex >= arraySize){          // here, it implies the node
                                                    // at 'index' has a left child)
// Check if the data for the
// node at          if (array[index] < array[leftChildIndex]){
// 'index' is less          int temp = array[index];
// than that of its         array[index] = array[leftChildIndex];
// left child. If so,       array[leftChildIndex] = temp;
// swap             }

                    return;
        }
```

// If the control reaches here, it means the node at ‘index’ has both left child
// and right child

// If the node at ‘index’ has data that is greater than or equal to both its left child and right child, then there is no need to reheapify for this index

```
if (array[index] >= array[leftChildIndex] &&
    array[index] >=  array[rightChildIndex])
        return;
```

// If the control reaches here, it implies the node at ‘index’ has data that is less than at least one of its two child nodes

```
int maxIndex = leftChildIndex;
if (array[leftChildIndex] < array[rightChildIndex])
        maxIndex = rightChildIndex;
```

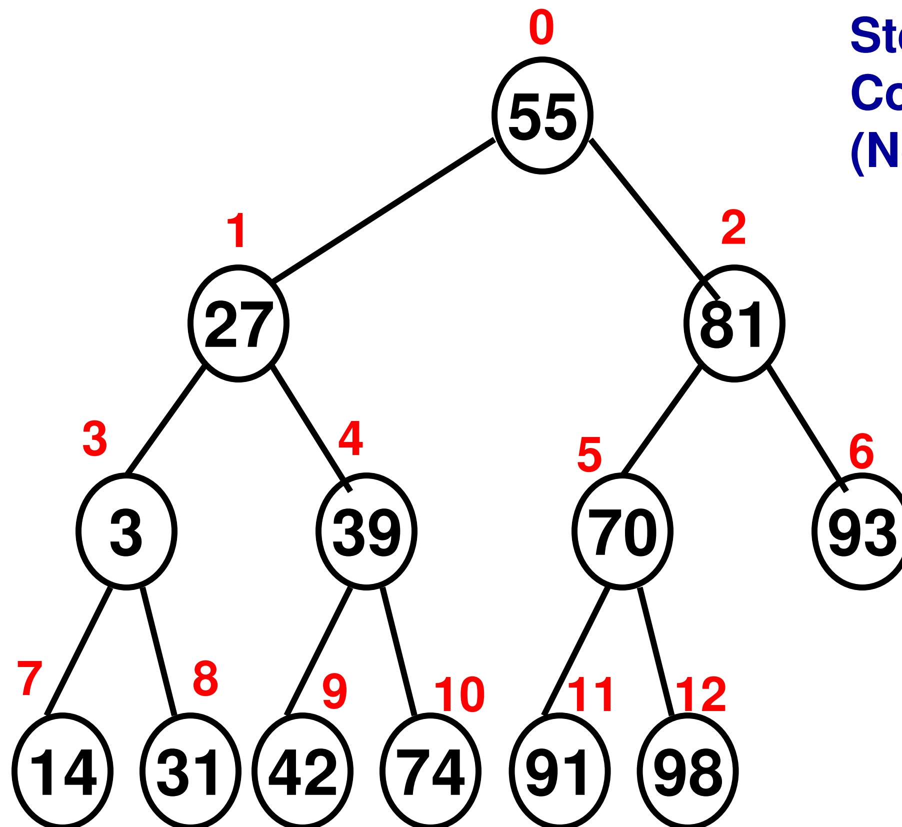
// Between the left and right
// child nodes, find the node
// that has relatively larger
// data, call the index of this
// as ‘maxIndex’ and swap
// its value with the node at
// ‘index’.

```
int temp = array[maxIndex];
array[maxIndex] = array[index];
array[index] = temp;

rearrangeHeapArray(array, arraySize, maxIndex);
```

// Call the rearrangeHeap function in a recursive fashion
// to see if further rearrangements need to be done starting
// from maxIndex

Heap Construction Example 2

0	1	2	3	4	5	6	7	8	9	10	11	12
55	27	81	3	39	70	93	14	31	42	74	91	98



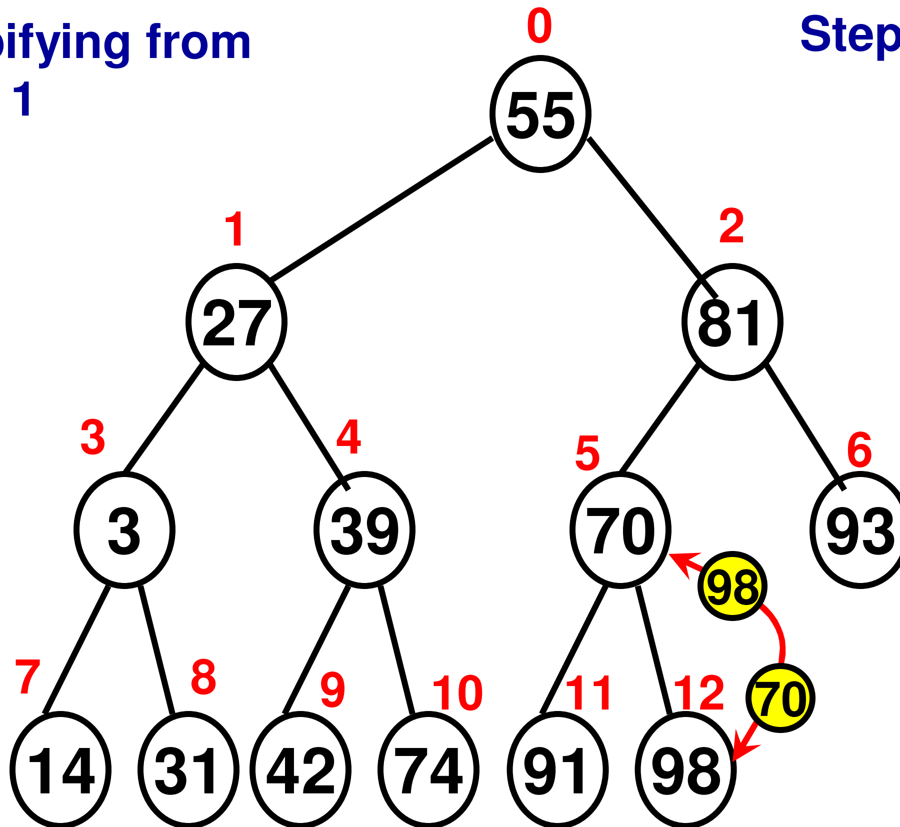
Step 1: Essentially Complete Binary Tree (Not a heap yet!)

Before (Reheapify at Index '5'):

0	1	2	3	4	5	6	7	8	9	10	11	12
