Module 5
Graph Algorithms

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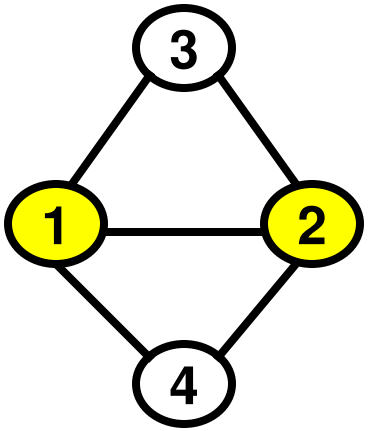
Number of Walks in a Graph

- An u-v walk between two vertices u and v is a sequence of zero or more intermediate vertices (that could be even repeated).
- The length of a walk is one plus the number of intermediate vertices
  - Example: **2 – 3 – 1 – 4 – 1 is a walk of length 4**.
- A walk is a path if the intermediate vertices, if any, are not repeated.
  - Example: **2 – 3 – 1 is a walk as well as a path, but the walk 2 – 3 – 1 – 4 – 1 is not a path.**
- The number of walks of length *k* between any two vertices in a graph could be determined by finding A^k where A is the binary adjacency matrix of the graph.



Adjacency Matrix (A)

	1	2	3	4
1	0	1	1	1
2	1	0	1	1
3	1	1	0	0
4	1	1	0	0

x

Adjacency Matrix (A)

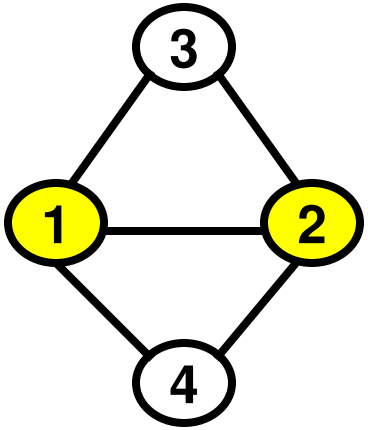
	1	2	3	4
1	0	1	1	1
2	1	0	1	1
3	1	1	0	0
4	1	1	0	0

=

 # Walks of Length 2 (A^2)

	1	2	3	4
1	3	2	1	1
2	2	3	1	1
3	1	1	2	2
4	1	1	2	2

Number of Walks in a Graph



 # Walks of Length 2 (A^2)

	1	2	3	4
1	3	2	1	1
2	2	3	1	1
3	1	1	2	2
4	1	1	2	2

x

Adjacency Matrix (A)

	1	2	3	4
1	0	1	1	1
2	1	0	1	1
3	1	1	0	0
4	1	1	0	0

=

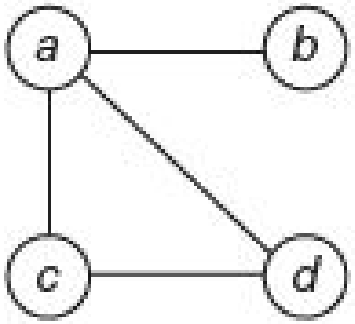
 # Walks of Length 3 (A^3)

	1	2	3	4
1	4	5	5	5
2	5	4	5	5
3	5	5	2	2
4	5	5	2	2

- Basic Rules for Matrix Multiplication
- To multiply two matrices A and B and get a product matrix P = A * B:
- (1) The number of columns in the first matrix A should be equal to the number of rows in the second matrix B
- (2) To get the value of a cell (i, j) in the product matrix P, do a pair-wise multiplication of the elements in row i of the first matrix with the elements in column j of the second matrix.

To find # Walks of Length ‘n’

 # Walks of Length 4: Find A^4.



$$A^2 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \end{array} \quad \mathbf{x} \quad A^2 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

$A^4 =$

	a	**b**	**c**	**d**
a	11	2	6	6
b	2	3	4	4
c	6	4	5	6
d	6	4	6	7

To find the number of walks length 4 between vertices b and c, just simply do a pair-wise multiplication and addition of the elements corresponding to the row for vertex ‘b’ in A^2 with the elements corresponding to the column for vertex ‘c’ in A^2.

Note: Rule for Matrix Multiplication
To find the value of an entry in cell (i, j) in the product matrix P = A * B,
Do a pair-wise multiplication and addition of the elements in row ‘i’ of the first matrix A and the elements in column ‘j’ of the second matrix B.

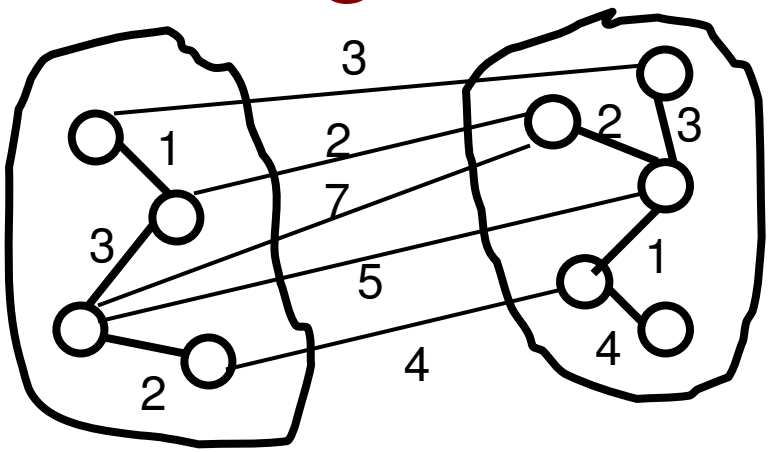
Minimum Spanning Trees

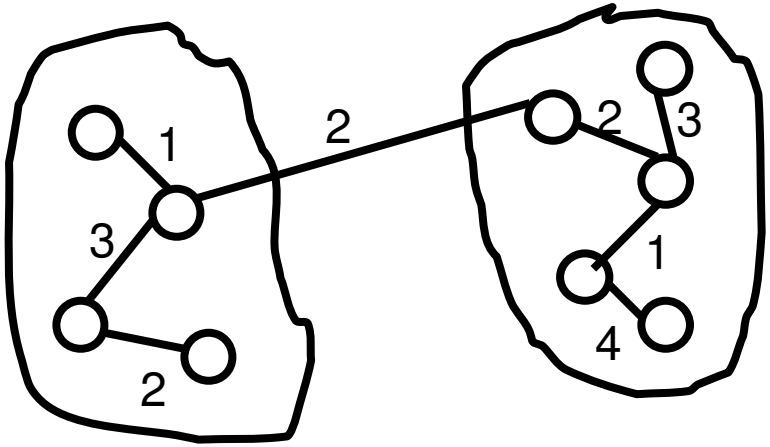
Minimum Spanning Tree Problem

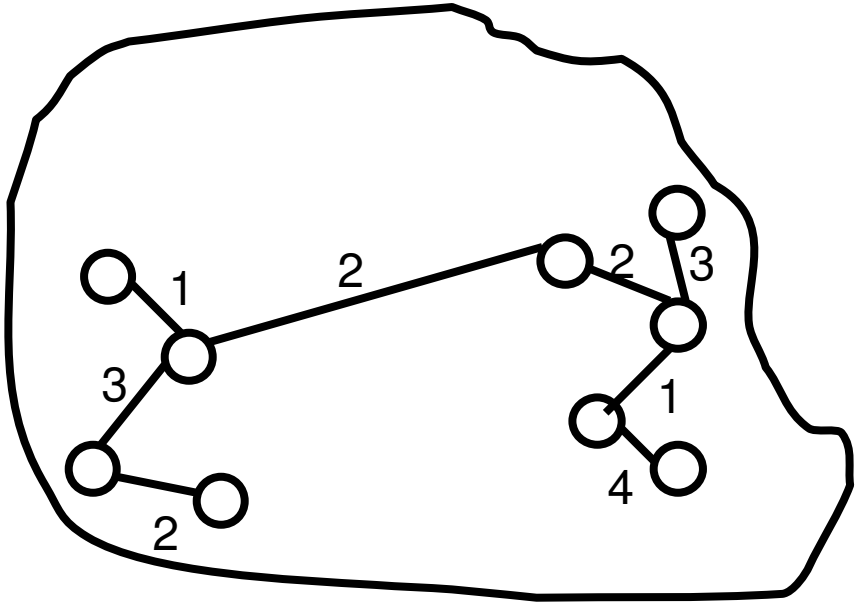
- Given a weighted graph, we want to determine a tree that spans all the vertices in the tree and the sum of the weights of all the edges in such a spanning tree should be minimum.
- Kruskal algorithm: Consider edges in the increasing order of their weights and include an edge in the tree, if and only if, by including the edge in the tree, we do not create a cycle!!
  - For a graph of E edges, we spend $\Theta(E*\log E)$ time to sort the edges and this is the most time consuming step of the algorithm.
- To start with, each vertex is in its own component.
- In each iteration, we merge two components using an edge of minimum weight connecting the vertices across the two components.
  - The merged component does not have a cycle and the sum of all the edge weights within a component is the minimum possible.
- To detect a cycle, the vertices within a component are identified by a component ID. If the edge considered for merging two components comprises of end vertices with the same component ID, then the edge is not considered for the merger.
  - An edge is considered for merging two components only if its end vertices are identified with different component IDs.

