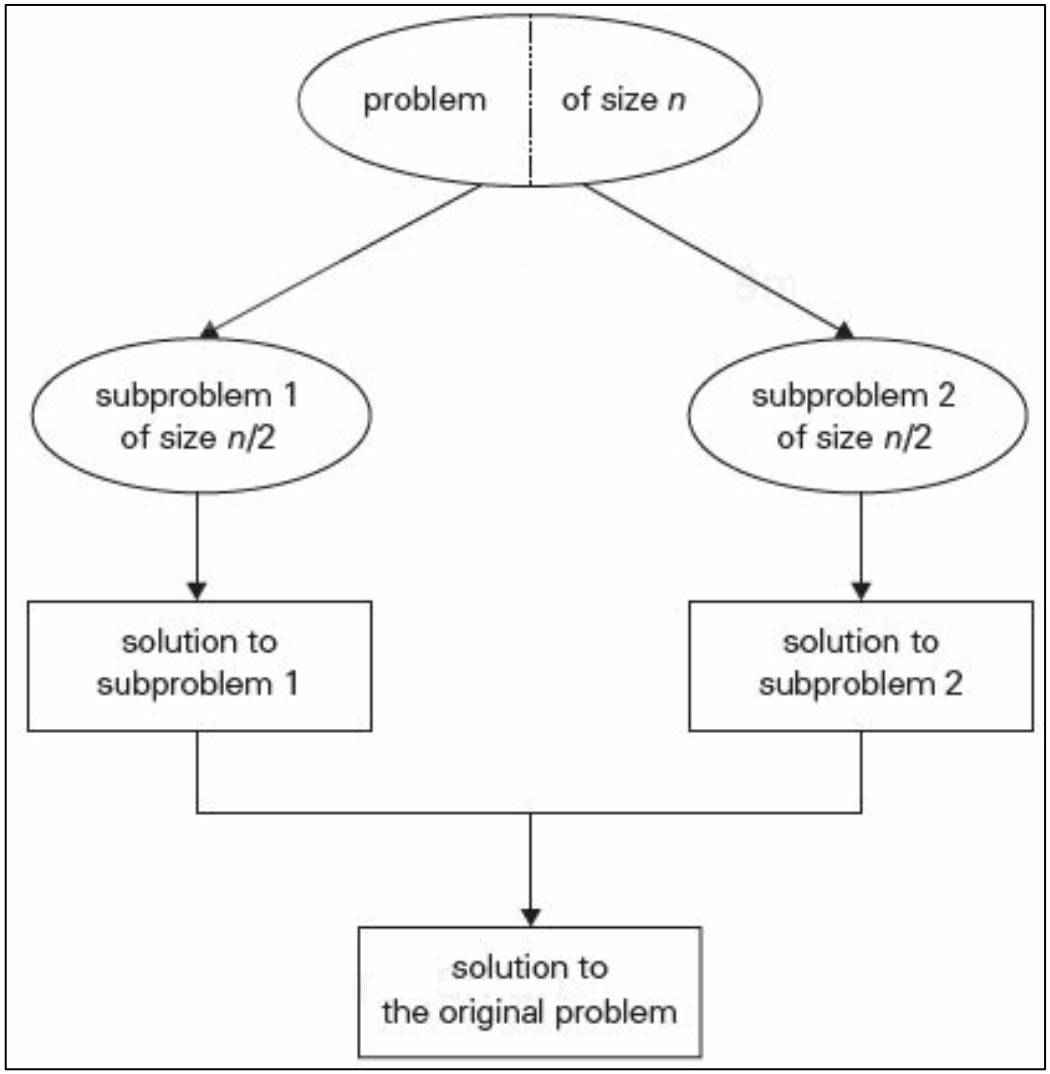
Module 2:
Divide and Conquer

Dr. Natarajan Meghanathan
Professor of Computer Science
Jackson State University
Jackson, MS 39217
E-mail: natarajan.meghanathan@jsums.edu

Divide-and-Conquer

The most-well known algorithm design strategy:

1. We divide a problem of instance size ‘n’ into several sub problems (each of size n/b);
2. Solve ‘a’ of these sub problems ($a \geq 1$; $b > 1$) recursively and
3. Combine the solutions to these sub problems to obtain a solution for the larger problem.



Typical Case of Divide and Conquer Problems

Master Theorem to Solve Recurrence Relations

- Assuming that size n is a power of b to simplify analysis, we have the following recurrence for the running time, T(n) = a T(n/b) + f(n)
  - where f(n) is a function that accounts for the time spent on dividing an instance of size n into instances of size n/b and combining their solutions.
- Master Theorem:

If $f(n) \in \Theta(n^d)$ where $d \geq 0$, then

$$T(n) \in \begin{cases} \Theta(n^d) & \text{if } a < b^d, \\ \Theta(n^d \log n) & \text{if } a = b^d, \\ \Theta(n^{\log_b a}) & \text{if } a > b^d. \end{cases}$$

The same results hold good for O and Ω too.

Examples:

1) T(n) = 4T(n/2) + n
a = 4; b = 2; d = 1 ➔ a > b^d

$$T(n) = \Theta\left(n^{\log_2 4}\right) = \Theta(n^2)$$

2) T(n) = 4T(n/2) + n^2
a = 4; b = 2; d = 2 ➔ a = b^d

$$T(n) = \Theta\left(n^2 \log n\right)$$

3) T(n) = 4T(n/2) + n^3
a = 4; b = 2; d = 3 ➔ a < b^d

$$T(n) = \Theta\left(n^3\right)$$

4) T(n) = 2T(n/2) + 1
a = 2; b = 2; d = 0 ➔ a > b^d

$$T(n) = \Theta\left(n^{\log_2 2}\right) = \Theta(n)$$

Master Theorem: More Problems

$T(n) = 3T(n/3) + \sqrt{n}$

$T(n) = 3T(n/3) + n^{(1/2)}$

$a = 3;\ b = 3;\ d = 1/2$

$b^d = 3^{1/2} = 1.732$

$a = 3 > b^d = 1.732$

$T(n) = \Theta(n^{\log_3 3}) = \Theta(n)$

$T(n) = 4T(n/2) + \log n$

$a = 4;\ b = 2;\ d < 1$, because $\log n < n^1$

$b^d = 2^{<1} < 2$

$a > b^d$

$T(n) = \Theta(n^{\log_2 4}) = \Theta(n^2)$

$T(n) = 6T(n/3) + n^2 \log n$

$a = 6;\ b = 3;\ 2 < d < 3$, because $\log n < n$ and hence $n^2 \log n < n^3$

$b^d = 3^{2<d<3} > 9 > a$

$a < b^d$

Hence, $T(n) = \Theta(n^d) = \Theta(n^2 \log n)$

Merge Sort

- Split array A[0..*n*-1] in two about equal halves and make copies of each half in arrays B and C
- Sort arrays B and C recursively
- Merge sorted arrays B and C into array A as follows:
  - Repeat the following until no elements remain in one of the arrays:
    - compare the first elements in the remaining unprocessed portions of the arrays
    - copy the smaller of the two into A, while incrementing the index indicating the unprocessed portion of that array
  - Once all elements in one of the arrays are processed, copy the remaining unprocessed elements from the other array into A.