55	27	81	3	39	70	93	14	31	42	74	91	98

Start reheapifying from Index ⌊13/2⌋ – 1

Step 2: Reheapify at index 5



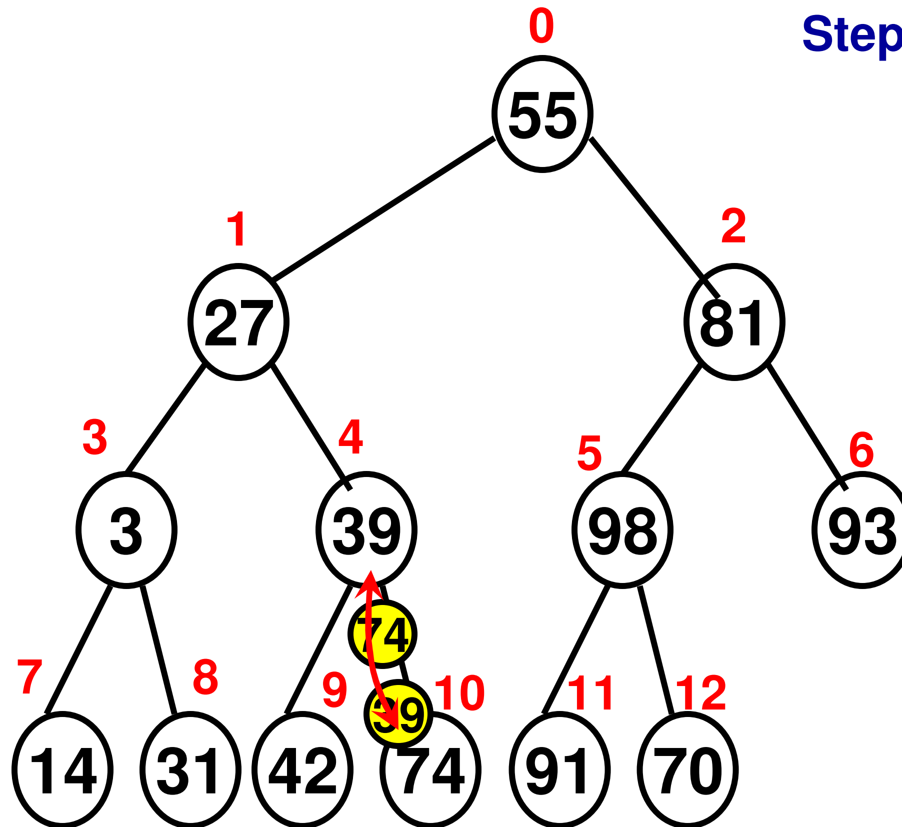
After (Reheapify at Index '5'):

0	1	2	3	4	5	6	7	8	9	10	11	12
55	27	81	3	39	98	93	14	31	42	74	91	70

Before (Reheapify at Index ‘4’):

0	1	2	3	4	5	6	7	8	9	10	11	12
55	27	81	3	39	98	93	14	31	42	74	91	70

Step 2: Reheapify at index 4



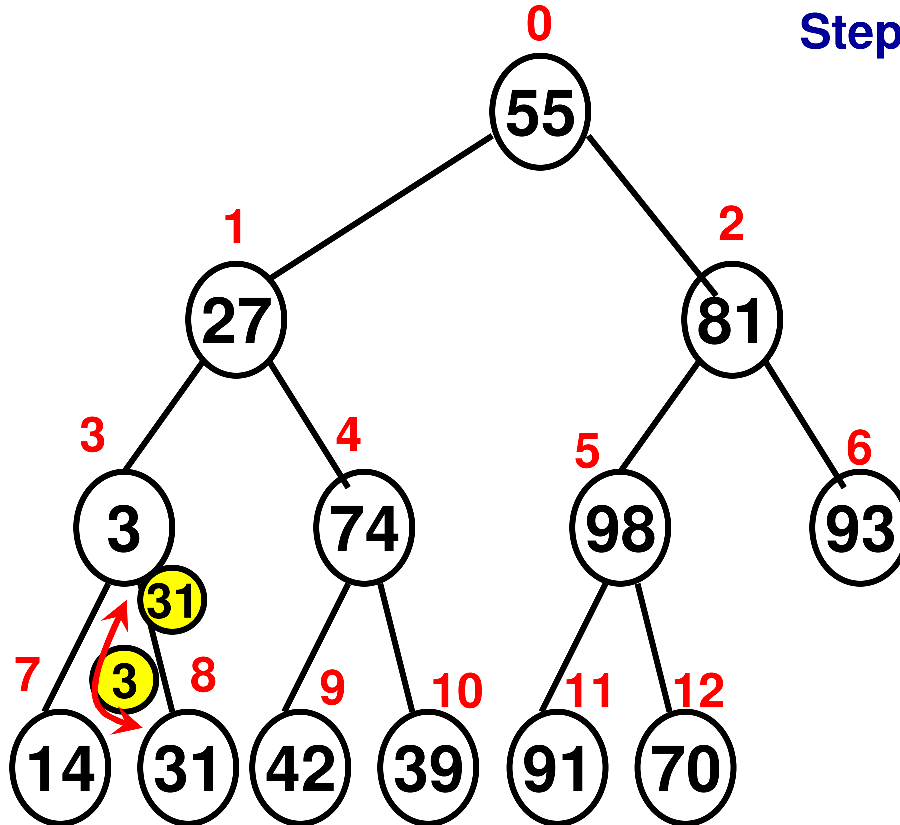
After (Reheapify at Index ‘4’):

0	1	2	3	4	5	6	7	8	9	10	11	12
55	27	81	3	74	98	93	14	31	42	39	91	70

Before (Reheapify at Index '3'):

0	1	2	3	4	5	6	7	8	9	10	11	12
55	27	81	3	74	98	93	14	31	42	39	91	70

Step 2: Reheapify at index 3



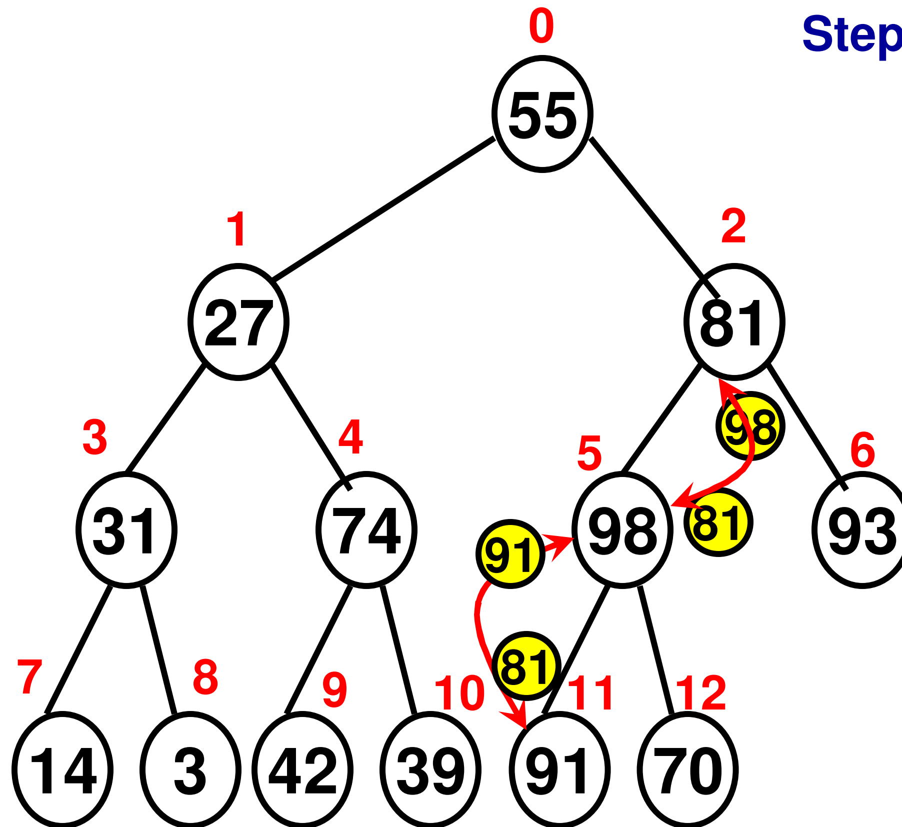
After (Reheapify at Index '3'):

0	1	2	3	4	5	6	7	8	9	10	11	12
55	27	81	31	74	98	93	14	3	42	39	91	70

Before (Reheapify at Index '2'):

0	1	2	3	4	5	6	7	8	9	10	11	12
55	27	81	31	74	98	93	14	3	42	39	91	70

Step 2: Reheapify at index 2



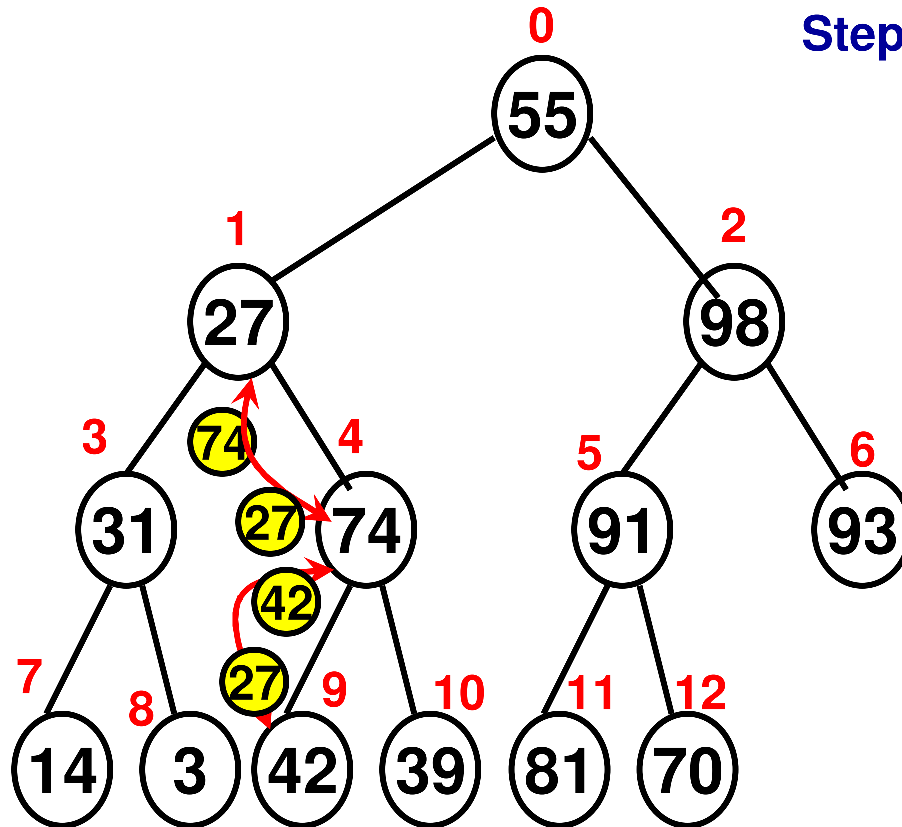
After (Reheapify at Index '2'):

0	1	2	3	4	5	6	7	8	9	10	11	12
55	27	98	31	74	91	93	14	3	42	39	81	70

Before (Reheapify at Index '1'):

0	1	2	3	4	5	6	7	8	9	10	11	12
55	27	98	31	74	91	93	14	3	42	39	81	70

Step 2: Reheapify at index 1



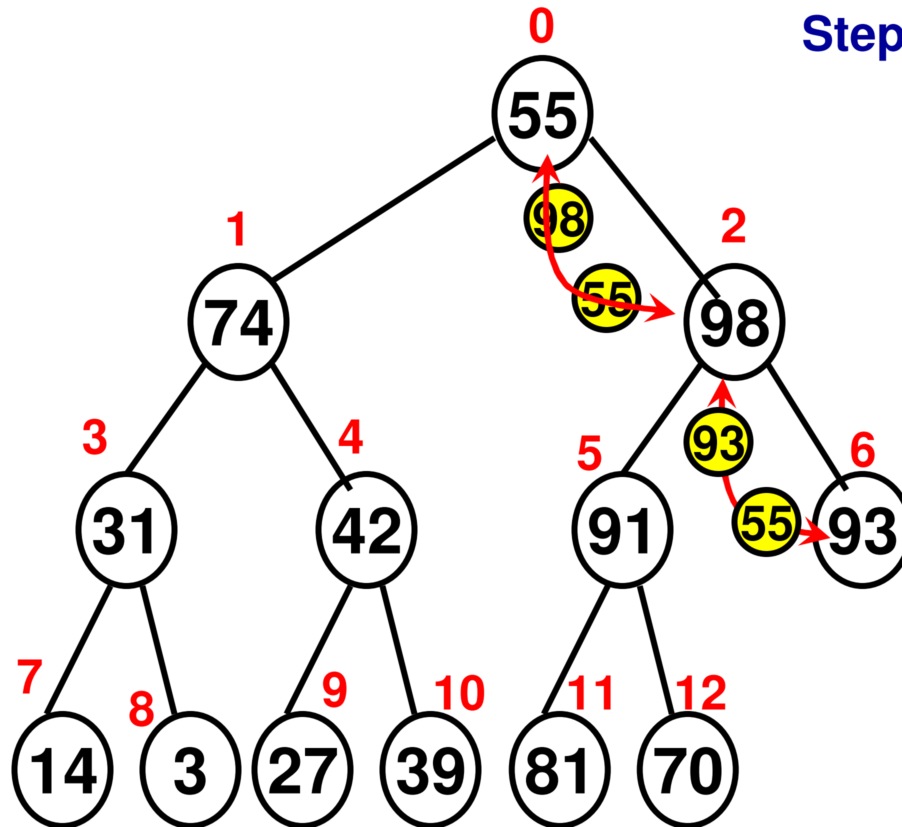
After (Reheapify at Index '1'):

0	1	2	3	4	5	6	7	8	9	10	11	12
55	74	98	31	42	91	93	14	3	27	39	81	70

Before (Reheapify at Index ‘0’):

0	1	2	3	4	5	6	7	8	9	10	11	12
55	74	98	31	42	91	93	14	3	27	39	81	70

Step 2: Reheapify at index 0



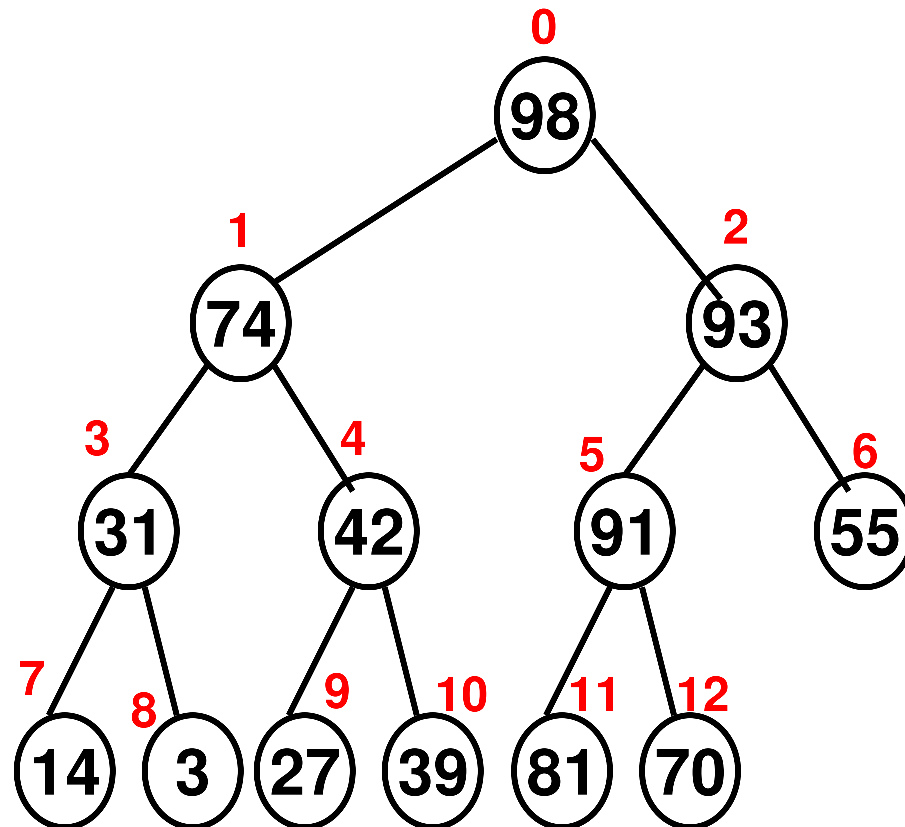
After (Reheapify at Index ‘0’):

0	1	2	3	4	5	6	7	8	9	10	11	12
98	74	93	31	42	91	55	14	3	27	39	81	70

Heap Construction Example 2

0	1	2	3	4	5	6	7	8	9	10	11	12
98	74	93	31	42	91	55	14	3	27	39	81	70

Final Array Representing Max Heap



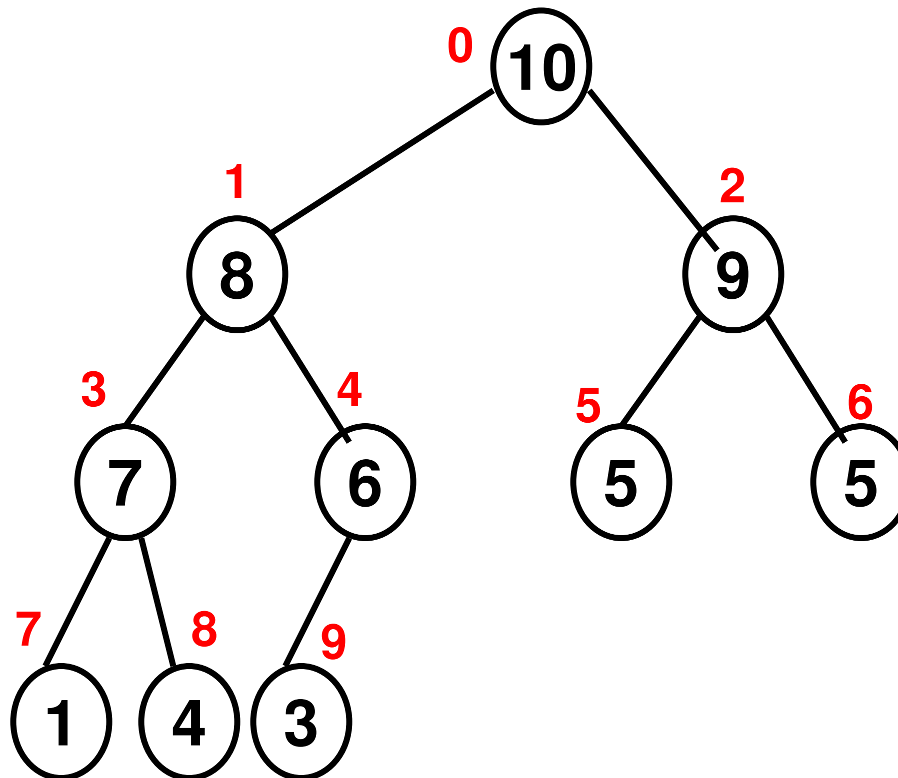
Heap Sort

- Given an array of size ‘n’, first construct a max-heap version of the array.