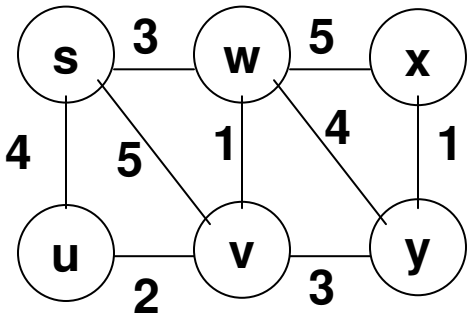
Property of any MST Algorithm

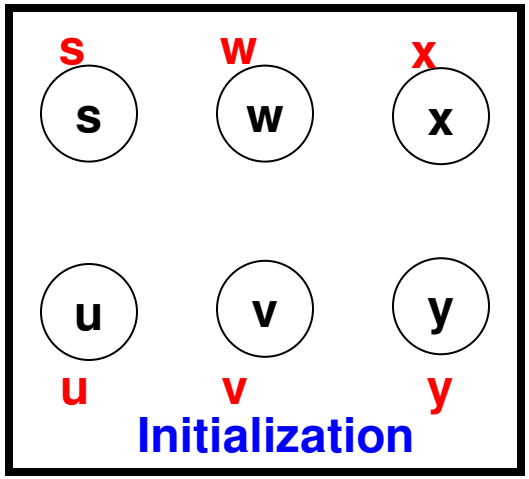
- Given two components of vertices (that are a tree by themselves of the smallest possible weights), any MST algorithm would choose an edge of the smallest weight that could connect the two components such that the merger of the two components is also a tree and is of the smallest possible weight.