Merge Sort

ALGORITHM *Mergesort*$(A[0..n-1])$

//Sorts array $A[0..n-1]$ by recursive mergesort
//Input: An array $A[0..n-1]$ of orderable elements
//Output: Array $A[0..n-1]$ sorted in nondecreasing order
if $n > 1$
    copy $A[0..\lfloor n/2 \rfloor - 1]$ to $B[0..\lfloor n/2 \rfloor - 1]$
    copy $A[\lfloor n/2 \rfloor..n-1]$ to $C[0..\lceil n/2 \rceil - 1]$
    Mergesort$(B[0..\lfloor n/2 \rfloor - 1])$
    Mergesort$(C[0..\lceil n/2 \rceil - 1])$
    Merge(B, C, A)

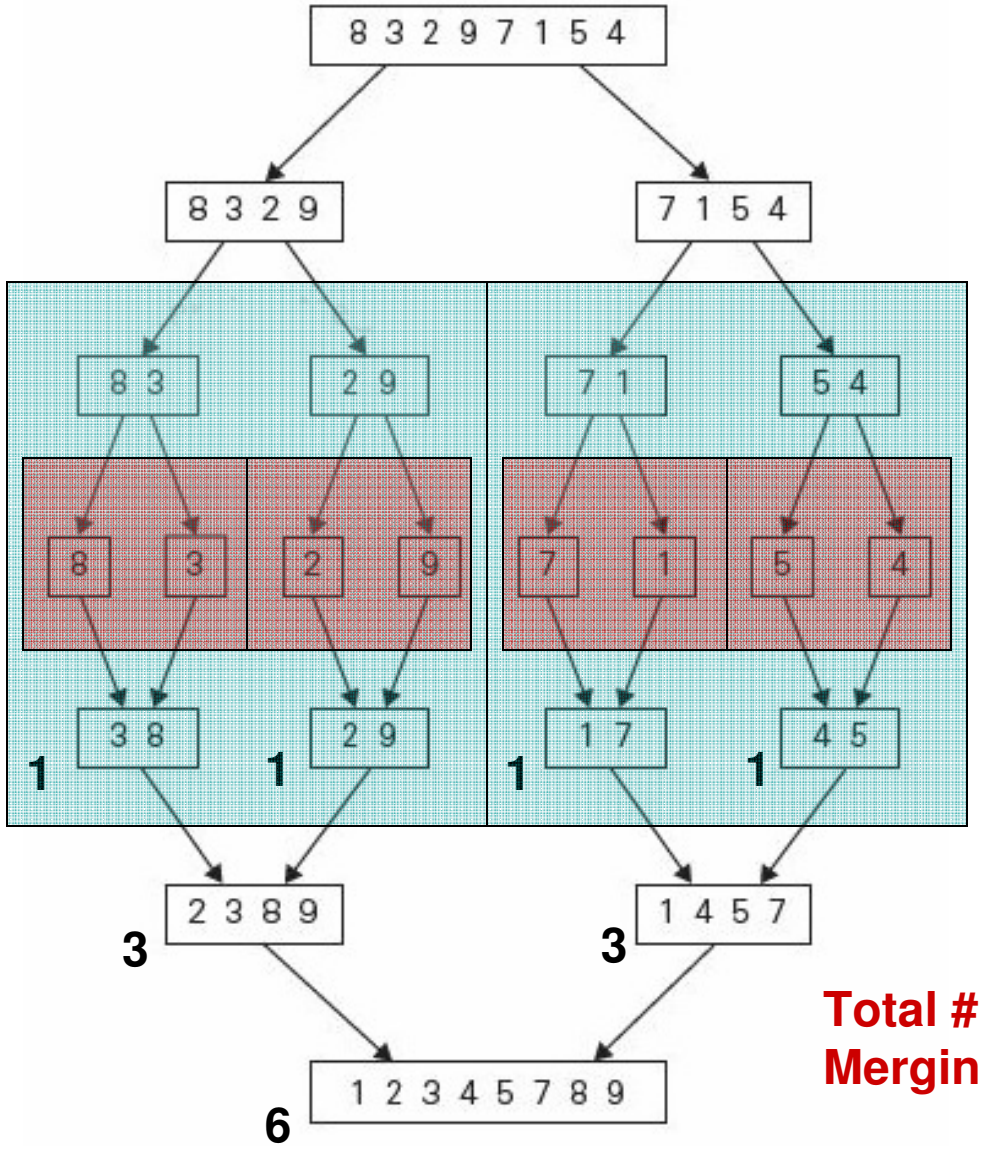
Merge Algorithm

ALGORITHM *Merge*$(B[0..p-1], C[0..q-1], A[0..p+q-1])$

//Merges two sorted arrays into one sorted array
//Input: Arrays $B[0..p-1]$ and $C[0..q-1]$ both sorted
//Output: Sorted array $A[0..p+q-1]$ of the elements of B and C
$i \leftarrow 0;\ \ j \leftarrow 0;\ \ k \leftarrow 0$
while $i < p$ **and** $j < q$ **do**
    if $B[i] \leq C[j]$
        $A[k] \leftarrow B[i];\ \ i \leftarrow i+1$
    else $A[k] \leftarrow C[j];\ \ j \leftarrow j+1$
    $k \leftarrow k+1$
if $i = p$
    copy $C[j..q-1]$ to $A[k..p+q-1]$
else copy $B[i..p-1]$ to $A[k..p+q-1]$

Example for Merge Sort

 # Comparisons (Merging)



Total # Comparisons for Merging: 1 + 1 + 1 + 1 + 3 + 3 + 6 = 14

Analysis of Merge Sort

The recurrence relation for the number of key comparisons C(n) is

$$C(n) = 2C(n/2) + C_{merge}(n) \quad \text{for } n > 1,\ C(1) = 0$$

At each step, exactly one comparison is made, after which the total number of elements in the two arrays still needed to be processed is reduced by one. In the worst case, neither of the two arrays becomes empty before the other one contains just one element (e.g., smaller elements may come from the alternating arrays). Hence, the worst case of $C_{merge}(n) = n - 1$ for $n > 1$.

$$C(n) = 2C(n/2) + (n-1) \qquad f(n) = n - 1 = \Theta(n)$$

Hence, according to Master Theorem,

$$a = 2, b = 2, d = 1$$

$$a = b^d$$

$$C(n) \in \Theta(n \log n)$$

Merge Sort: Space-time Tradeoff

- Unlike the sorting algorithms (insertion sort, bubble sort, selection sort) we saw in Module 1, Merge sort incurs a worst-case time-complexity of $\Theta(n\log n)$, whereas the other sorting algorithms we have seen incur a worst-case time complexity of $O(n^2)$.

- The tradeoff is Merge sort requires additional space proportional to the size of the array being sorted. That is, the space-complexity of merge sort is $\Theta(n)$, whereas the other sorting algorithms we have seen incur a space-complexity of $\Theta(1)$.

Finding the Maximum Integer in an Array: Recursive Divide and Conquer

```
Algorithm FindMaxIndex(Array A, int leftIndex, int rightIndex)
// returns the index of the maximum left in the array A for //index
    positions ranging from leftIndex to rightIndex

    if (leftIndex = rightIndex)
        return leftIndex
    middleIndex = (leftIndex + rightIndex)/2
leftMaxIndex = FindMaxIndex(A, leftIndex, middleIndex)
rightMaxIndex = FindMaxIndex(A, middleIndex + 1, rightIndex)
if A[leftMaxIndex] ≥ A[rightMaxIndex]
        return leftMaxIndex
else
        return rightMaxIndex
```

Divide part

Conquer part

Since we keep track of the index positions of the maximum element in the sub arrays, We do not need to create additional space. So, this algorithm is in-place.