- Run ‘n-1’ iterations (iteration index 0 to n-1)
  - Swap element at index “0” with element at index “n-1-iteration index”
  - Element at index “0” has now moved to its final location “n-1-iteration index” in the sorted array
  - Reheapify the array as a result of this swap with the array index values ranging from “0” to “n-1-iteration index – 1”.
- Each iteration would require “logn” swappings at the worst case, across the entire height of the binary tree.
- For a total of ‘n-1’ iterations, the time complexity of heap sort is O(nlogn).

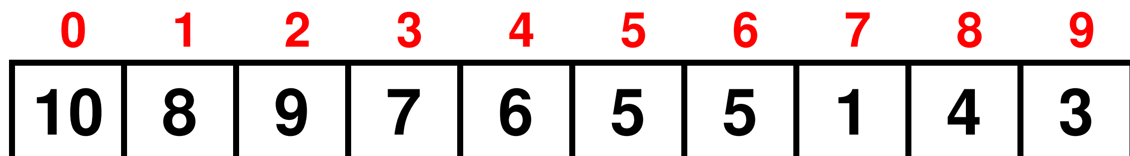
Heap Sort: Example 1

	0	1	2	3	4	5	6	7	8	9
Original Array	5	6	5	4	3	10	9	1	7	8
Max-Heap Version	10	8	9	7	6	5	5	1	4	3

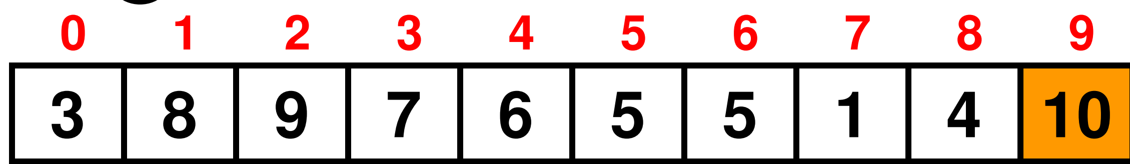
Max-Heap



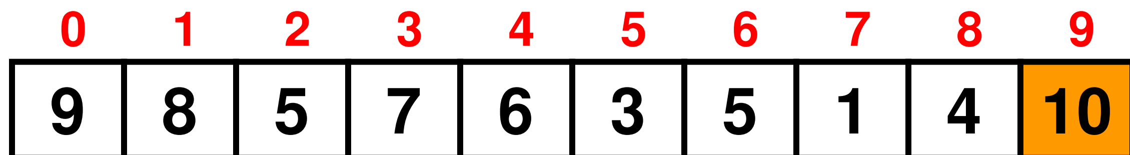
Max-Heap Version



After Swap

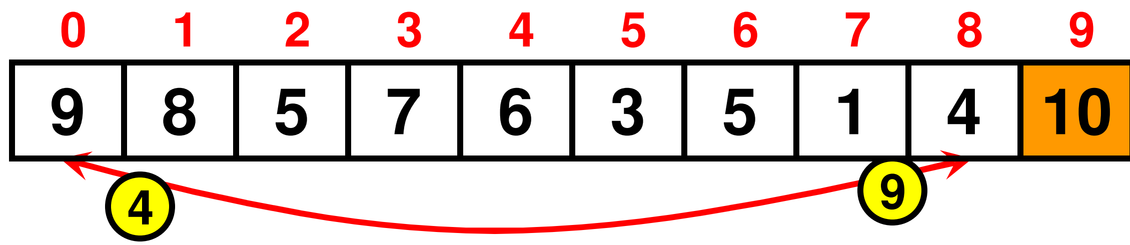


After Reheapify

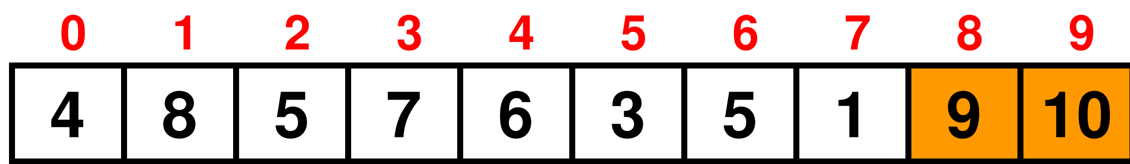


Iteration 0

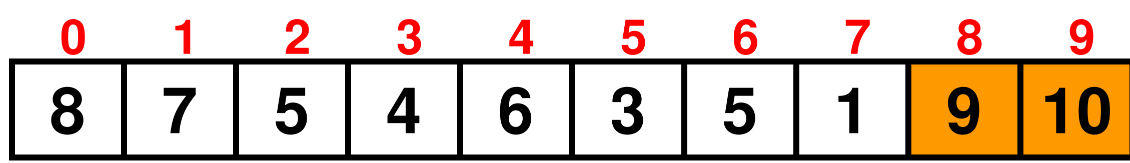
Max-Heap Version

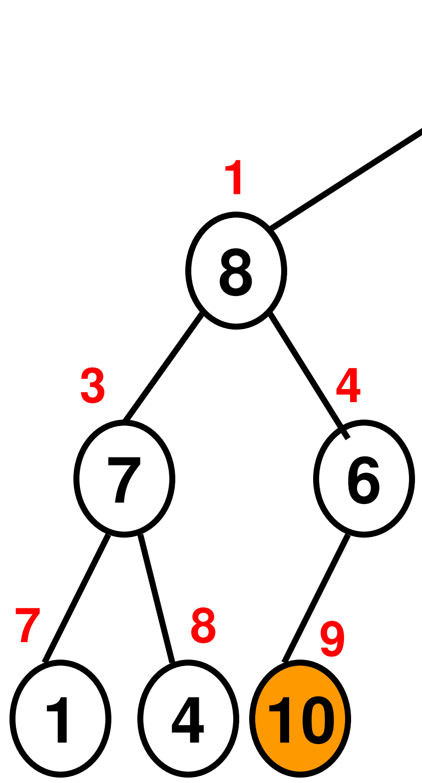


After Swap

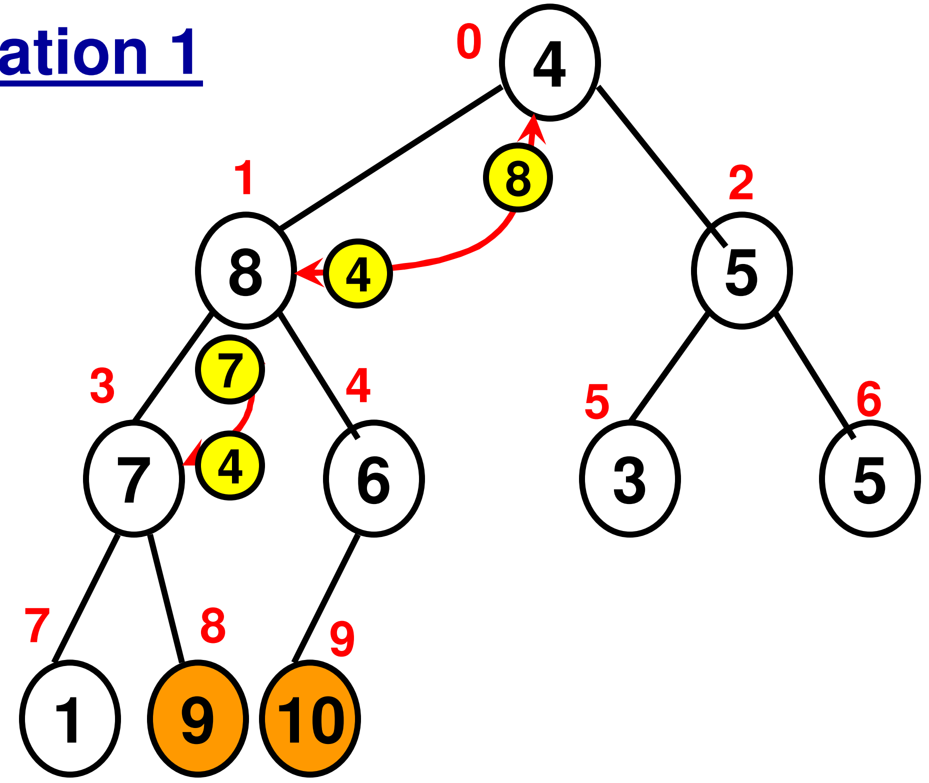


After Reheapify