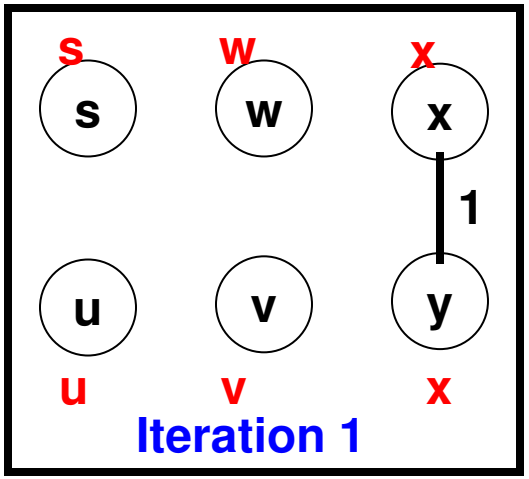


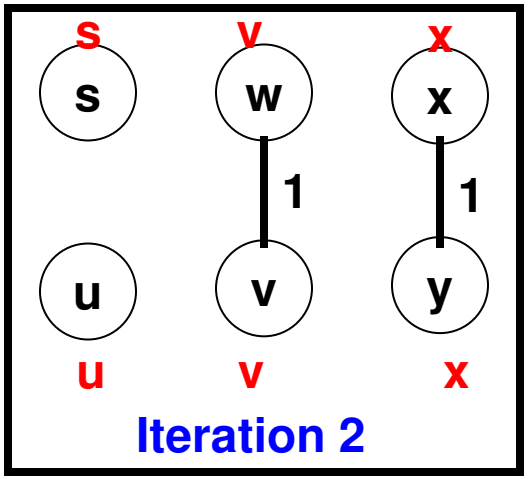


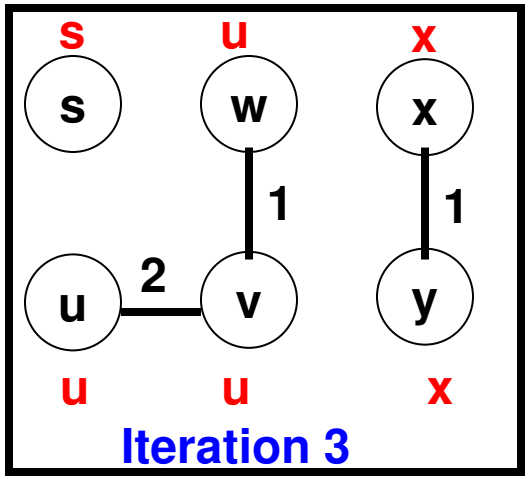


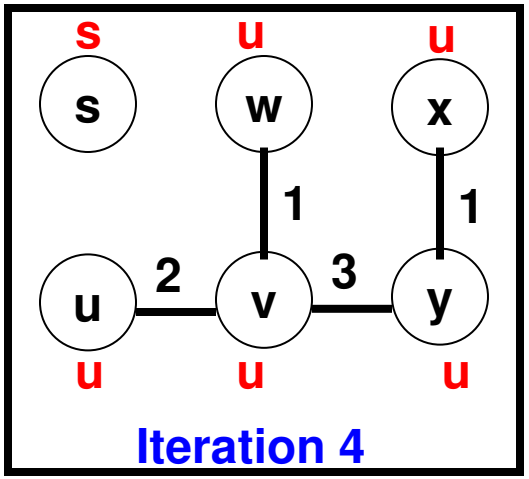


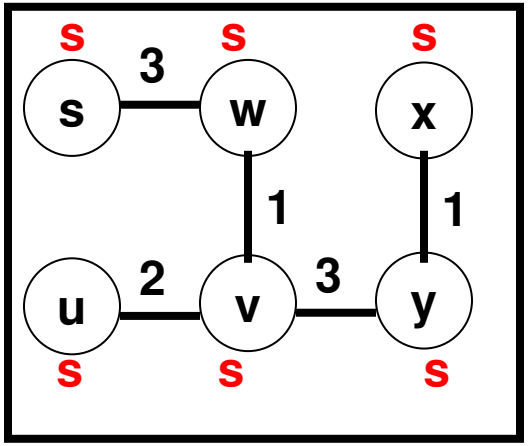






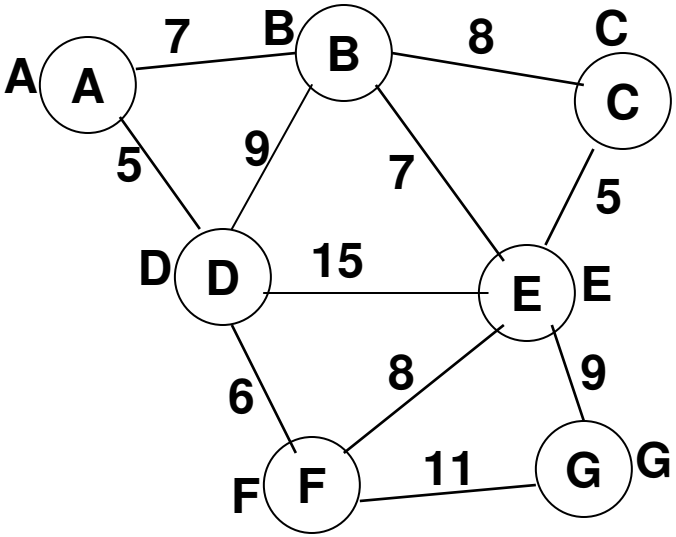


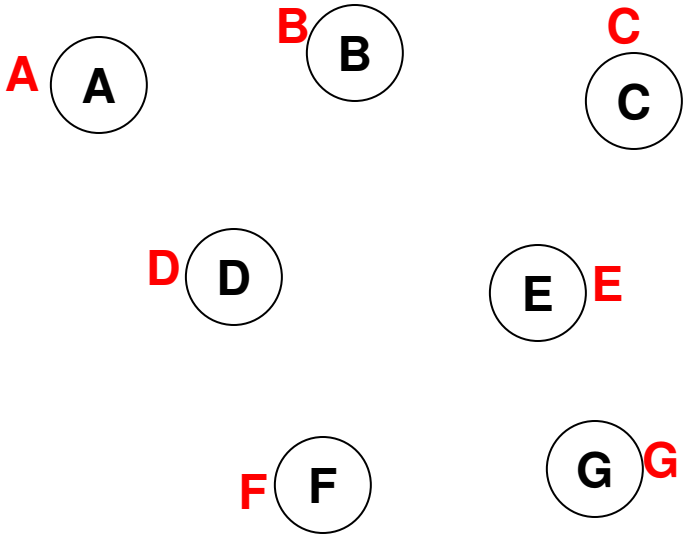




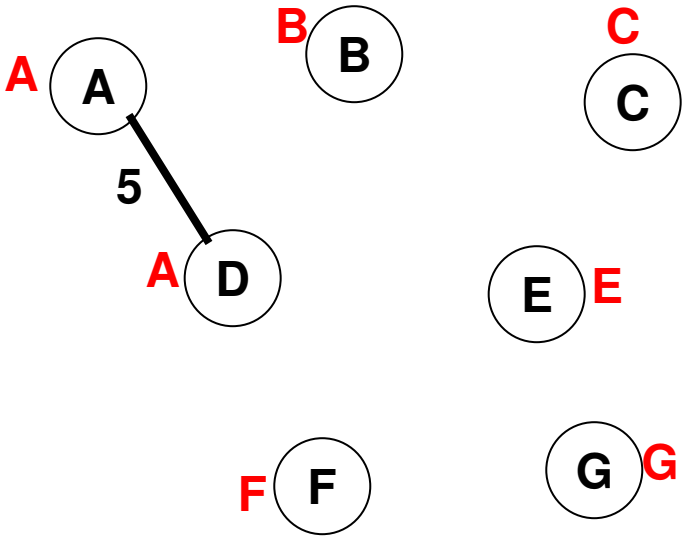
Iteration 5
Min. Spanning Tree

MST
Weight
10

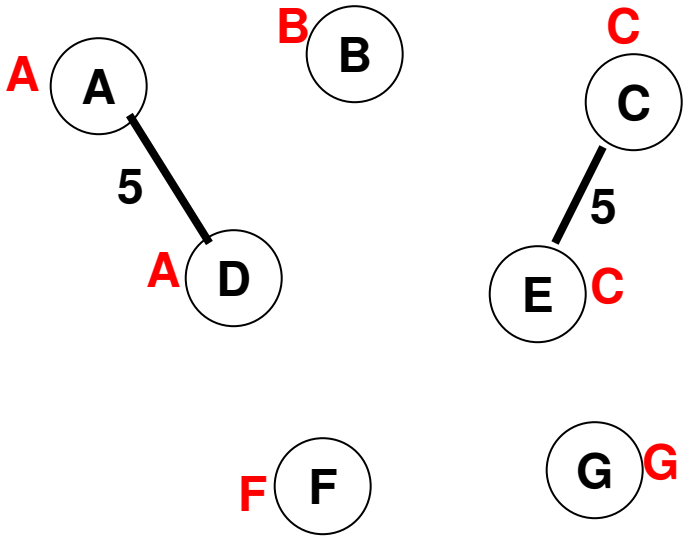




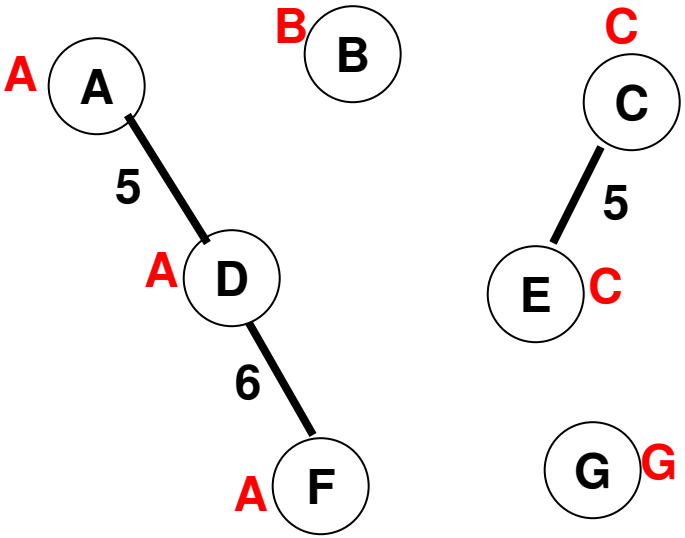
Initialization



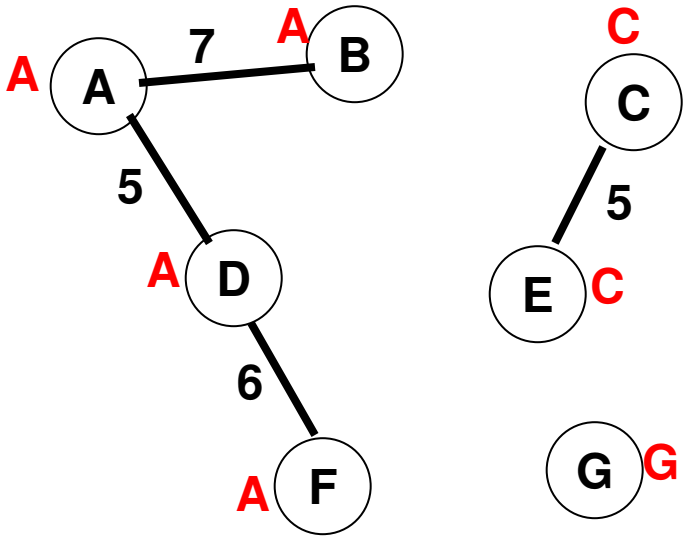
Iteration 1



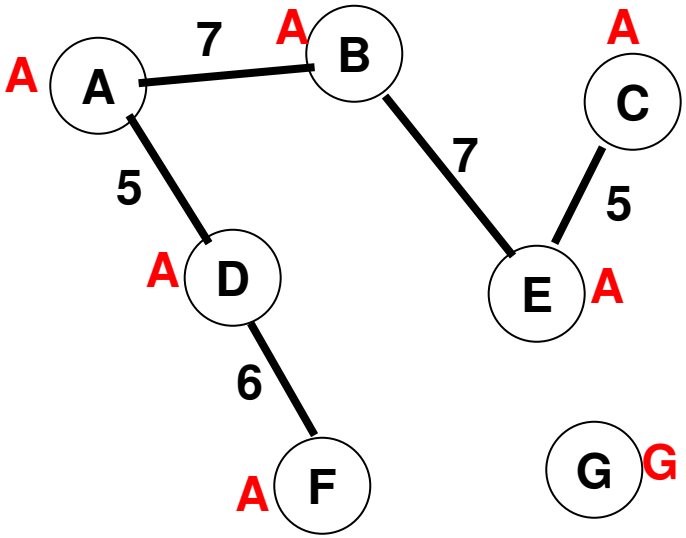
Iteration 2



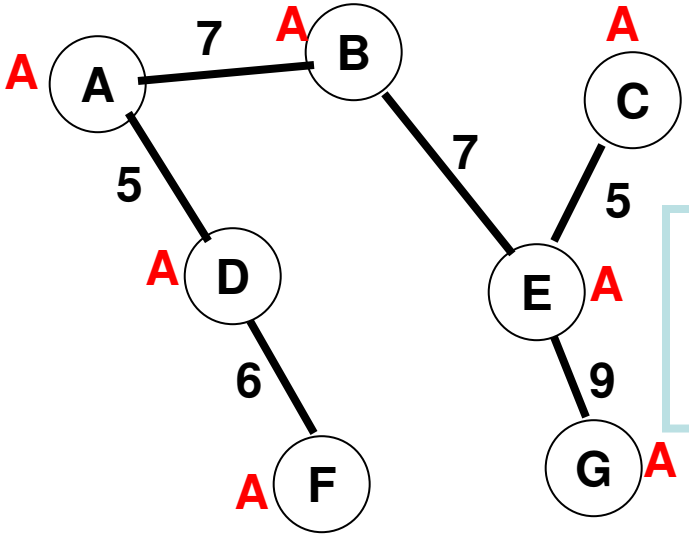
Iteration 3



Iteration 4



Iteration 5



MST Weight 39

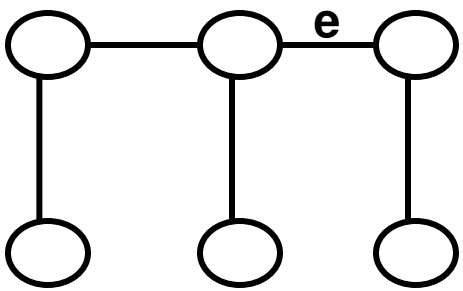
Iteration 6: Min. Sp Tree

Proof of Correctness: Kruskal's Algorithm

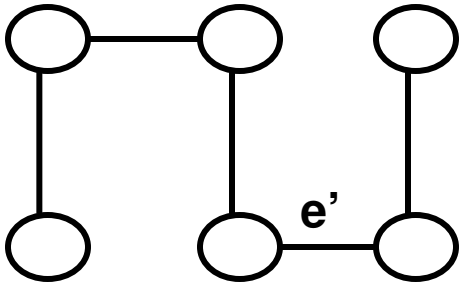
- Let *T* be the spanning tree generated by Kruskal's algorithm for a graph *G*. Let *T'* be a minimum spanning tree for *G*. We need to show that both *T* and *T'* have the same weight.
- Assume that wt(*T'*) < wt(*T*).
- Hence, there should be an edge *e* in *T* that is not in *T* ' and likewise there should be an edge e' in T' that is not in T. Because, if every edge of *T* is in *T'*, then *T* = *T'* and wt(*T*) = wt(*T'*).
- Remove the edge e' that is in T'. This would disconnect the T' to two components. The edge e that was in T and not in T' should be one of the edges (along with e') that cross the two split components of T'.
- Depending on how Kruskal's algorithm works, wt(e) ≤ wt(e'). Hence, the two components of T' could be merged using edge e (instead of e') and this would only lower the weight of T' from what it was before (and not increase it).
- That is, wt(modified T') = wt(T' – {e'} U {e}) ≤ wt(T').
- We could repeat the above procedure for all edges that are in T' and not in T, and eventually transform *T'* to *T*, without increasing the cost of the spanning tree.
- Hence, *T* is a minimum spanning tree.