Max Integer Index Problem: Time Complexity

$T(n) = 2*T(n/2) + 1$

i.e., $T(n) = 2*T(n/2) + \Theta(n^0)$

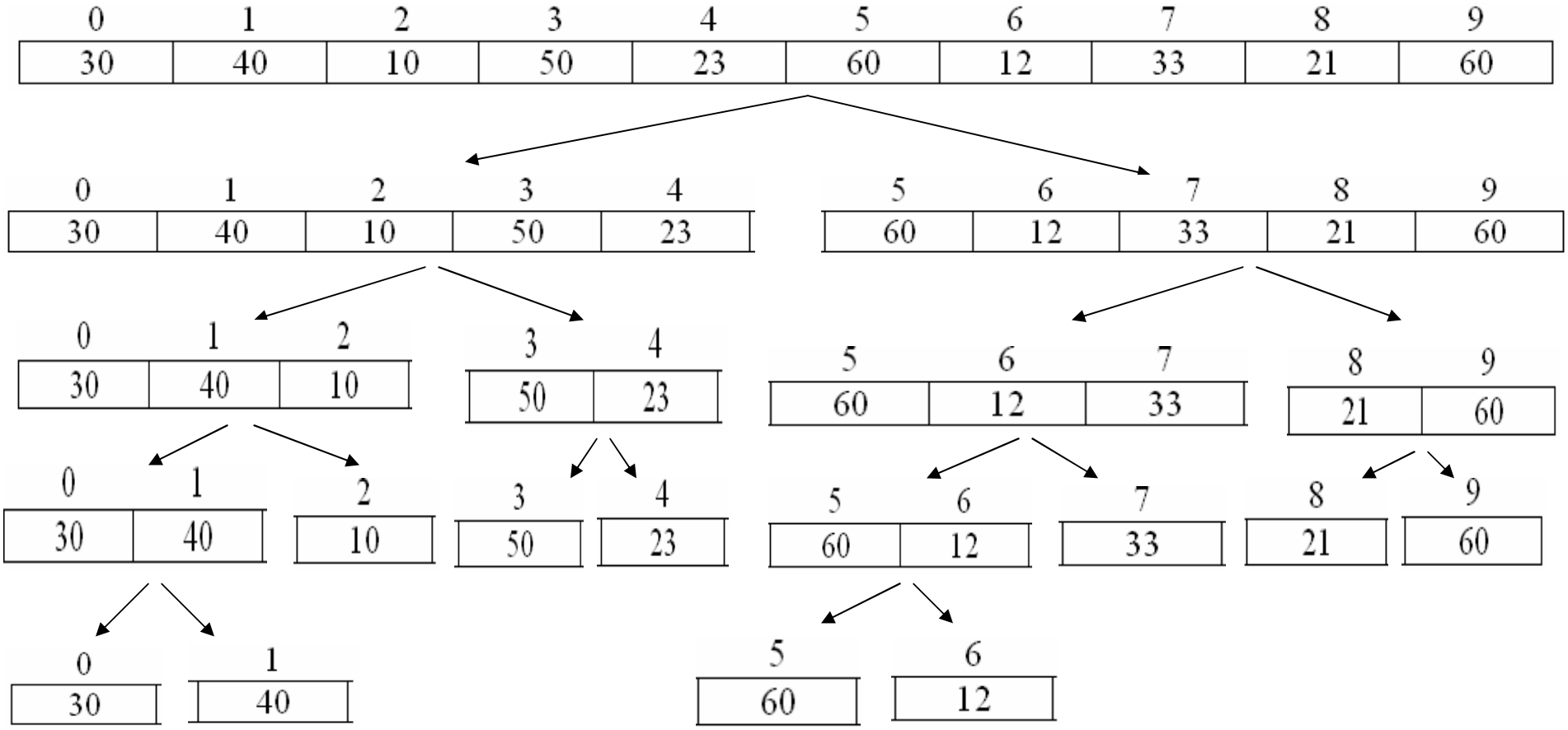
$a = 2, b = 2, d = 0$

$b^d = 2^0 = 1$. Hence, $a > b^d$

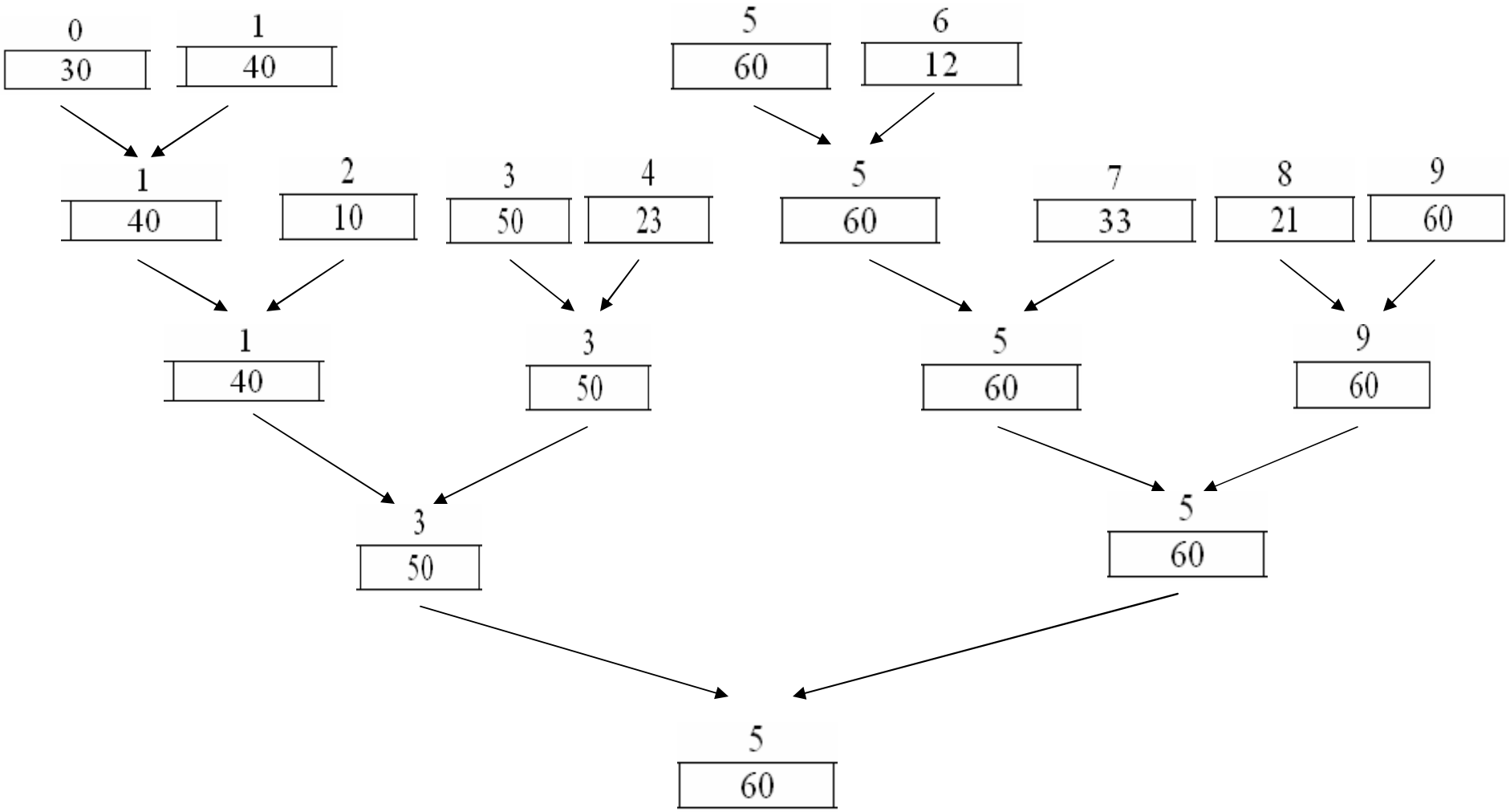
$$\mathbf{T(n) = \Theta(n^{\log_b(a)}) = = \Theta(n^{\log2(2)}) = \Theta(n)}$$

Note that even an iterative approach would take Θ(n) time to compute the time-complexity. The overhead comes with recursion.