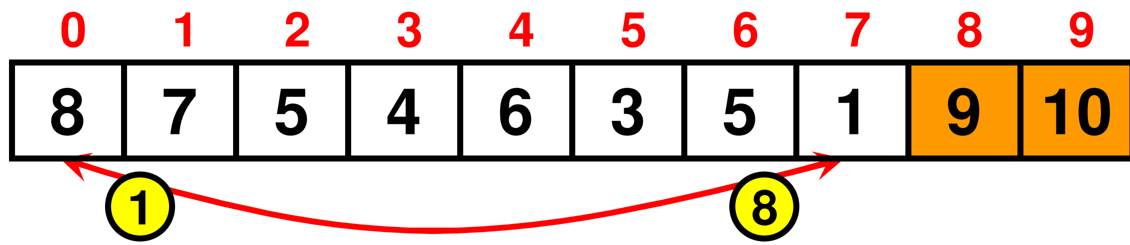




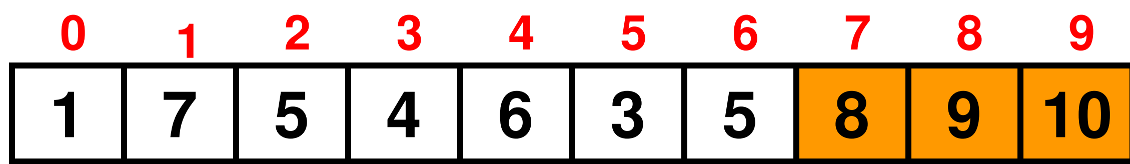
Iteration 1



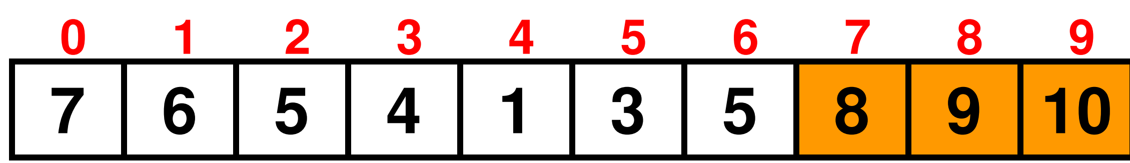
Max-Heap Version



After Swap

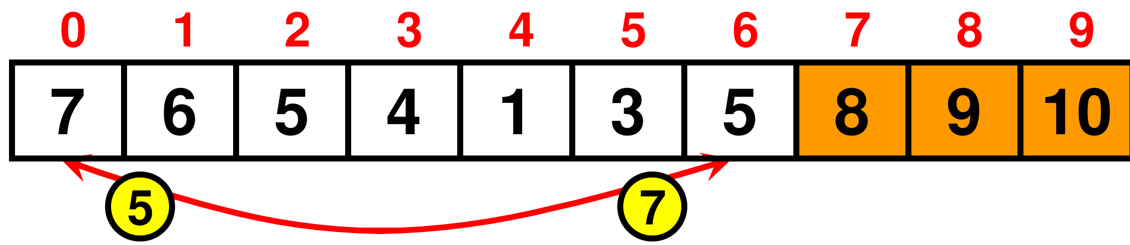


After Reheapify

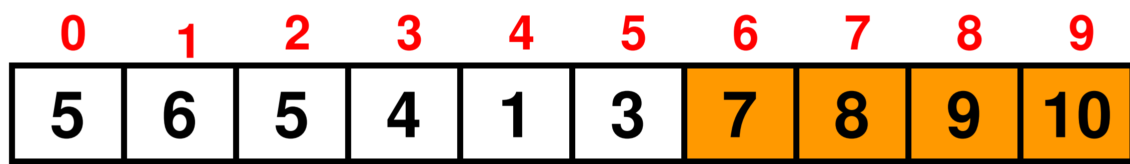


Iteration 2

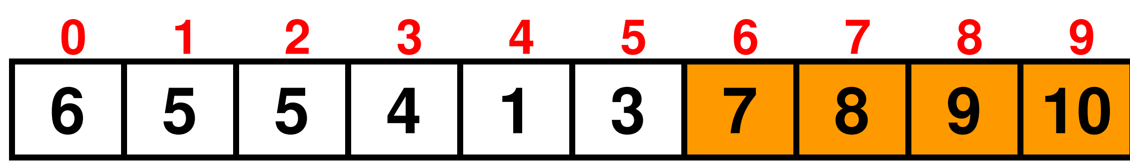
Max-Heap Version



After Swap

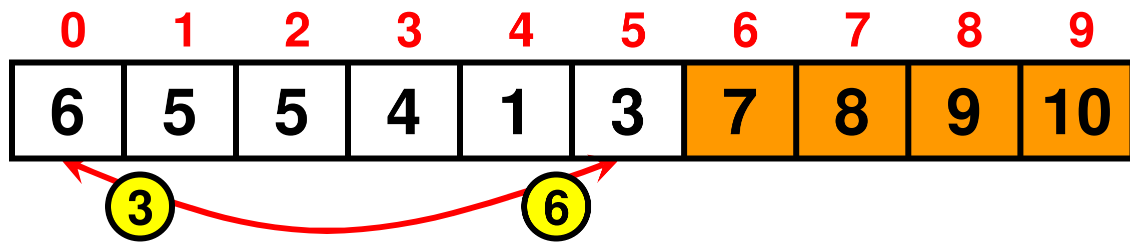


After Reheapify

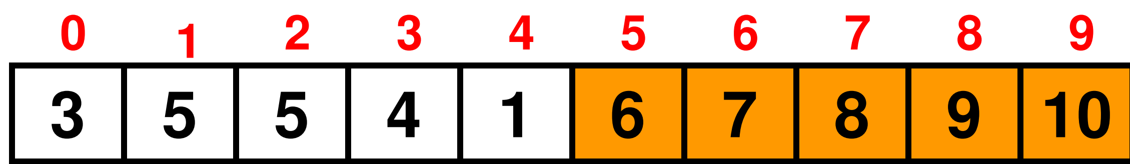


Iteration 3

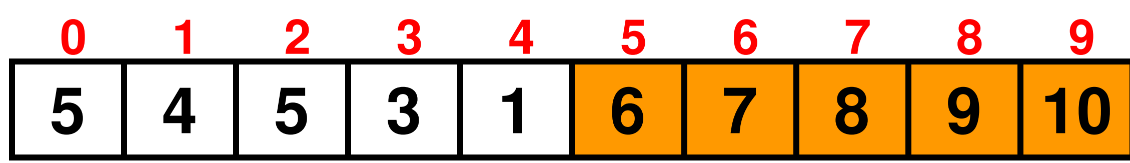
Max-Heap Version



After Swap

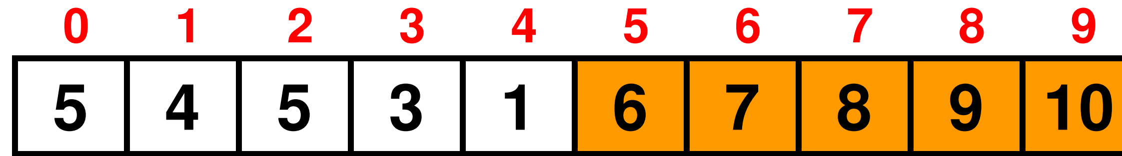


After Reheapify

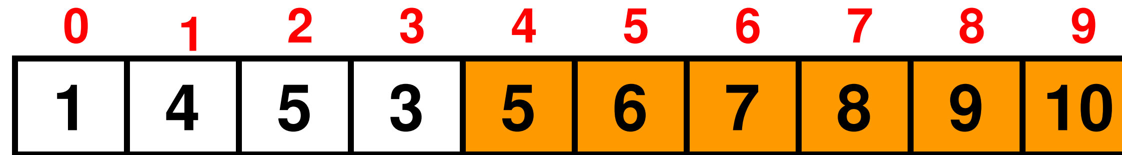


Iteration 4

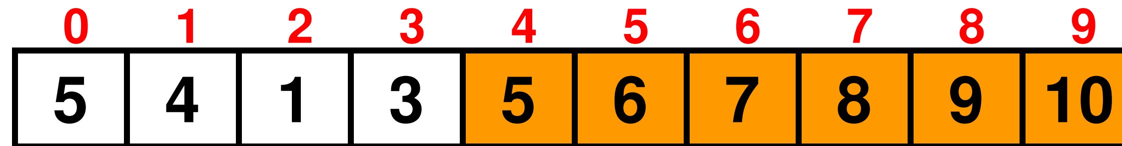
Max-Heap Version



After Swap



After Reheapify

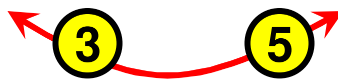


Iteration 5

Iteration 6

Max-Heap Version

0	1	2	3	4	5	6	7	8	9
5	4	1	3	5	6	7	8	9	10



After Swap

0	1	2	3	4	5	6	7	8	9
3	4	1	5	5	6	7	8	9	10

After Reheapify

0	1	2	3	4	5	6	7	8	9
4	3	1	5	5	6	7	8	9	10

Iteration 7

Max-Heap Version

0	1	2	3	4	5	6	7	8	9
4	3	1	5	5	6	7	8	9	10

After Swap

0	1	2	3	4	5	6	7	8	9