Proof of Correctness

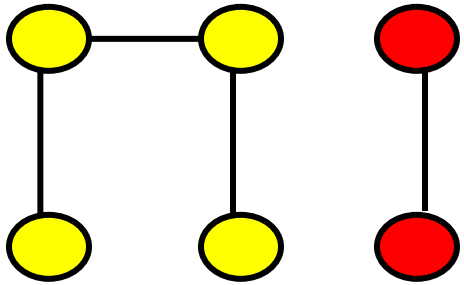
Let T be the spanning tree determined using Kruskal's
Let T' be a hypothetical spanning tree that is a MST such that W(T') < W(T)



T

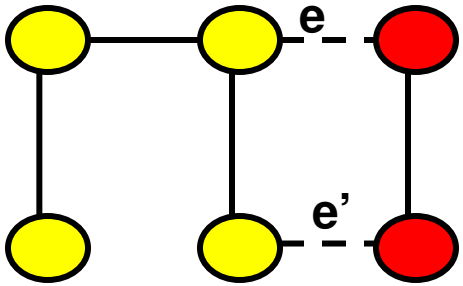


T'

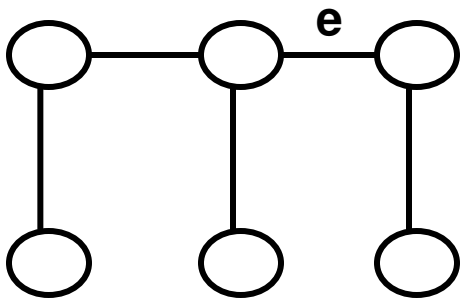


Wt(e) ≤ Wt(e')

Wt(T' – {e'} U {e}) ≤ Wt(T'). Hence, by reducing the edge difference and making T' approach T, we are able to only decrease the weight of T' further, if possible, making T' not a MST to start with, a contradiction.



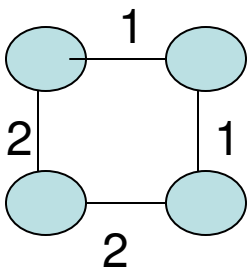
Candidate edges to merge the two components



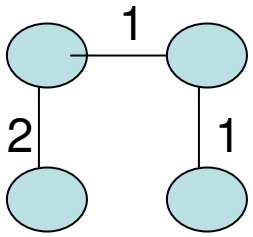
Modified T ' = T ' – {e'} U {e}

Properties of Minimum Spanning Tree

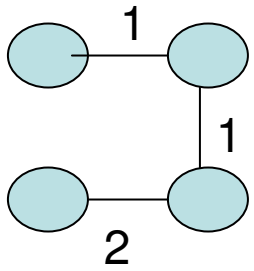
- **Property 2:** If a graph does not have unique edge weights, there could be more than one minimum spanning tree for the graph.
- **Proof (by Example)**



Graph



One Min. Spanning Tree



Another Min. Spanning Tree

- **Property 3:** If all the edges in a weighted graph have unique weights, then there can be only one minimum spanning tree of the graph.
- **Proof:** Consider a graph G whose edges are of distinct weights. Assume there are two different spanning trees *T* and T', both are of minimum weight; but have at least one edge difference. Let e' be an edge in T' that is not in T. Removing e' from T' will split the latter into two components. There should be an edge e that is not part of T' but part of T and should also be a candidate edge to connect the two components of the split T'.

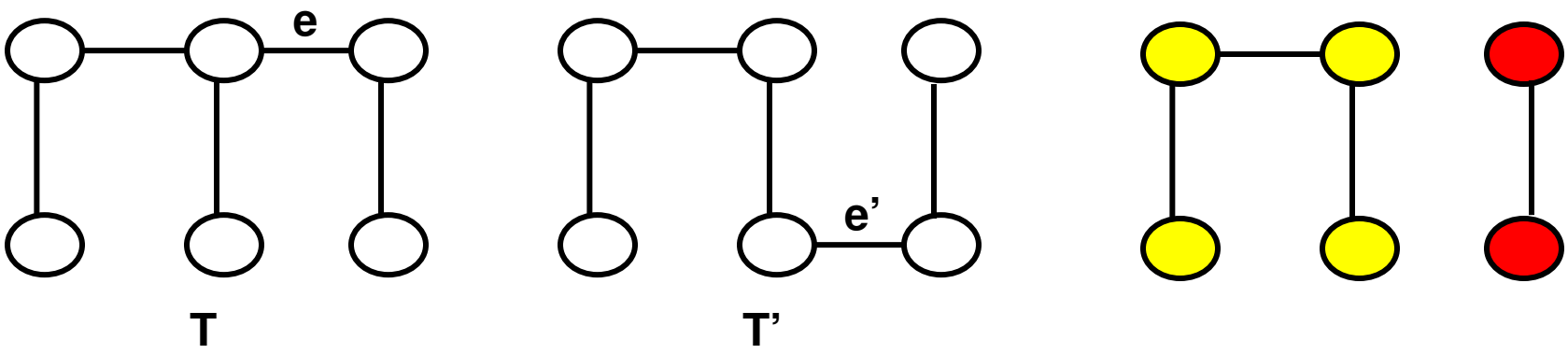
Properties of Minimum Spanning Tree

- **Property 3:** If all the edges in a weighted graph have unique weights, then there can be only one minimum spanning tree of the graph.
- **Proof (continued..):** If wt(e) < wt(e'), then we could merge the two components of T' using e and this would lower the weight of T' from what it was before. Hence, wt(e) ≥ wt(e').
- However, since the graph has unique edge weights, wt(e) > wt(e'). But, if this is the case, then we could indeed remove e from T and have e' to merge the two components of T resulting from the removal of e. This would only lower the weight of T from what it was before.
- So, if T and T' have to be two different MSTs ➔ wt(e) = wt(e').
  - This is a contradiction to the given statement that the graph has unique edge weights.
- Not (wt(e) = wt(e')) ➔ Not (T and T' have to be two different MSTs)
- That is, wt(e) ≠ wt(e') ➔ T and T' have to be the same MST.

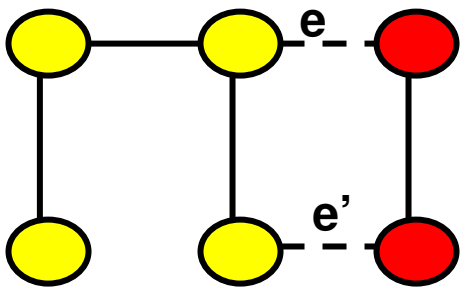
- Hence, if a graph has unique edge weights, there can be only one MST for the graph.

Property 3

Assume that both T and T' are MSTs, but different MSTs to start with.



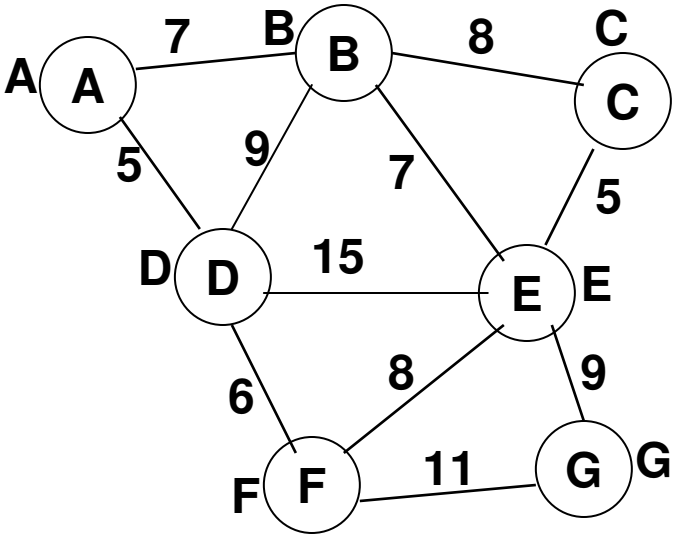
W(e) < W(e') => T' is not a MST
W(e) > W(e') => T is not a MST
Hence, for both T and T' to be two different MSTs ➔ W(e) = W(e').
But the graph has unique edge weights.
W(e) ≠ W(e) ➔ Both T and T' have to be the same.