FindMaxIndex: Example

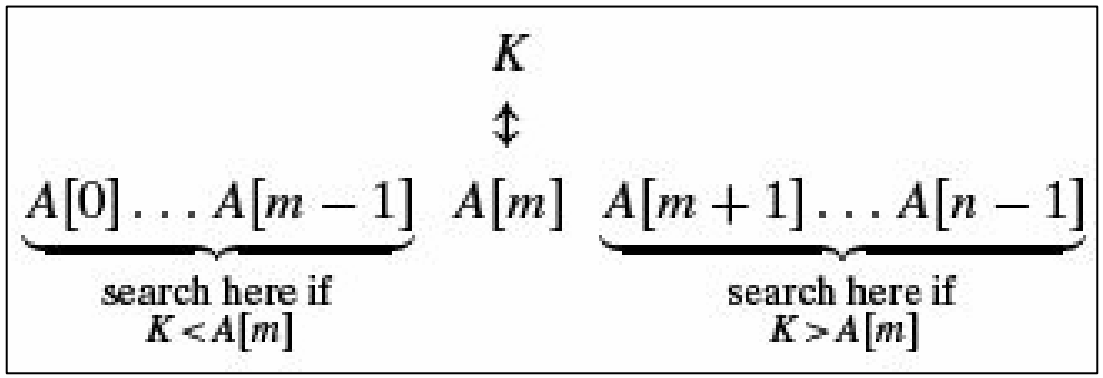


FindMaxIndex: Example (contd…)



Binary Search

- Binary search is a Θ(log n), highly efficient search algorithm, in a sorted array.
- It works by comparing a search key K with the array's middle element A[m]. If they match, the algorithm stops; otherwise, the same operation is repeated recursively for the first half of the array if K < A[m], and for the second half if K > A[m].
- Though binary search in based on a recursive idea, it can be easily implemented as a non-recursive algorithm.



Binary Search

Example

	Search Key K = 70	
l=0	r=12	m=6
l=7	r=12	m=9
l=7	r=8	m=7

index	0	1	2	3	4	5	6	7	8	9	10	11	12
value	3	14	27	31	39	42	55	70	74	81	91	93	98
iteration 1	*l*						*m*						*r*
iteration 2								*l*		*m*			*r*
iteration 3								*l*,*m*	*r*				

ALGORITHM *BinarySearch*$(A[0..n-1], K)$

```
//Implements nonrecursive binary search
//Input: An array A[0..n − 1] sorted in ascending order and
//       a search key K
//Output: An index of the array's element that is equal to K
//        or −1 if there is no such element
l ← 0;   r ← n − 1
while l ≤ r do
    m ← ⌊(l + r)/2⌋
    if K = A[m] return m
    else if K < A[m] r ← m − 1
    else l ← m + 1
return −1
```

Worst-case # Key Comparisons

$$C_{worst}(n) = C_{worst}(\lfloor n/2 \rfloor) + \Theta(1) \quad \text{for } n > 1, \quad C_{worst}(1) = 1$$

$$C_{worst}(2^k) = k + 1 = \log_2 n + 1.$$

$$C_{worst}(n) = \Theta(\log n)$$

Unsuccessful Search

Search K = 10

l=0 r=12 m=6

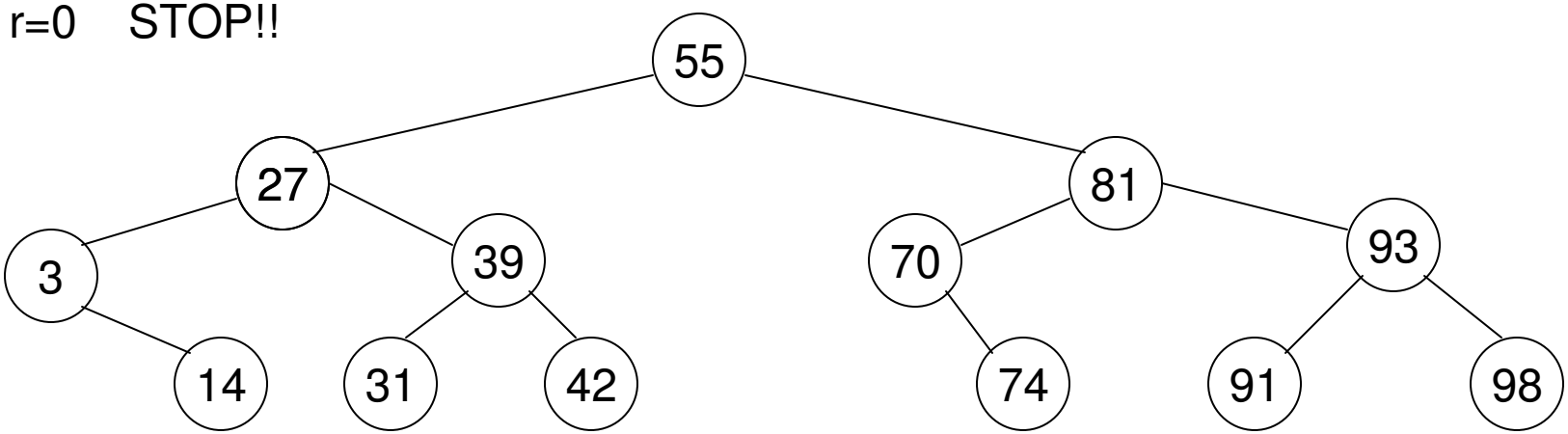
l=0 r=5 m=2

l=0 r=1 m=0

l=1 r=1 m=1

l=1 r=0 STOP!!

index	0	1	2	3	4	5	6	7	8	9	10	11	12
value	3	14	27	31	39	42	55	70	74	81	91	93	98
iteration 1	*l*						*m*						*r*
iteration 2								*l*		*m*			*r*
iteration 3								*l,m*	*r*				