1	3	4	5	5	6	7	8	9	10

After Reheapify

0	1	2	3	4	5	6	7	8	9
3	1	4	5	5	6	7	8	9	10

Max-Heap Version

0	1	2	3	4	5	6	7	8	9
3	1	4	5	5	6	7	8	9	10

After Swap

0	1	2	3	4	5	6	7	8	9
1	3	4	5	5	6	7	8	9	10

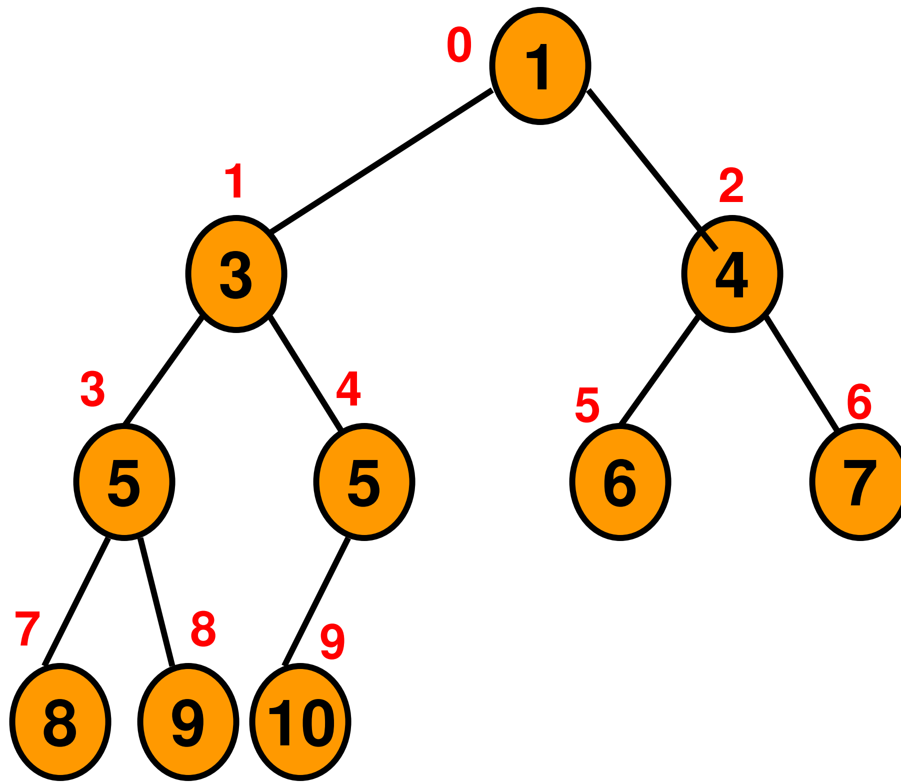
After Reheapify

0	1	2	3	4	5	6	7	8	9
1	3	4	5	5	6	7	8	9	10

Iteration 8

Heap Sort: Example
Final Sorted Array

0	1	2	3	4	5	6	7	8	9
1	3	4	5	5	6	7	8	9	10



Heap Sort (Code 8.1: C++)

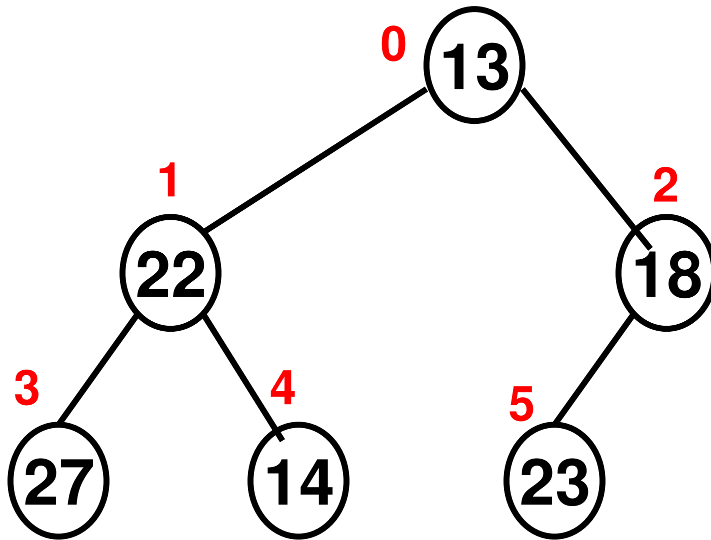
```
for (int iterationIndex = 0; iterationIndex < arraySize; iterationIndex++){

    int temp = array[0];
    array[0] = array[arraySize-1-iterationIndex];
    array[arraySize-1-iterationIndex] = temp;

    rearrangeHeapArray(array, arraySize-1-iterationIndex, 0);

    cout << "Iteration " << iterationIndex << " : ";
    for (int index = 0; index < arraySize; index++)
        cout << array[index] << " ";

    cout << endl;

}
```

Swap the element at the top of the heap with the element at the last index (arraySize-1-iterationIndex) in the active portion of the array

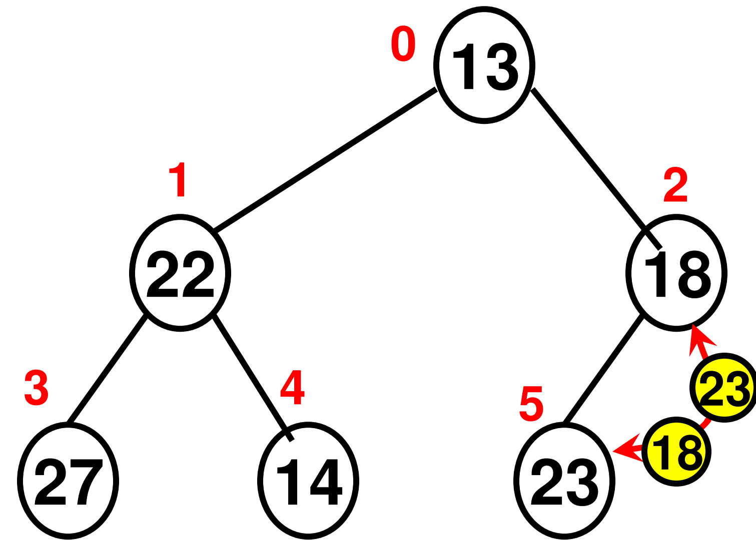