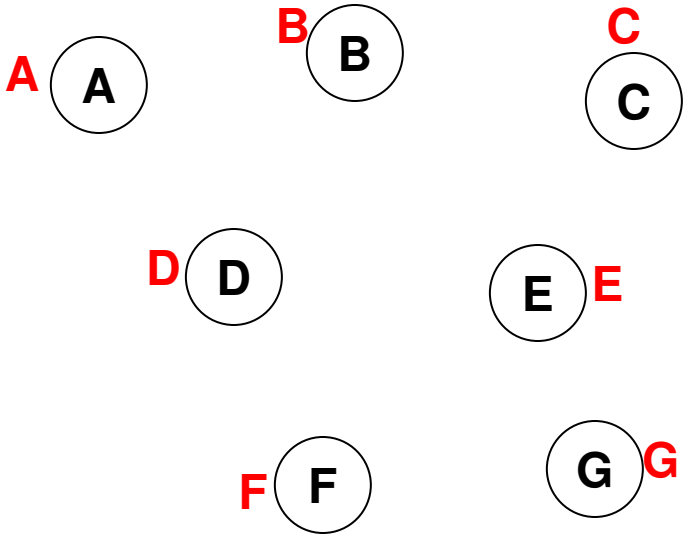


Candidate edges to merge the two components

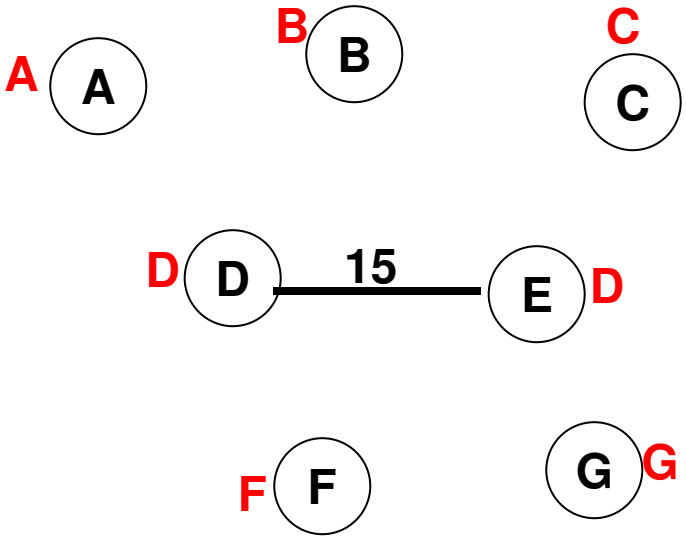
Maximum Spanning Tree

- A Maximum Spanning Tree is a spanning tree such that the sum of its edge weights is the maximum.
- We can find a Maximum Spanning Tree through any one of the following ways:
  - Straightforward approach: Run Kruskal's algorithm by selecting edges in the decreasing order of edge weights (i.e., edge with the largest weight is chosen first) as long as the end vertices of an edge are in two different components
  - Alternate approach (Example for Transform and Conquer): Given a weighted graph, set all the edge weights to be negative, run a minimum spanning tree algorithm on the negative weight graph, then turn all the edge weights to positive on the minimum spanning tree to get a maximum spanning tree.

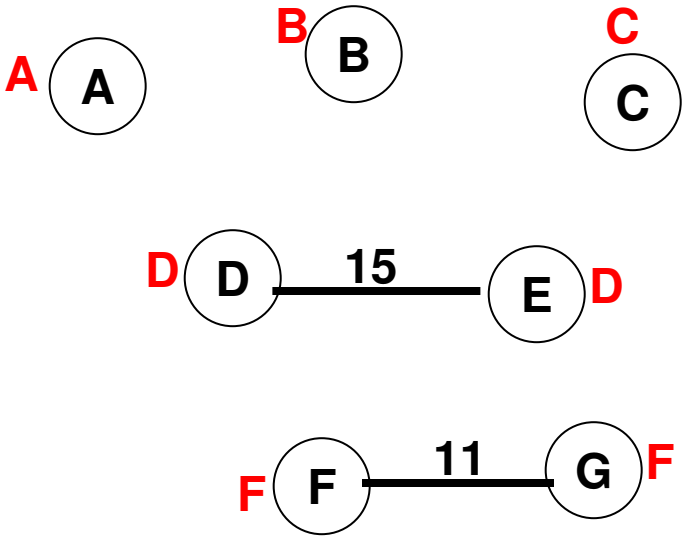




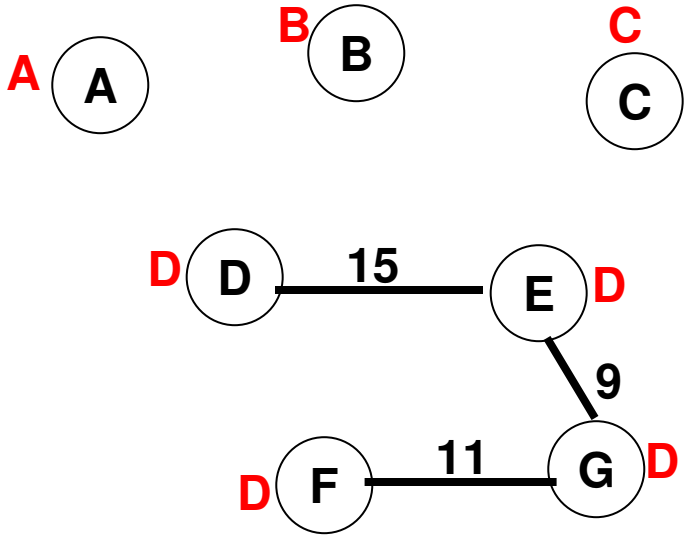
Initialization



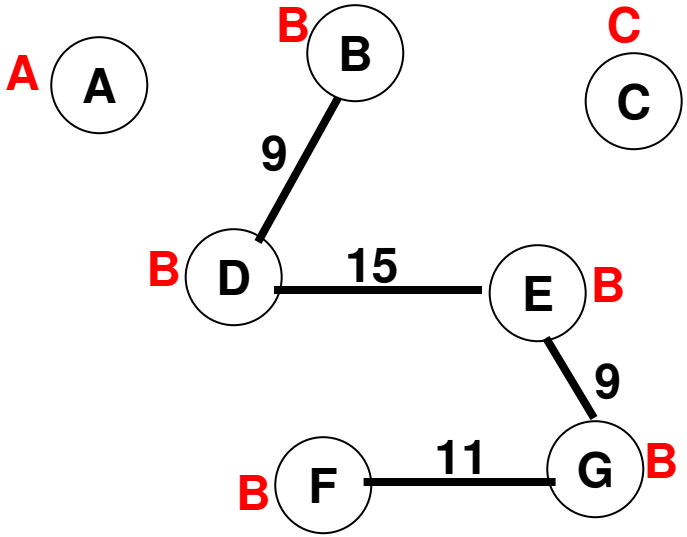
Iteration 1



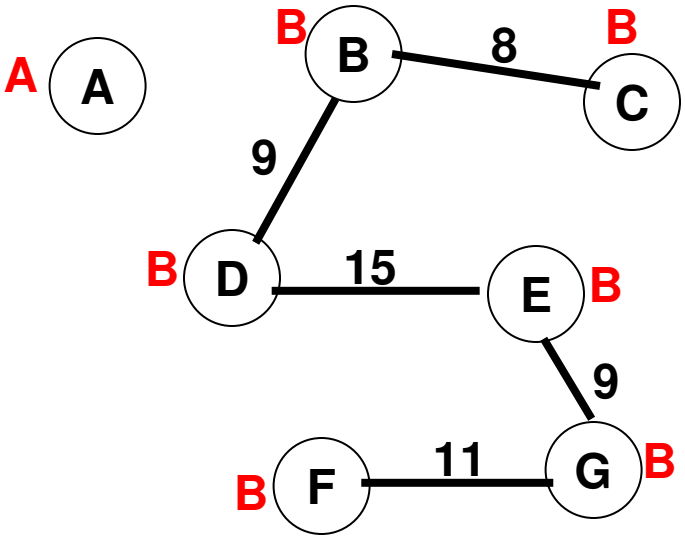
Iteration 2



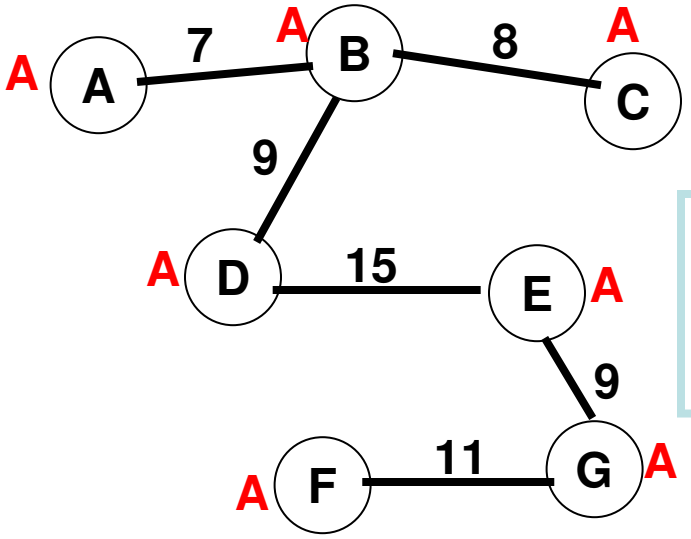
Iteration 3



Iteration 4



Iteration 5



MST Weight 59

Iteration 6: Max. Sp Tree

Practice Proofs

- Similar to the proof of correctness that we saw for the Minimum Spanning Trees, write the proof of correctness for the Kruskal's algorithm to find Maximum Spanning Trees.