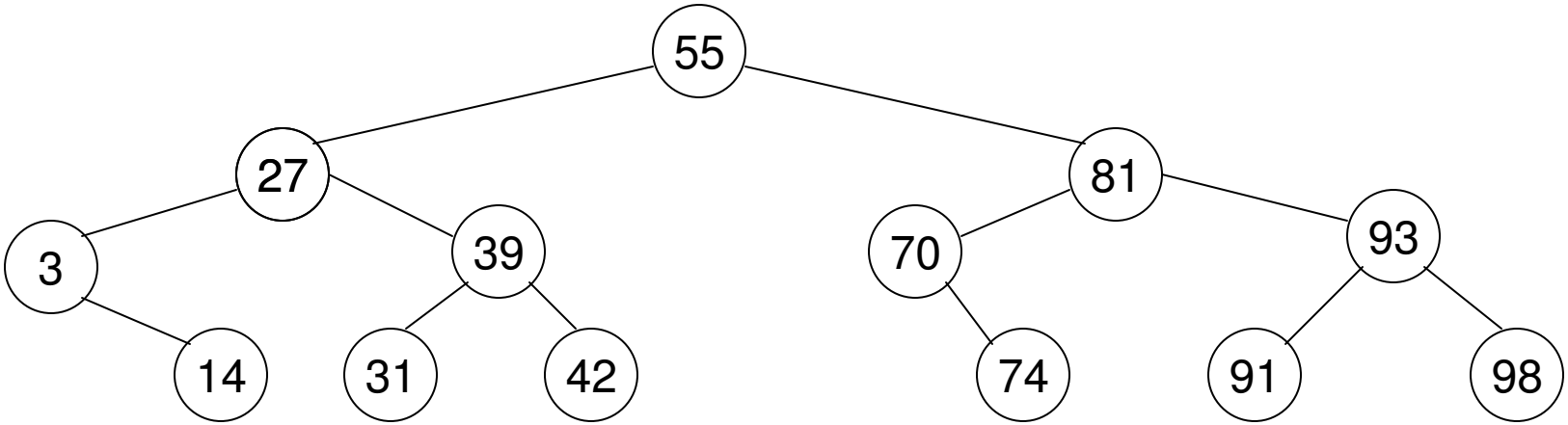


The keys that will require the largest number of comparisons: 14, 31, 42, 74, 91, 98

Average # Comparisons for Successful Search

Keys	# comparisons
55	1
27, 81	2
3, 39, 70, 93	3
14, 31, 42, 74, 91, 98	4

Avg # comparisons
= [Sum of the product of the # keys with certain # comparisons] / [Total Number of keys]
= [(1)(1) + (2)(2) + (3)(4) + (4)(6)] /13
= 3.15



Average # Comparisons for Unsuccessful Search

Range of Keys for Unsuccessful search	# comparisons
< 3	3
> 3 and < 14	4
> 14 and < 27	4
> 27 and < 31	4
> 31 and < 39	4
> 39 and < 42	4
> 42 and < 55	4
> 55 and < 70	3
> 70 and < 74	4
> 74 and < 81	4
> 81 and < 91	4
> 91 and < 93	4
> 93 and < 98	4
> 98	4

Avg = [4*12 + 3*2] / 14
= 3.86

Applications of Binary Search (1)

Finding the Maximum Element in a Unimodal Array

- A unimodal array is an array that has a sequence of monotonically increasing integers followed by a sequence of monotonically decreasing integers.
- All elements in the array are unique
- Examples
  - {4, 5, 8, 9, 10, 11, 7, 3, 2, 1}: Max. Element: 11
    - There is an increasing seq. followed by a decreasing seq.
  - {11, 9, 8, 7, 5, 4, 3, 2, 1}: Max. Element: 11
    - There is no increasing seq. It is simply a decreasing seq.
  - {1, 2, 3, 4, 5, 7, 8, 9, 11}: Max. Element: 11
    - There is an increasing seq., but there is no decreasing seq.
- Algorithm: Modified binary search.

Applications of Binary Search (1)

Finding the Maximum Element in a Unimodal Array

```
L = 0; R = n-1
while (L < R) do
    m = (L+R)/2
   if A[m] < A[m+1]
        L = m+1  // max. element is from m+1 to R
   else if A[m] > A[m+1]
        R = m // max. element is from L to m
end while
return A[L]
```

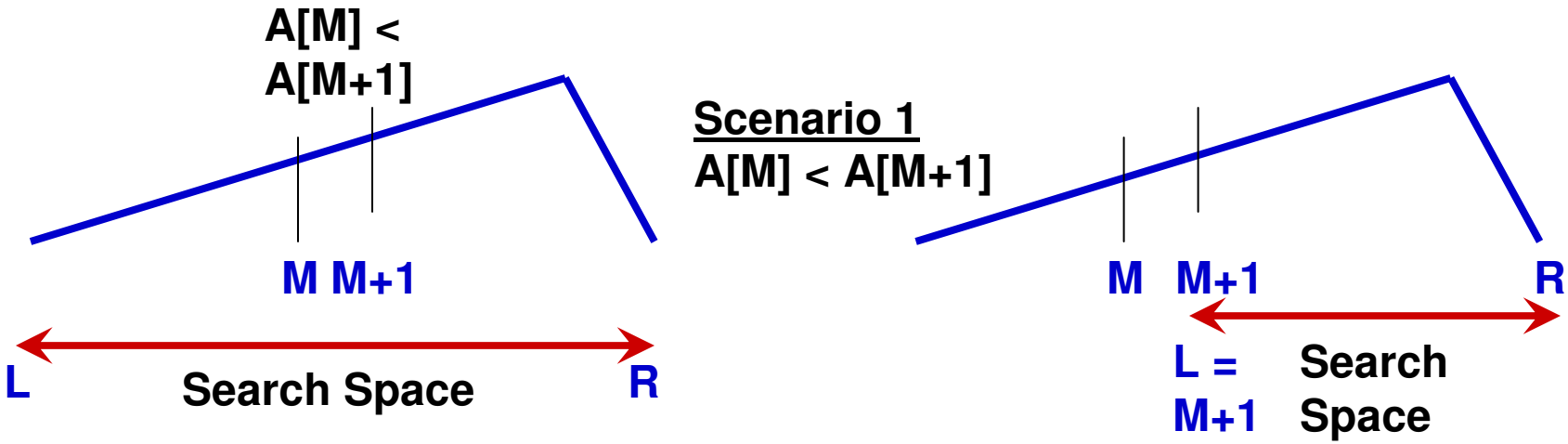
C(n) = C(n/2) + 2
Using Master Theorem,
C(n) = Θ(logn)

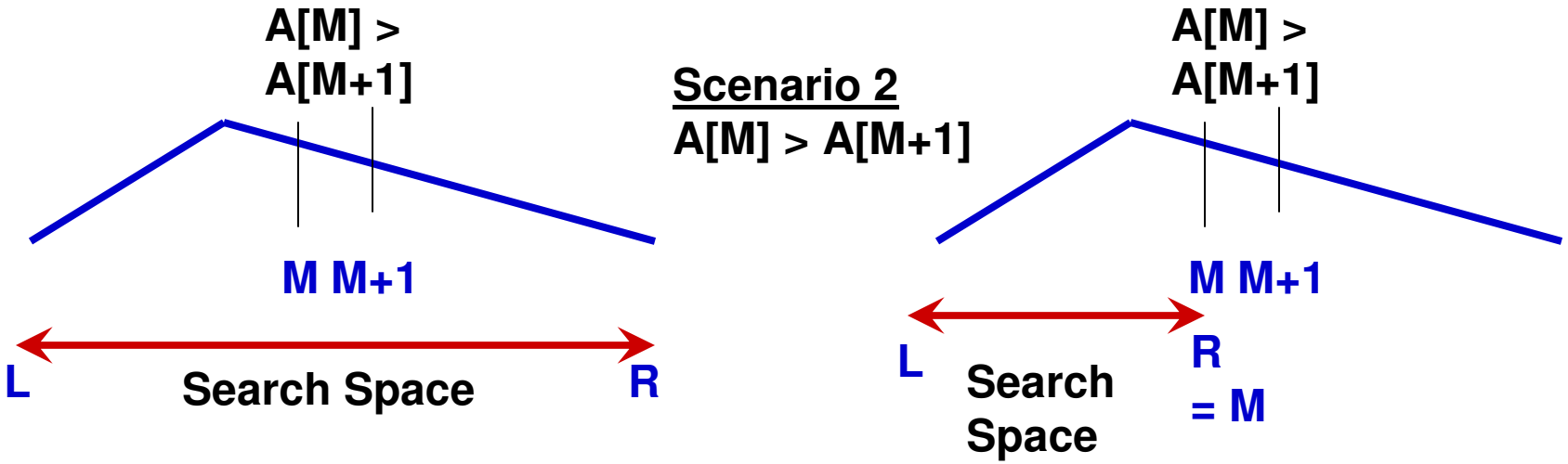
Space complexity: Θ(1)