arraySize-1-iterationIndex is also the number of elements in the Unsorted portion of the array (in otherwords, the size of the active portion of the array)

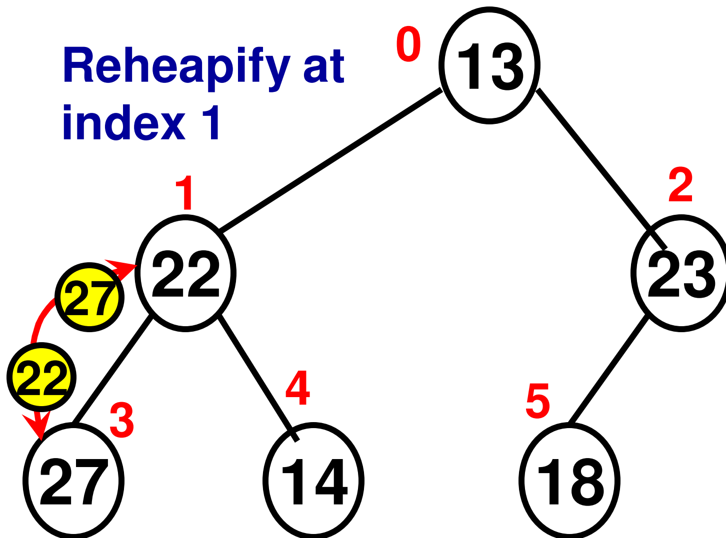
Original Array

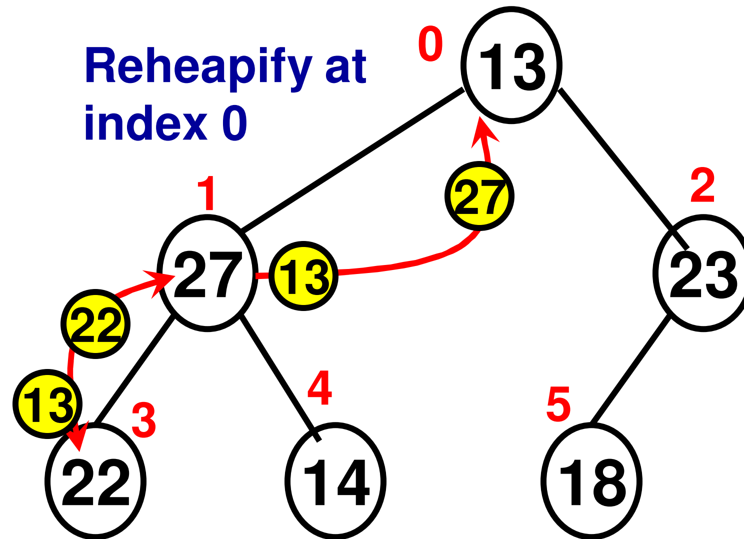
13	22	18	27	14	23



Reheapify at index 2



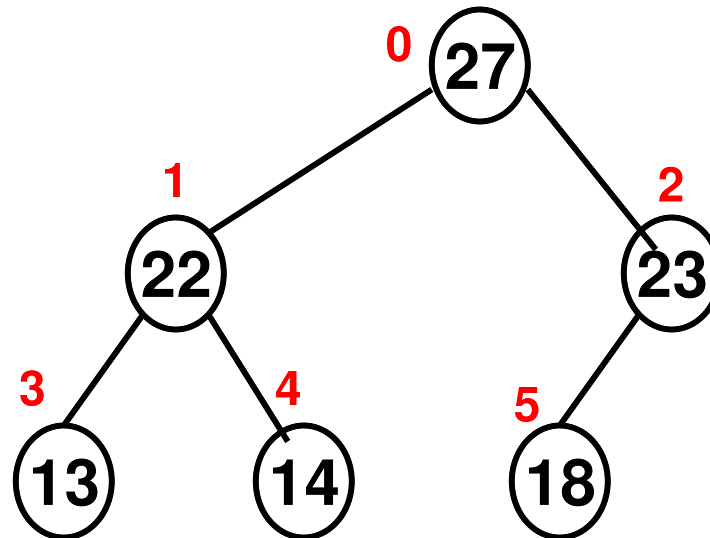




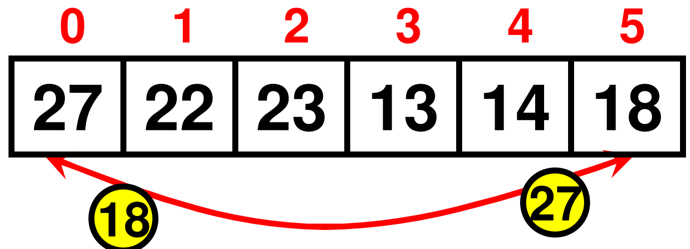
Heap Sort: Example 2

	0	1	2	3	4	5
Original Array	13	22	18	27	14	23
Max-Heap Version	27	22	23	13	14	18

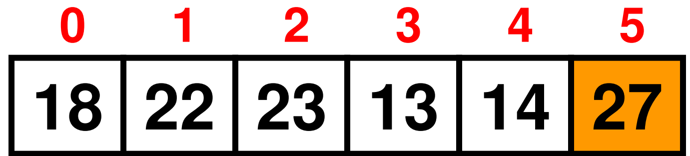
Max-Heap



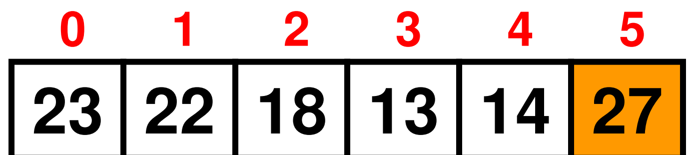
Max-Heap Version



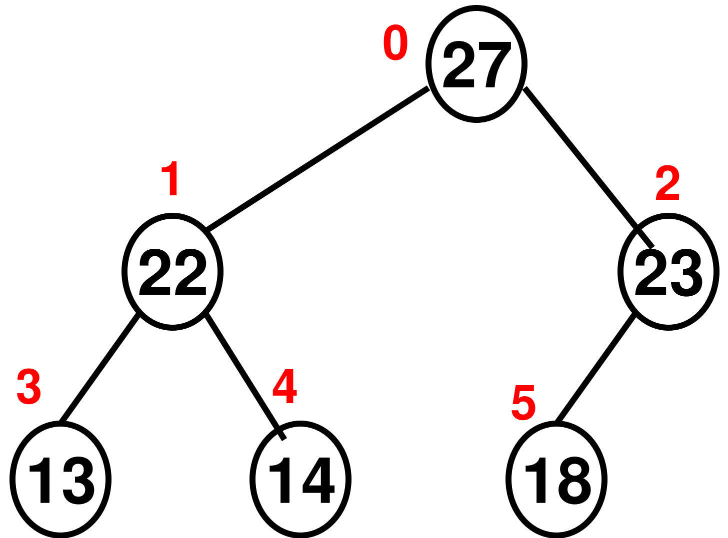
After Swap

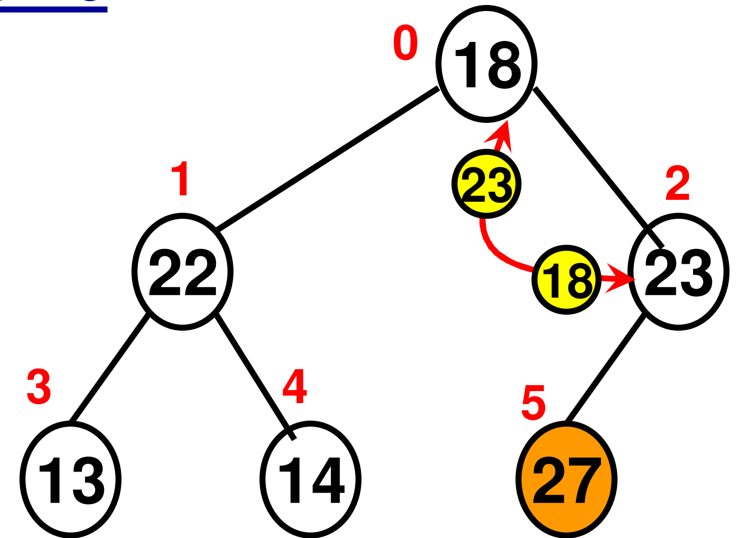


After Reheapify

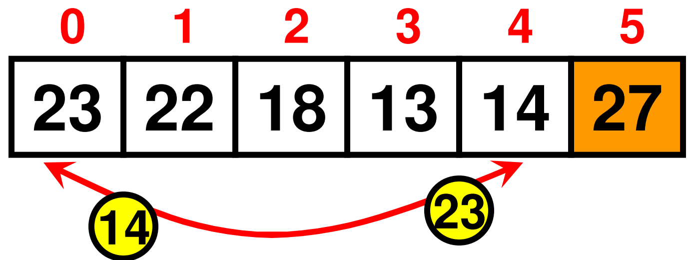


Iteration 0

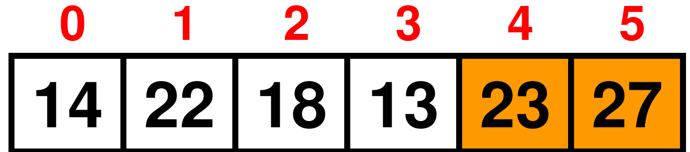




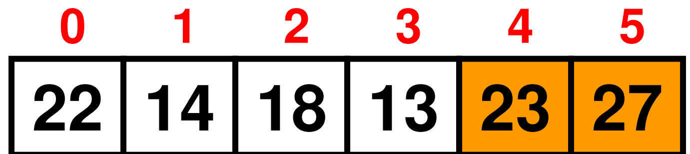
Max-Heap Version



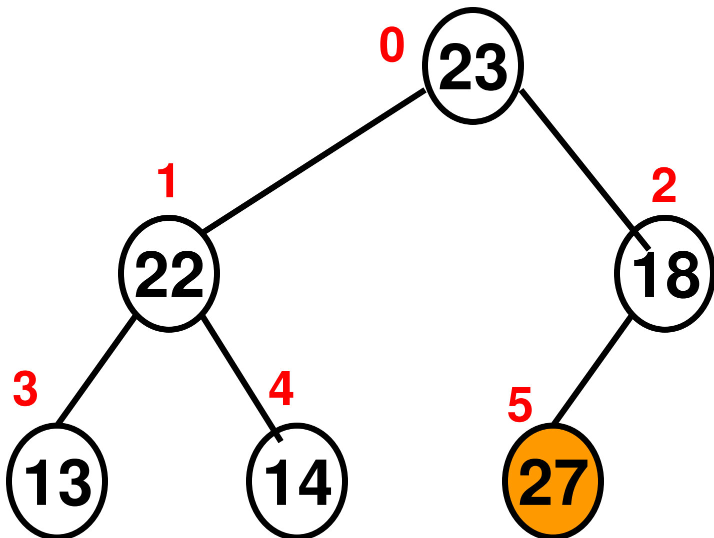
After Swap

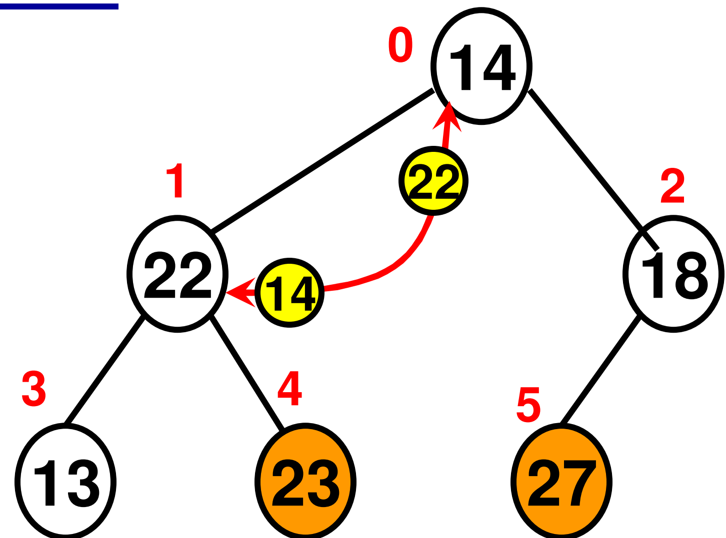


After Reheapify

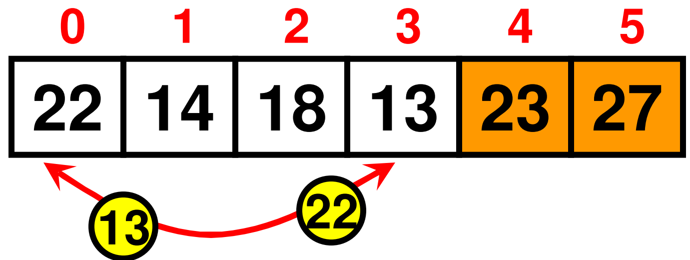


Iteration 1

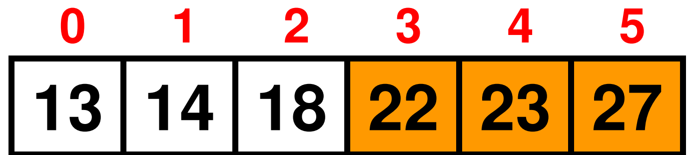




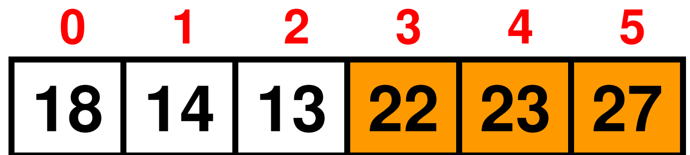
Max-Heap Version