- Prove the following property: If all the edges in a weighted graph have unique weights, then there can be only one maximum spanning tree of the graph.

Dijkstra’s Shortest Path Algorithm

Shortest Path (Min. Wt. Path) Problem

- Path p of length k from a vertex s to a vertex d is a sequence $(v_0, v_1, v_2, \ldots, v_k)$ of vertices such that $v_0 = s$ and $v_k = d$ and $(v_{i-1}, v_i) \in E$, for $i = 1, 2, \ldots, k$

- Weight of a path $p = (v_0, v_1, v_2, \ldots, v_k)$ is $w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$

- The weight of a shortest path from s to d is given by

  $\delta(s, d) = \min \{w(p): s \xrightarrow{p} d$ if there is a path from s to $d\}$

  $= \infty$ otherwise

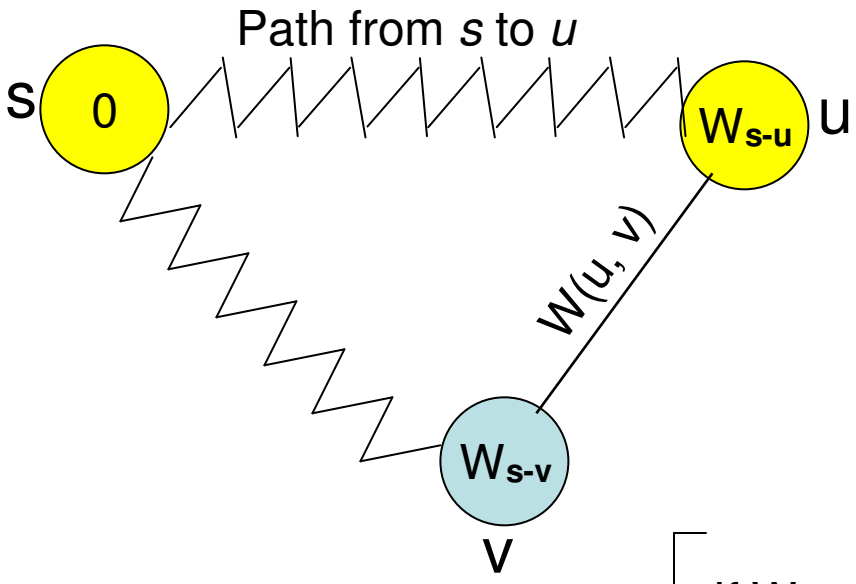
:

Dijkstra Algorithm

- **Assumption:** $w(u, v) > 0$ for each edge $(u, v) \in E$ (i.e., the edge weights are positive)
- **Objective:** Given $G = (V, E, w)$, find the shortest weight path between a given source s and destination d
- **Principle:** Greedy strategy
- Maintain a minimum weight path estimate $d[v]$ from s to each other vertex v.
- At each step, pick the vertex that has the smallest minimum weight path estimate
- **Output:** After running this algorithm for $|V|$ iterations, we get the shortest weight path from s to all other vertices in G
- **Time Complexity:** Dijkstra algorithm – $\Theta(|E|*log|V|)$

Dr. Meg's YouTube Video Explanation:
https://www.youtube.com/watch?v=V8VxK1cr0x0

Principle of Dijkstra Algorithm

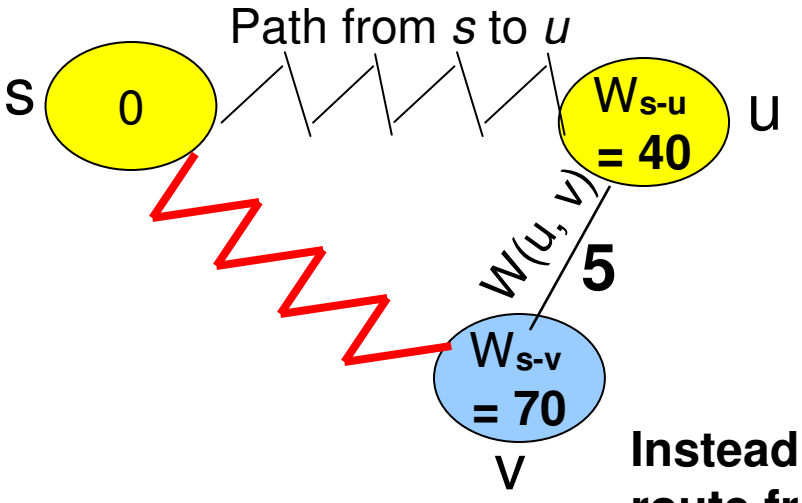


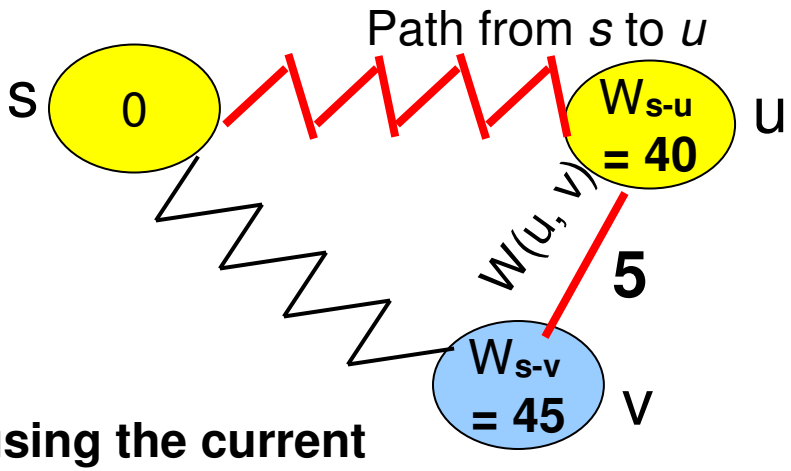
Principle in a nutshell

During the beginning of each iteration we will pick a vertex *u* that has the minimum weight path to *s*. We will then explore the neighbors of *u* for which we have not yet found a minimum weight path. We will try to see if by going through *u*, we can reduce the weight of path from *s* to *v*, where *v* is a neighbor of *u*.

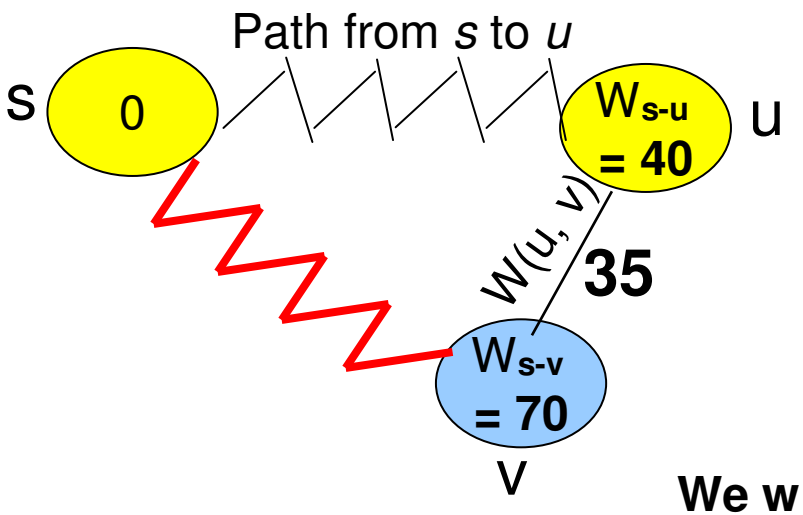
Relaxation Condition

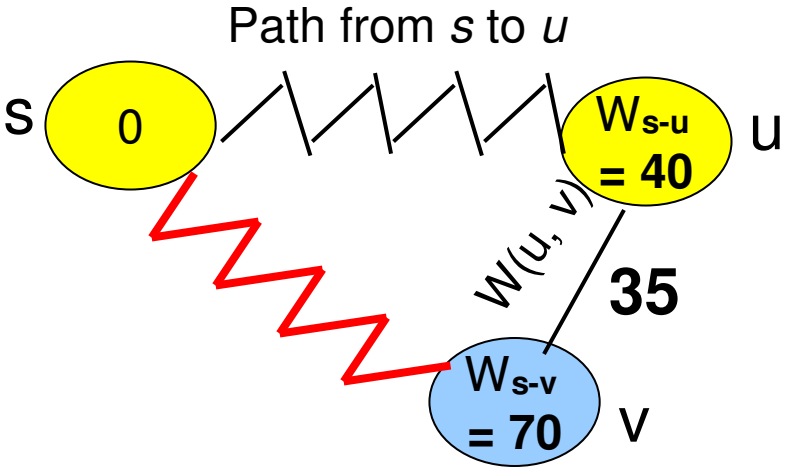
If $W_{s-v} > W_{s-u} + W(u, v)$ then
    $W_{s-v} = W_{s-u} + W(u, v)$
    Predecessor (v) = u
else
    Retain the current path from *s* to *v*