0	1	2	3	4	5	6	7	8	9
3	5	8	9	10	14	11	4	2	1

L = 0; R = 9; m = 4: A[m] < A[m+1]
L = 5; R = 9; m = 7: A[m] > A[m+1]
L = 5; R = 7; m = 6: A[m] > A[m+1]
L = 5; R = 6; m = 5: A[m] > A[m+1]
L = 5; R = 5; return A[5] = 14

Two Scenarios





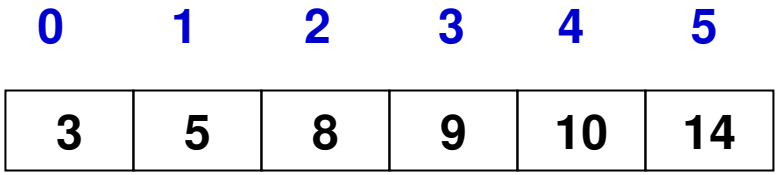
Applications of Binary Search (1)

Finding the Maximum Element in a Unimodal Array

- Proof of Correctness
  - We always maintain the invariant that the maximum element lies in the range of indexes: L…R.
  - If A[m] < A[m+1], then, the maximum element has to be either at index m+1 or to the right of index m+1. Hence, we set L = m+1 and retain R as it is, maintaining the invariant that the maximum element is in the range L…R.
  - If A[m] > A[m+1], then, the maximum element is either at index m or before index m. Hence, we set R = m and retain L as it is, maintaining the invariant that the maximum element is in the range L...R.
  - The loop runs as long as L < R. Once L = R, the loop ends and we return the maximum element.

Applications of Binary Search (1)

Finding the Maximum Element in a Unimodal Array



L = 0; R = 5; m = 2: A[m] < A[m+1]
L = 3; R = 5; m = 4: A[m] < A[m+1]
L = 5; R = 5; return A[5] = 14

```
L = 0; R = n-1
while (L < R) do
    m = (L+R)/2
   if A[m] < A[m+1]
        L = m+1  // max. element is from m+1 to R
   else if A[m] > A[m+1]
        R = m // max. element is from L to m
end while
return A[L]
```

Applications of Binary Search (2)

Local Minimum in an Array

- Problem: Given an array A[0,…, n-1], an element at index i (0 < i < n-1) is a local minimum if A[i] < A[i-1] as well as A[i] < A[i+1]. That is, the element is lower than the element to the immediate left as well as to the element to the immediate right.
- Constraints:
  - The arrays has at least three elements
  - The first two numbers are decreasing and the last two numbers are increasing.
  - The numbers are unique
- Example:
  - Let A = {8, **5**, 7, **2**, 3, 4, **1**, 9}; the array has several local minimum. These are: 5, 2 and 1.
- Algorithm: Do a binary search and see if every element we index into is a local minimum or not.
  - If the element we index into is not a local minimum, then we search on the half corresponding to the smaller of its two neighbors.

Applications of Binary Search (2)

Local Minimum in an Array

Examples

1)

0	1	2	3	4	5	6	7
8	5	7	2	3	4	1	9

Iteration 1: L = 0; R = 7; M = (L+R)/2 = 3 Element at A[3] is a local minimum.

2)

0	1	2	3	4	5	6	7
8	5	2	7	3	4	1	9

Iteration 1: L = 0; R = 7; M = (L+R)/2 = 3 Element at A[3] is NOT a local minimum.
Search in the space [0…2] corresponding to the smaller neighbor '2'

Iteration 2: L = 0; R = 2; M = (L+R)/2 = 1 Element at A[1] is NOT a local minimum.
Search in the space [2…2] corresponding to the smaller neighbor '2'

Iteration 3: L = 2; R = 2; M = (L+R)/2 = 2. Element at A[2] is a local minimum.

Applications of Binary Search (2)

Local Minimum in an Array

Examples

3)

0	1	2	3	4	5	6	7	8	9	10
-2	-5	5	2	4	7	1	8	0	-8	10

Iteration 1: L = 0; R = 10; M = (L+R)/2 = 5 Element at A[5] is NOT a local minimum.
Search in the space [6…10] corresponding to the smaller neighbor '1'
Iteration 2: L = 6; R = 10; M = (L+R)/2 = 8 Element at A[8] is NOT a local minimum.
Search in the space [9…10] corresponding to the smaller neighbor '-8'
Iteration 3: L = 9; R = 10; M = (L+R)/2 = 9. Element at A[9] is a local minimum. STOP

Time-Complexity Analysis

Recurrence Relation: $T(n) = T(n/2) + 3$ for $n > 3$
Basic Condition: $T(3) = 2$
Using Master Theorem, we have
$a = 1, b = 2, d = 0 \rightarrow a = b^d$.
Hence, $T(n) = \Theta(n^d \log n) = \Theta(n^0 \log n) = \Theta(\log n)$

- One comparison for A[M] with A[M+1]
- One comparison for A[M] with A[M-1]
- One comparison for A[M-1] with A[M+1]

Space Complexity: As all evaluations are done on the input array itself, no extra space proportional to the input is needed. Hence, space complexity is $\Theta(1)$.