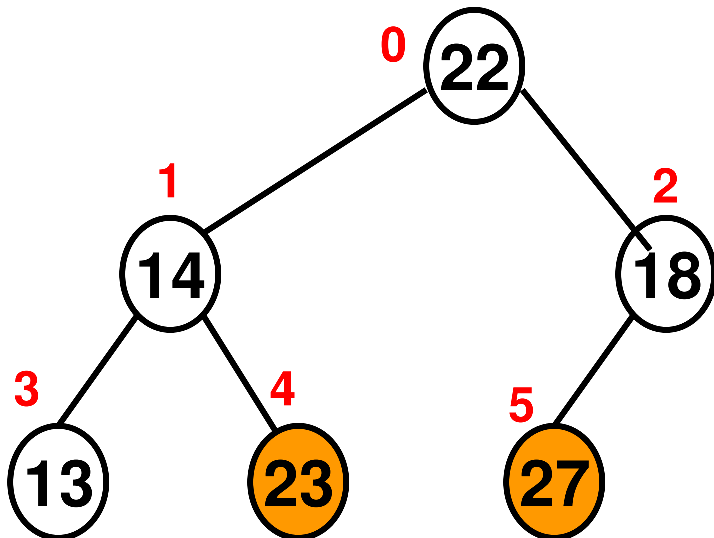
After Swap

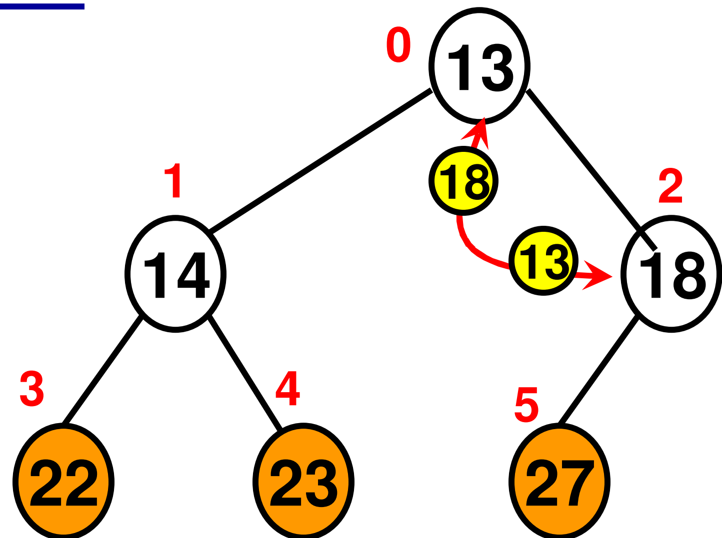


After Reheapify

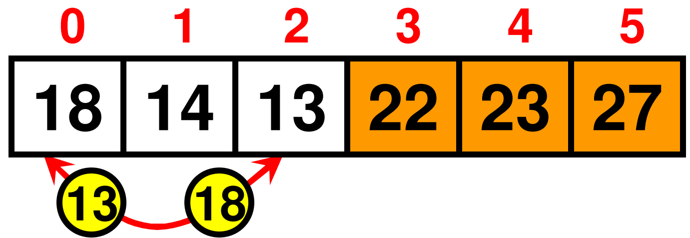


Iteration 2

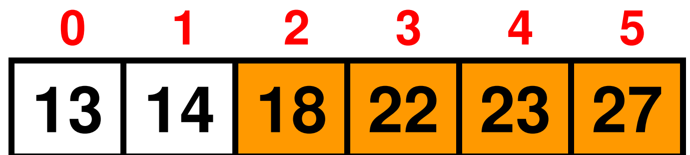




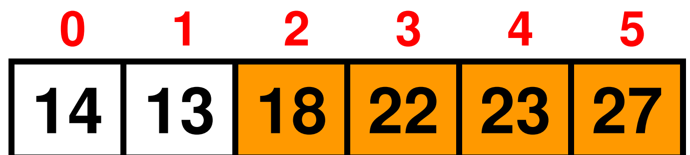
Max-Heap Version



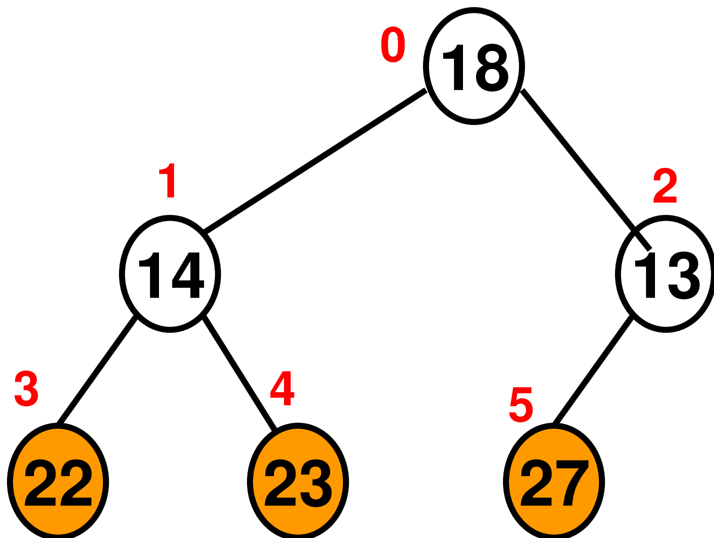
After Swap

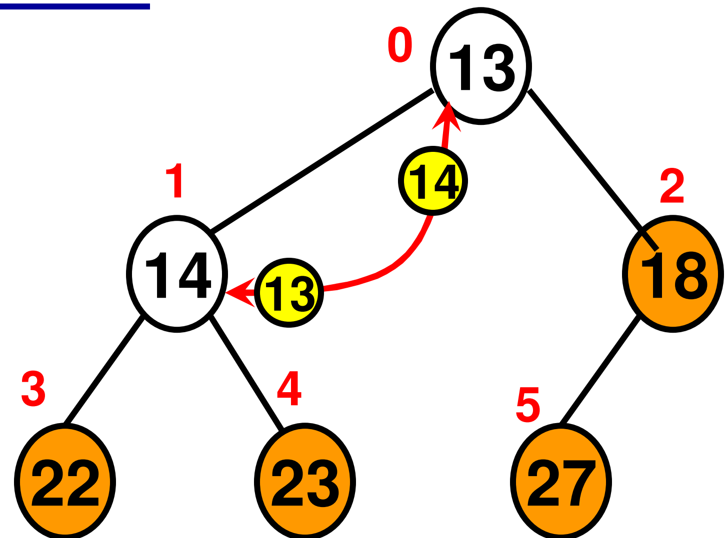


After Reheapify

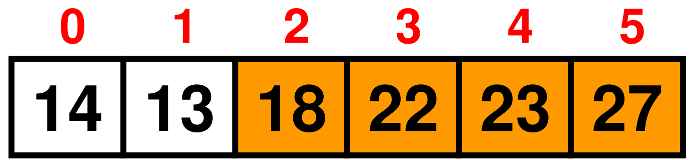


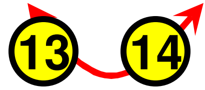
Iteration 3



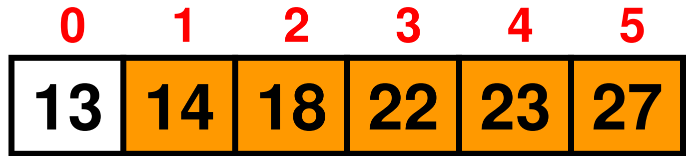


Max-Heap Version

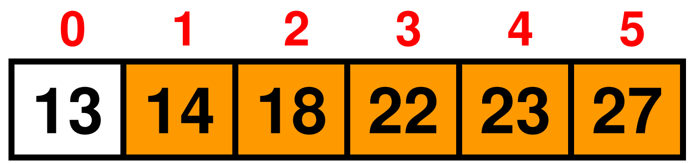




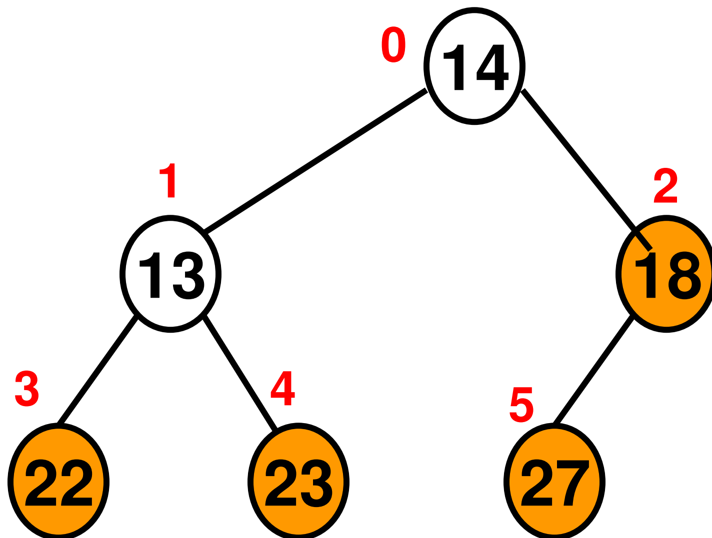
After Swap

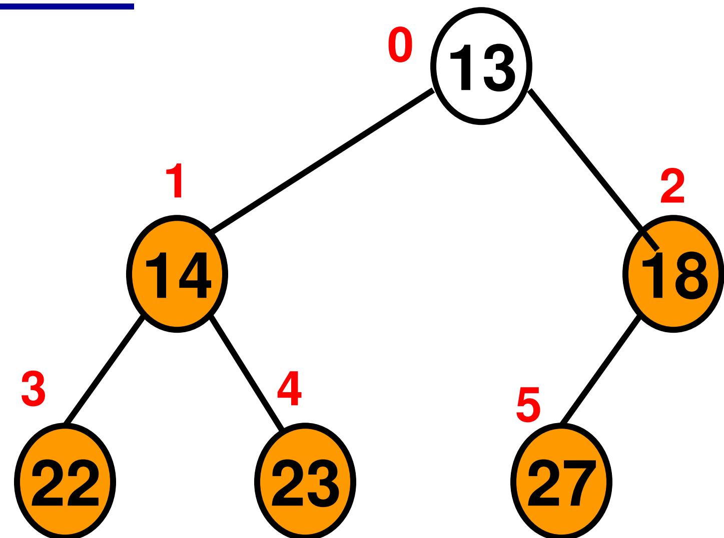


After Reheapify



Iteration 4

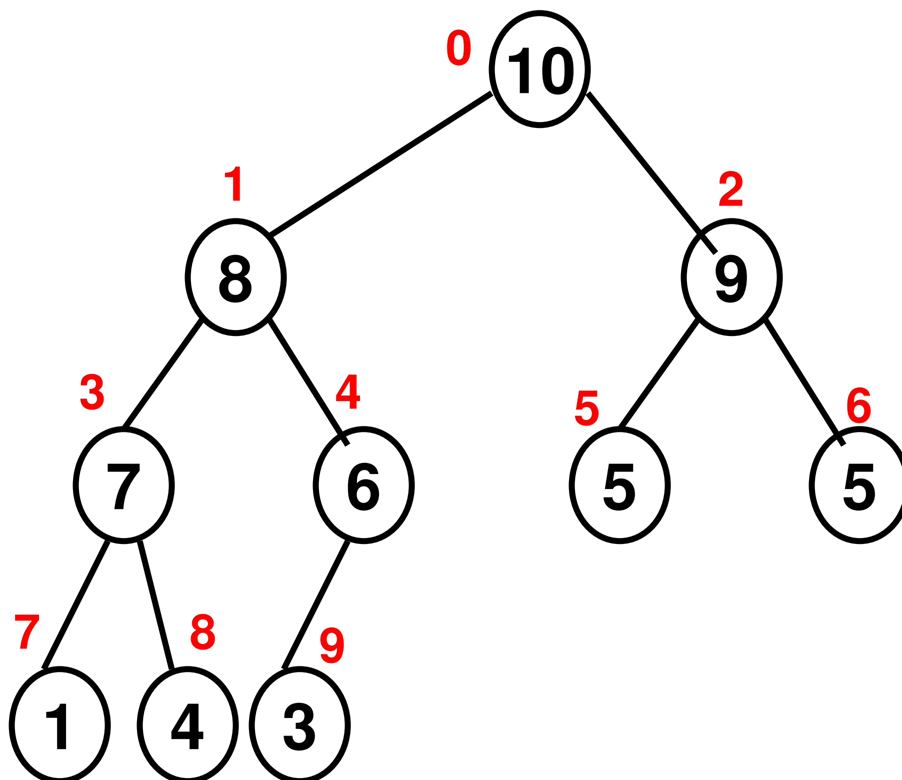


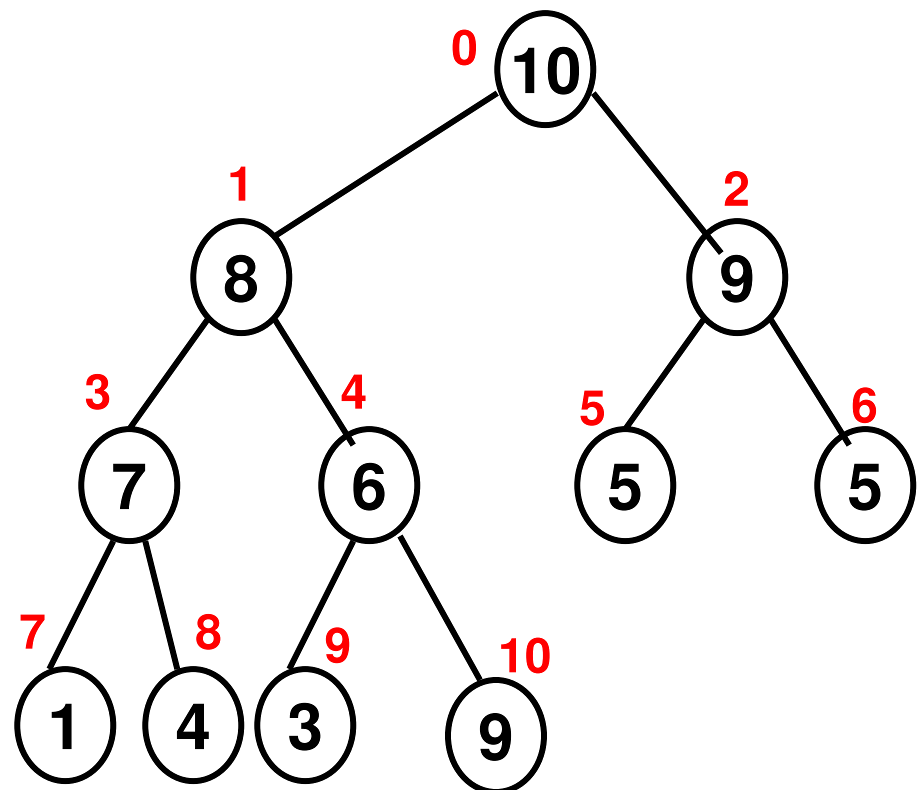


Inserting Data to a (Max) Heap

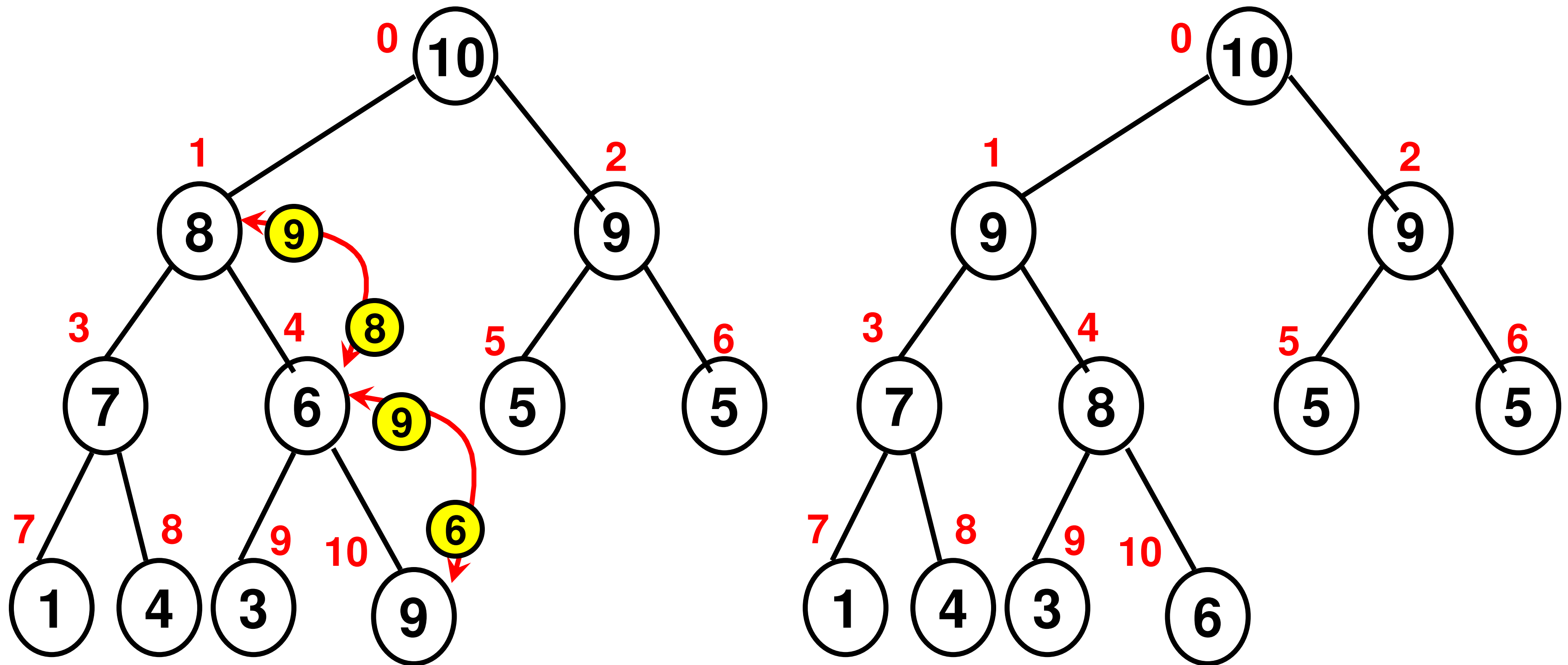
- Insert the data at the bottommost level at the leftmost position. Then reheapify starting from the parent node of the inserted node and recursively all the way to the root node or the internal node at which the heap property is satisfied.