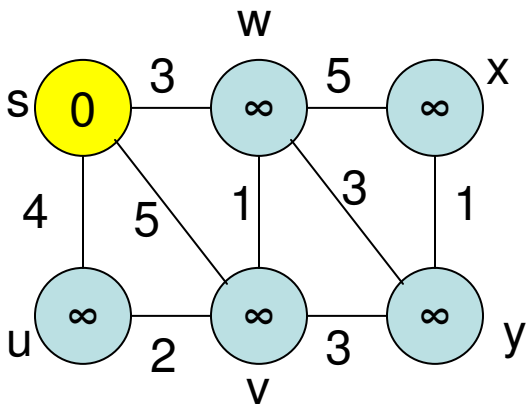


Instead of using the current route from s to v, we will go through u to reach v from s

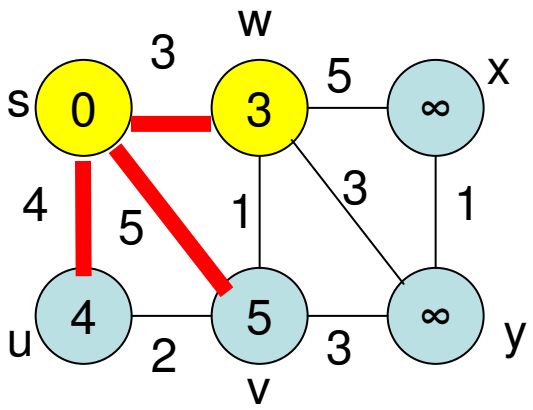




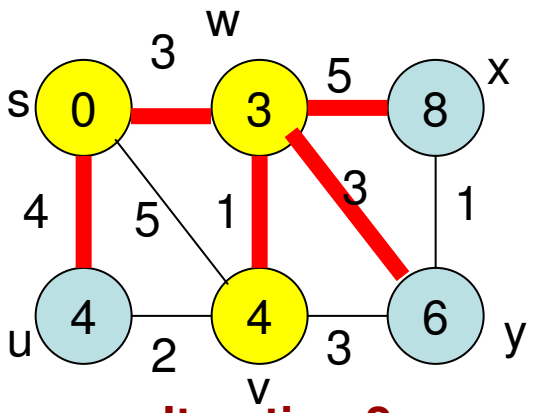
We will stay with the current route we know from s to v.