Applications of Binary Search (2)

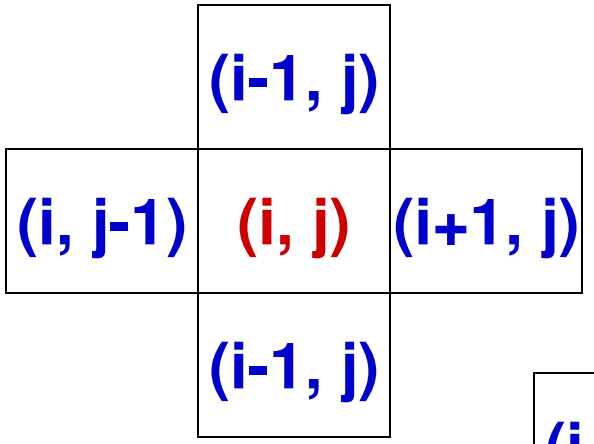
Local Minimum in an Array

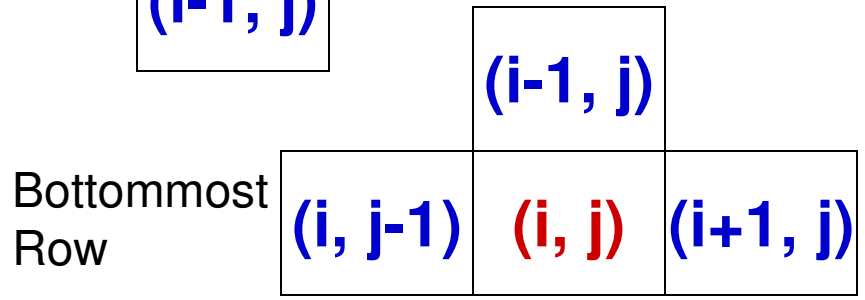
- Constraints:
  - The arrays has at least three elements
  - The first two numbers are decreasing and the last two numbers are increasing.
  - The numbers are unique
- Theorem: If the above three constraints are met for an array, then the array has to have at least one local minimum.
- Proof: Let us prove by contradiction.
  - If the second number is not to be a local minimum, then the third number in the array has to be less than the second number.
  - Continuing like this, if the third number is not to be a local minimum, then the fourth number has to be less than the third number and so on.
  - Again, continuing like this, if the penultimate number is not to be a local minimum, then the last number in the array has to be smaller than the penultimate number. This would mean the second constraint is violated (and also the array is basically a monotonically decreasing sequence). A contradiction.

Applications of Binary Search (3)

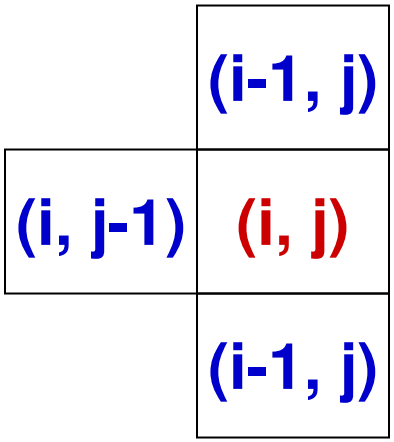
Local Minimum in a Two-Dimensional Array

- An element is a local minimum in a two-dim array if the element is the minimum compared to the elements to its immediate left and right as well as to the elements to its immediate top and bottom.
  - If an element is in the edge row or column, it is compared only to the elements that are its valid neighbors.

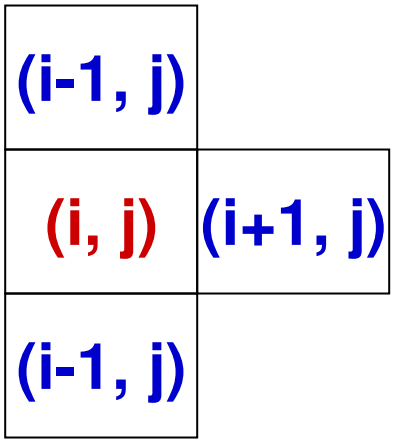




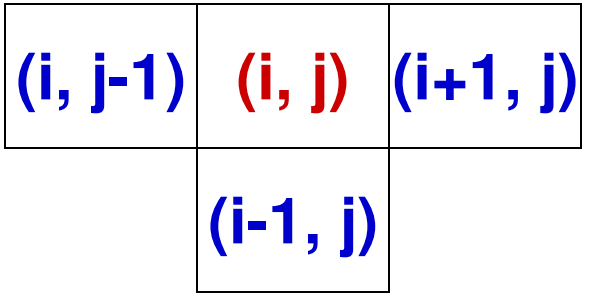
Bottommost Row



Rightmost column



Leftmost Column



Topmost Row

Applications of Binary Search (3)

Local Minimum in a Two-Dimensional Array

```
Given an array A[0…numRows-1][0…numCols-1]
TopRowIndex = 0
BottomRowIndex = numRows – 1
while (TopRowIndex ≤ BottomRowIndex) do
    MidRowIndex = (TopRowIndex + BottomRowIndex) / 2
    MinColIndex = FindMinColIndex( A[MidRowIndex][ ] )
        /* Finds the col index with the minimum element in the row
        corresponding to MidRowIndex */
    MinRowIndex = FindMinRowIndexNeighborhood (A, MidRowIndex,
                                                    MinColIndex)
        /* Finds the min entry in the column represented by MinColIndex
        and the rows MidRowIndex, MidRowIndex – 1,
        MidRowIndex + 1, as appropriate */
    if (MinRowIndex == MidRowIndex)
        return A[MinRowIndex][MinColIndex]
    else if (MinRowIndex < MidRowIndex)
        BottomRowIndex = MidRowIndex – 1
    else if (MinRowIndex > MidRowIndex)
        TopRowIndex = MidRowIndex + 1
end While
```

Local Minimum in a Two-Dim Array: Ex. 1

	0	1	2	3	4	5	6
0	30	19	18	40	16	45	13
1	43	14	15	12	25	34	17
2	24	1	32	33	31	36	11
3	44	6	48	46	39	27	8
4	29	20	49	26	28	22	7
5	38	4	47	5	10	23	3
6	42	41	37	2	9	35	21

Iteration 1

Use the function FindMinRowIndexNeighborhood

Use the FindMinColIndex function

Top Row Index →

Mid Row Index →

Bottom Row Index →

	0	1	2	3	4	5	6
0	30	19	18	40	16	45	13
1	43	14	15	12	25	34	17
2	24	1	32	33	31	36	11
3	44	6	48	46	39	27	8
4	29	20	49	26	28	22	7
5	38	4	47	5	10	23	3
6	42	41	37	2	9	35	21

Local Minimum in a Two-Dim Array: Ex. 1 (1)

Iteration 2

	0	1	2	3	4	5	6
Top Row Index → 0	30	19	18	40	16	45	13
1	43	14	15	12	25	34	17
Bottom Row Index → 2	24	1	32	33	31	36	11
3	44	6	48	46	39	27	8
4	29	20	49	26	28	22	7
5	38	4	47	5	10	23	3
6	42	41	37	2	9	35	21

	0	1	2	3	4	5	6
Top Row Index → 0	30	19	18	40	16	45	13
Mid Row Index → 1	43	14	15	12	25	34	17
Bottom Row Index → 2	24	1	32	33	31	36	11
3	44	6	48	46	39	27	8
4	29	20	49	26	28	22	7
5	38	4	47	5	10	23	3
6	42	41	37	2	9	35	21