Assume we want to insert data ‘9’
Initially, insert at index 10 and reheapify starting from index (10-1)/2 = 4.





Inserting Data to a (Max) Heap



Heap – Priority Queue

- A heap can be used to implement a priority queue.
- Each element in the queue has a priority (typically, the numerical value of the element is its priority).

- The elements in the queue are arranged as a max or min heap (depending on how we define priority: the element with the largest value has the highest priority – max heap; the element with the lowest value has the highest priority – min heap).

- A dequeue operation on the priority queue will remove the root node of the heap and it will take O(logn) time to reheapify the heap.
- An enqueue operation on the priority queue will insert the node initially at the last index and then reheapify all the way to the root node if needed: O(logn) time.

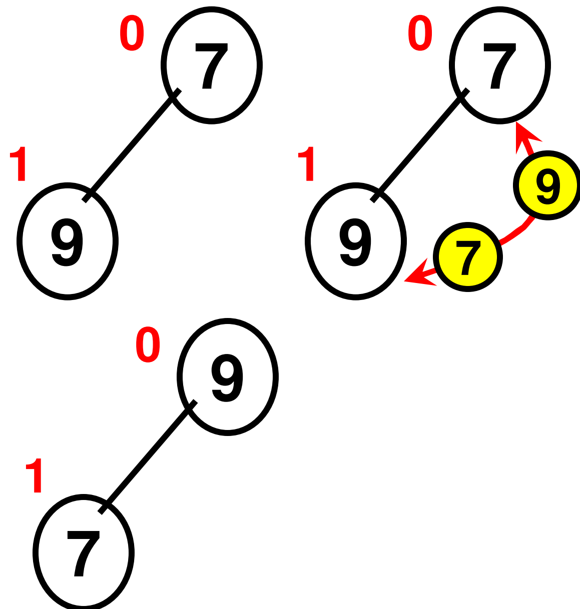
- <u>Tradeoff:</u> We saw earlier that a regular FIFO queue could be implemented as a doubly linked list with O(1) time for the enqueue and dequeue operations.

Priority Queue Construction: Example

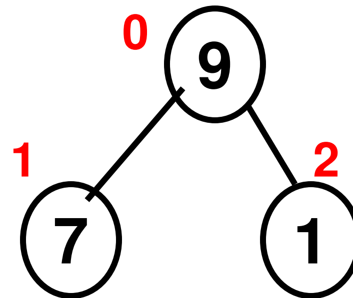
- Construct a sequence of priority queues (max heaps) with the joining of the elements 7, 9, 1, 10, 5, 8 one at a time.

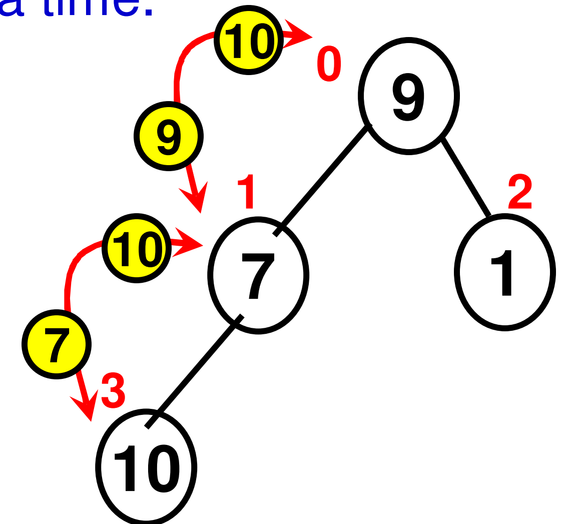
1. Enqueue of 7

2. Enqueue of 9

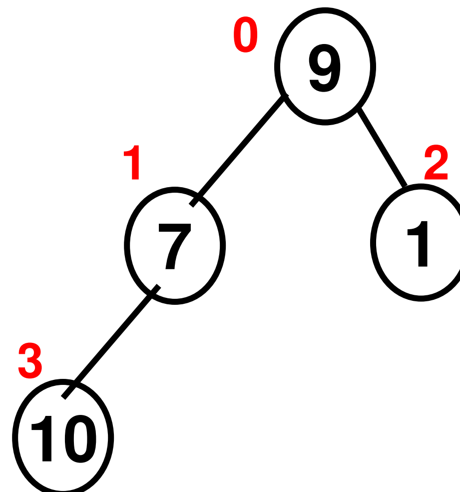


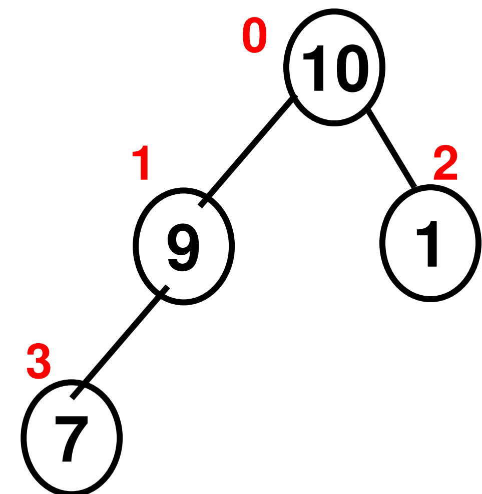
3. Enqueue of 1





4. Enqueue of 10

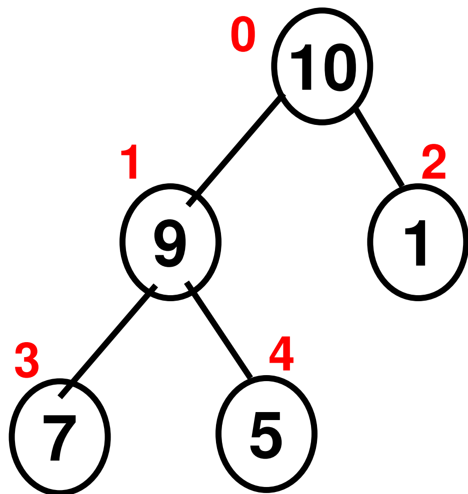




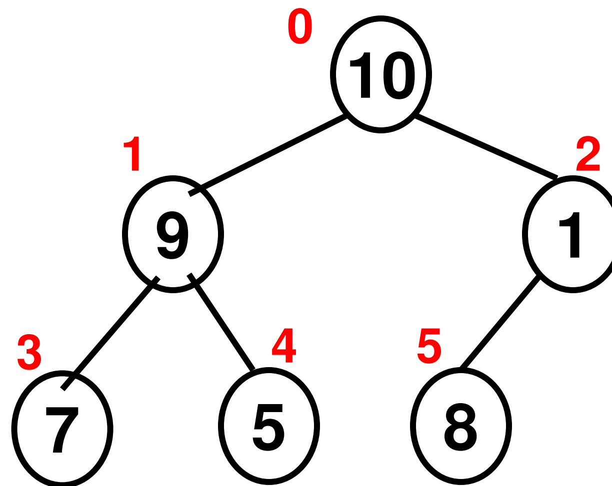
Priority Queue Construction: Example

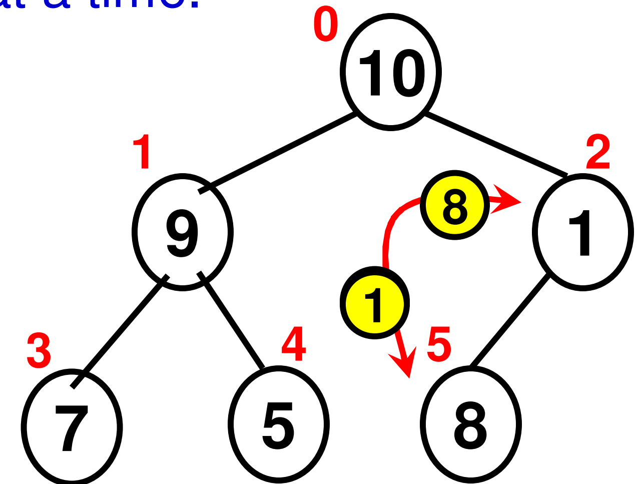
- Construct a sequence of priority queues (max heaps) with the joining of the elements 7, 9, 1, 10, 5, 8 one at a time.

5. Enqueue of 5

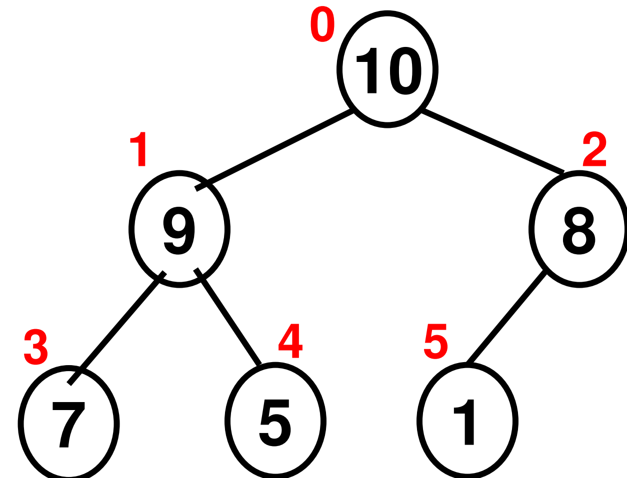


6. Enqueue of 8





Final Max Heap Priority Queue



Dequeue of a Priority Queue (Max Heap)

- Remove the root node.
- Replace the data for the root node with the data of the element at the rightmost leaf node at the bottommost level, and remove the latter.
- Reheapify starting from the root node.