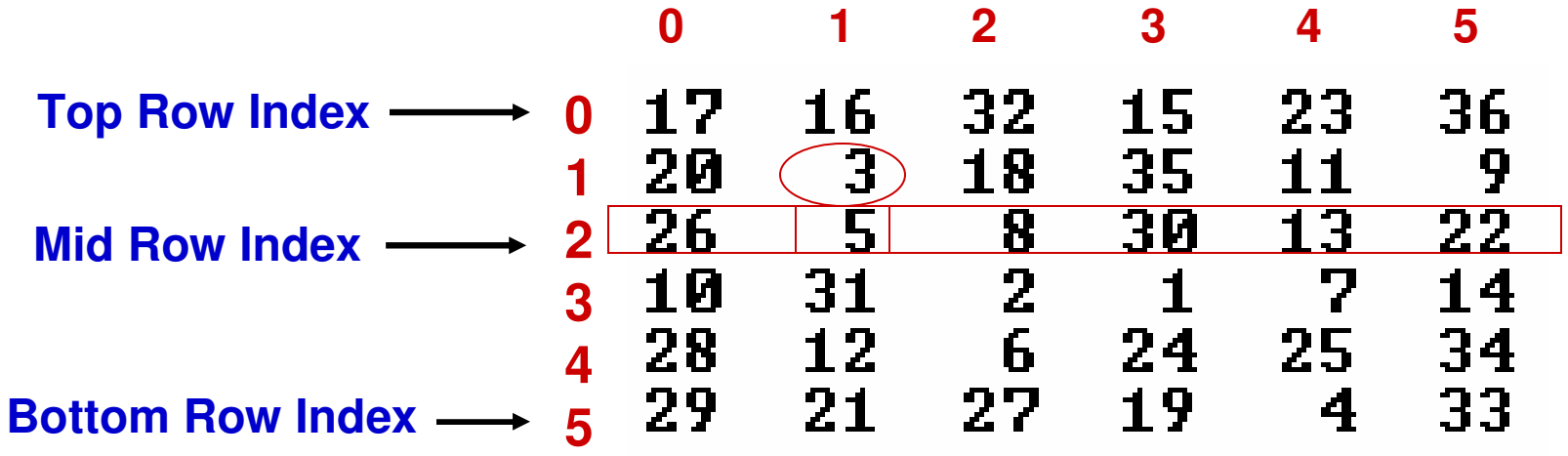
The minimum element 12 in Mid Row is smaller than its immediate top (40) and bottom (33) neighbors

12 at (1, 3) is a local minimum

Local Minimum in a Two-Dim Array: Ex. 2

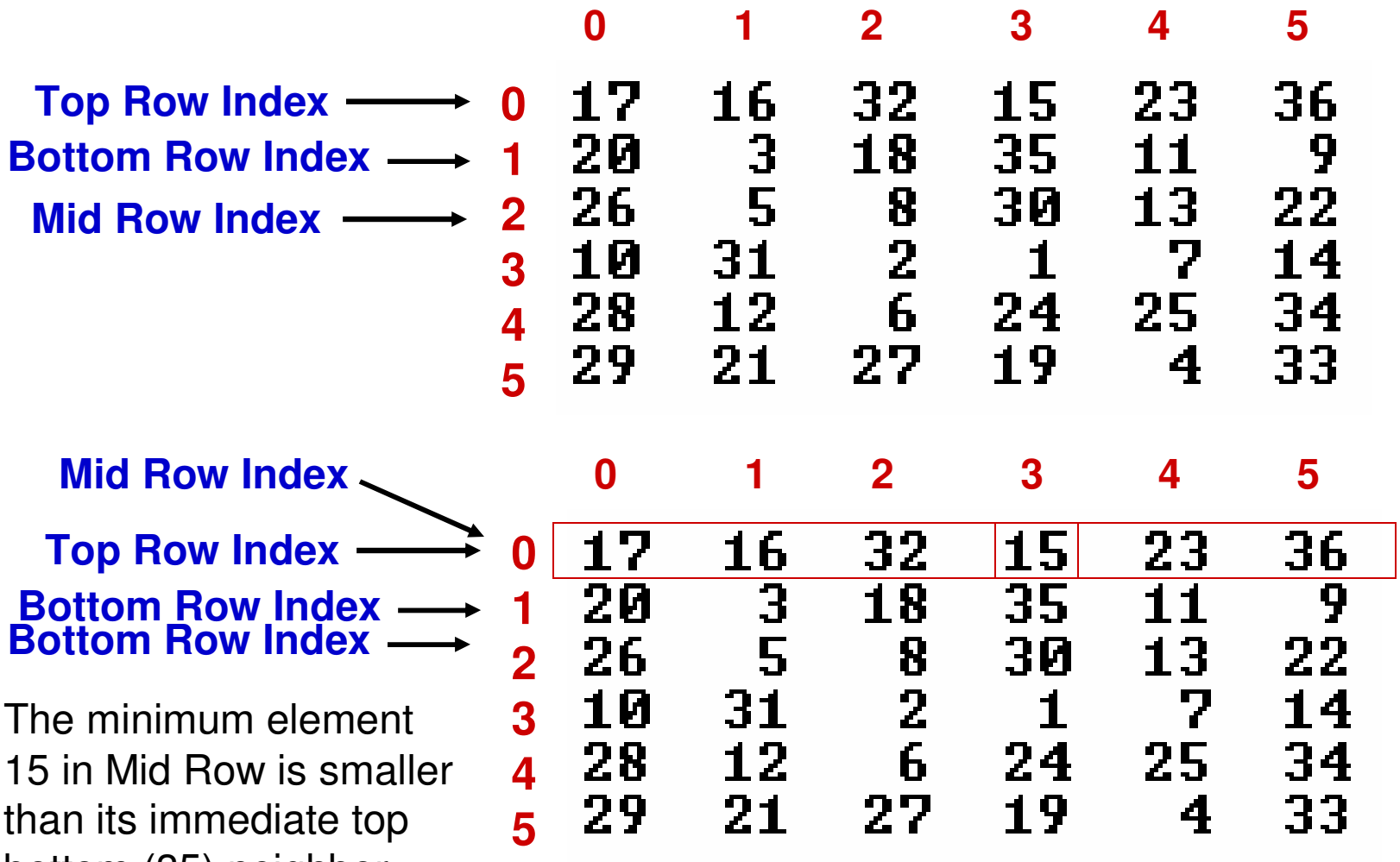
	0	1	2	3	4	5
0	17	16	32	15	23	36
1	20	3	18	35	11	9
2	26	5	8	30	13	22
3	10	31	2	1	7	14
4	28	12	6	24	25	34
5	29	21	27	19	4	33

Iteration 1



Local Minimum in a Two-Dim Array: Ex. 2 (1)

Iteration 2



The minimum element 15 in Mid Row is smaller than its immediate top bottom (35) neighbor

15 at (0, 3) is a local minimum

Applications of Binary Search (3)

Local Minimum in a Two-Dimensional Array

- Time Complexity Analysis

$T(n^2) = T(n^2/2) + \Theta(n)$

Time complexity to search for the minimum element in a row

The search space reduces by half

Let $N = n^2$.

$T(N) = T(N/2) + \Theta(N^{1/2})$

Use Master Theorem: $a = 1$, $b = 2$, $d = ½$

We have $a < b^d$. Hence, $T(N) = \Theta(N^{1/2}) = \Theta(n)$

Space Complexity: $\Theta(1)$

Proof of Correctness

- We will prove by contradiction.
- Assume the local minimum is not in the top half (as well as in the bottom half) and not in the middle row either.
- If the local minimum is not in the middle row and there is an element in the immediate top row of the middle row that is less than the minimum element in the middle row, then we move the search space to the top half (or likewise to the bottom half).
- If there is no local minimum in the top half, then for every row in the top half: for the minimum element in this row, there is an element that is lower than it in the immediate top row (recursively starting from the row above the middle row) or the immediate bottom row.
  - This implies, there should be an element above the topmost row that is less than the minimum element in the topmost row (or) there is an element in the initial middle row that is lower than the element in the immediate top row.
  - Such a row (that is above the topmost row or that is the initial middle row) does not exist. (A contradiction)
  - Hence, there should be some element in the top half (or the bottom half) that should be a local minimum, if a local minimum does not exist in the middle row.