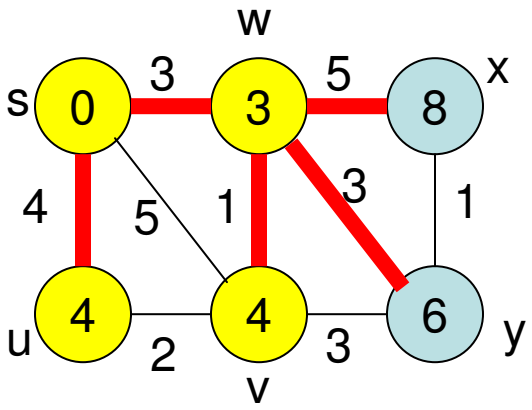
Given Graph, Initialization



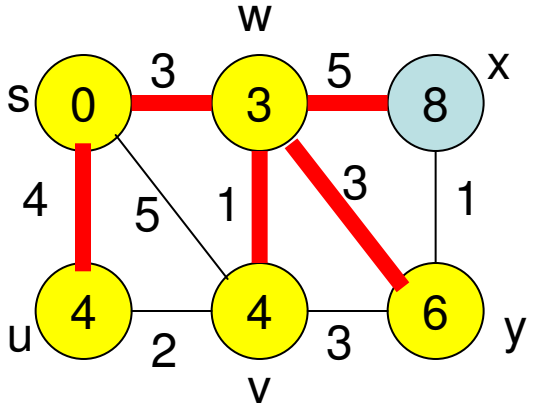
Iteration 1



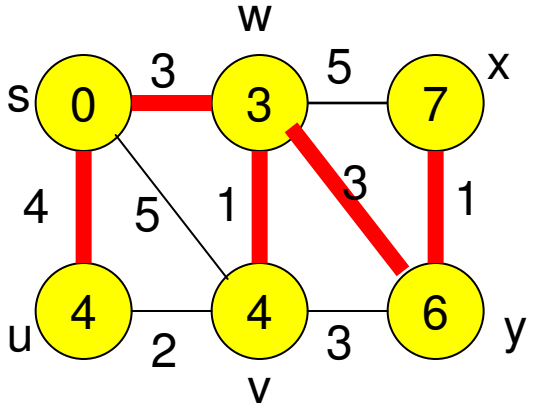
Iteration 2



Iteration 3



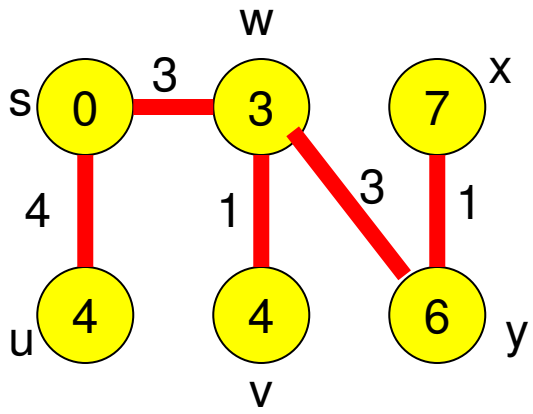
Iteration 4

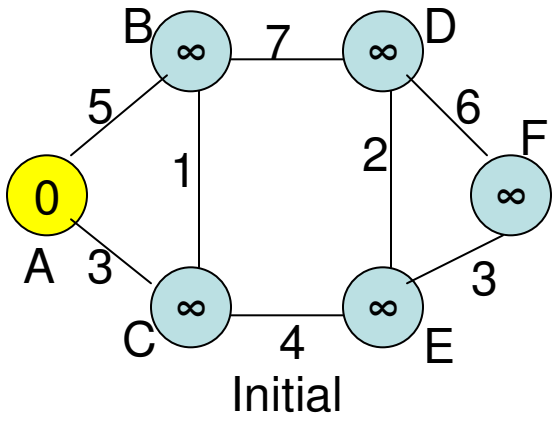


Iteration 5

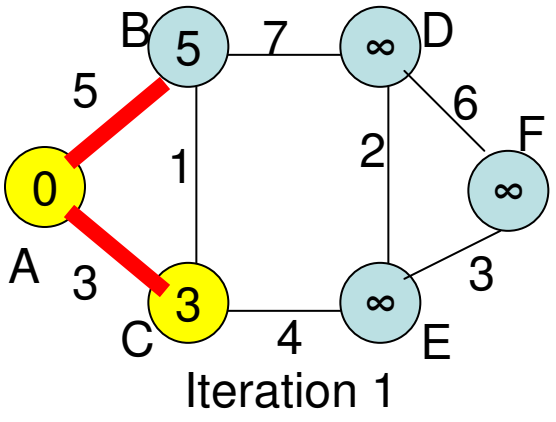
Dijkstra Algorithm Example 1

Shortest Path Tree

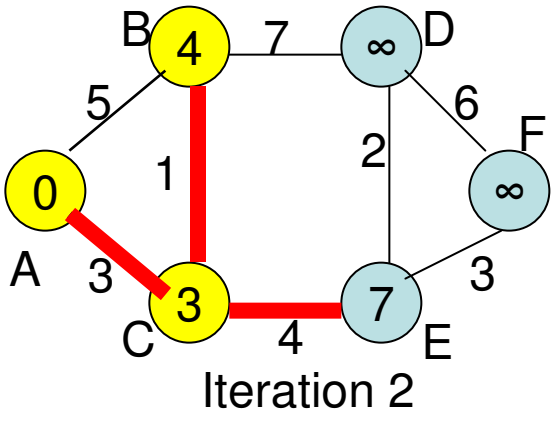




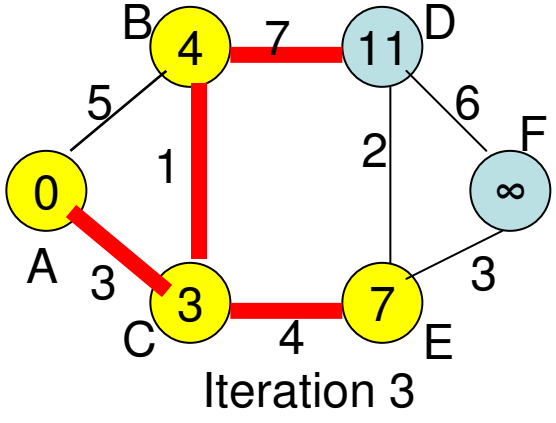
Initial



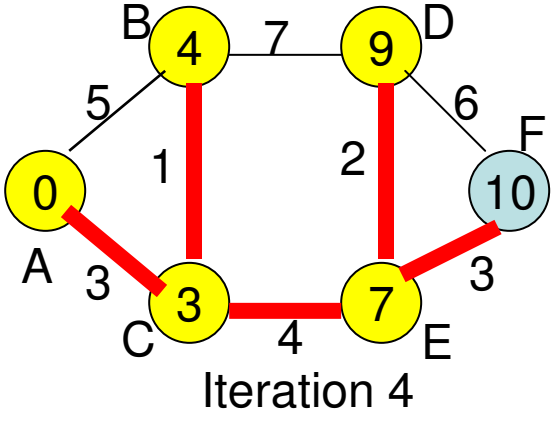
Iteration 1



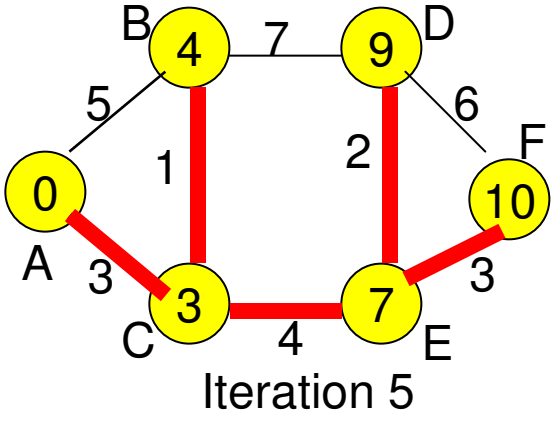
Iteration 2



Iteration 3

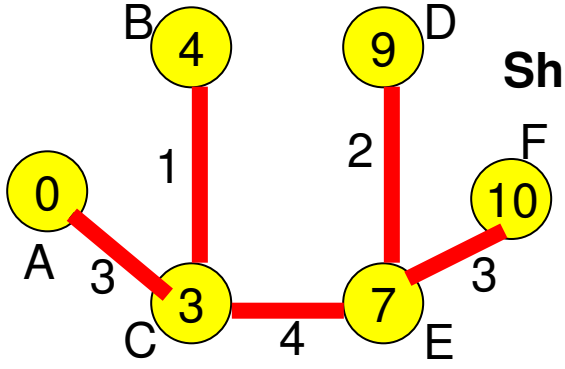


Iteration 4



Iteration 5

Dijkstra Algorithm Example 2



Shortest Path Tree

Dijkstra Algorithm

```
Begin Algorithm Dijkstra (G, s)
1    For each vertex v Є V
2        d [v] ← ∞ // an estimate of the min-weight path from s to v
3    End For
4    d [s] ← 0
5    S ← Φ  // set of nodes for which we know the min-weight path from s
6    Q ←  V // set of nodes for which we know estimate of min-weight path from s
7    While Q ≠ Φ
8        u ← EXTRACT-MIN(Q)
9        S ← S U {u}
10       For each vertex v such that (u, v) Є E
11           If v ∈ Q and d [v] > d [u] + w (u, v) then
12               d [v] ← d [u] + w (u, v)
13               Predecessor (v) = u
13           End If
14       End For
15   End While
16 End Dijkstra
```

Dijkstra Algorithm: Time Complexity

Begin Algorithm *Dijkstra* (*G*, *s*)

1 **For** each vertex $v \in V$

2 $d[v] \leftarrow \infty$ // an estimate of the min-weight path from *s* to *v*

3 **End For**

4 $d[s] \leftarrow 0$

5 $S \leftarrow \Phi$ // set of nodes for which we know the min-weight path from *s*

6 $Q \leftarrow V$ // set of nodes for which we know estimate of min-weight path from *s*

7 **While** $Q \neq \Phi$

8 $u \leftarrow$ EXTRACT-MIN(Q)

9 $S \leftarrow S \cup \{u\}$

10 **For** each vertex v such that $(u, v) \in E$

11 **If** $v \in Q$ and $d[v] > d[u] + w(u, v)$ then

12 $d[v] \leftarrow d[u] + w(u, v)$

13 Predecessor (v) = u

13 **End If**

14 **End For**

15 **End While**

16 **End** *Dijkstra*

Θ(V) time

Θ(V) time to Construct a Min-heap

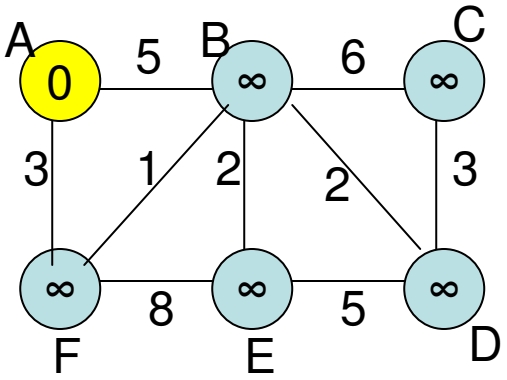
done |V| times = Θ(V) time

Each extraction takes Θ(logV) time

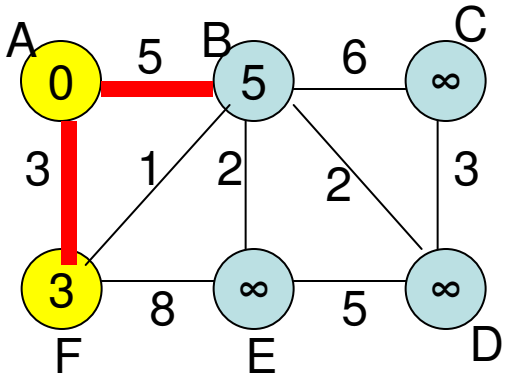
done Θ(E) times totally

It takes Θ(logV) time when done once

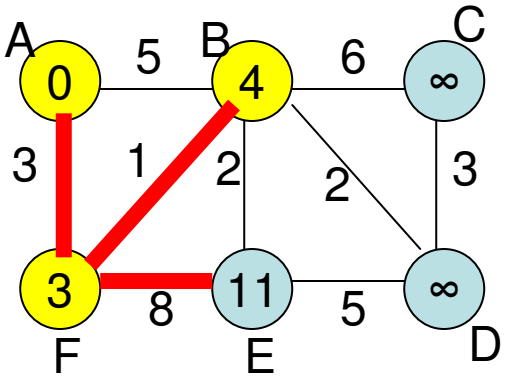
Overall Complexity: Θ(V) + Θ(V) + Θ(VlogV) + Θ(ElogV)
Since the |E| ≥ |V|-1, the VlogV term is dominated by the ElogV term. Hence, overall complexity = Θ(|E|*log|V|)



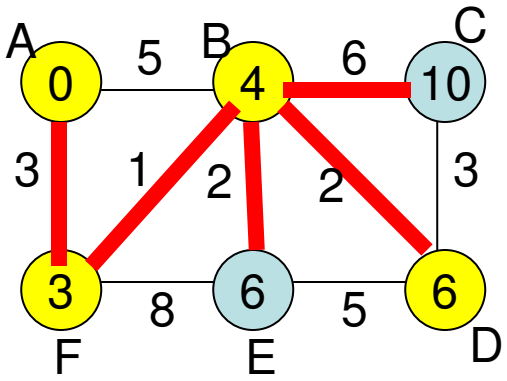
Initial



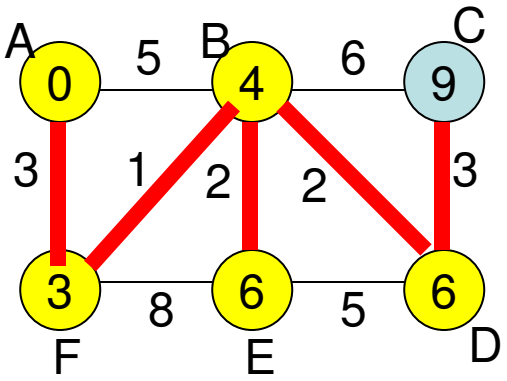
Iteration 1



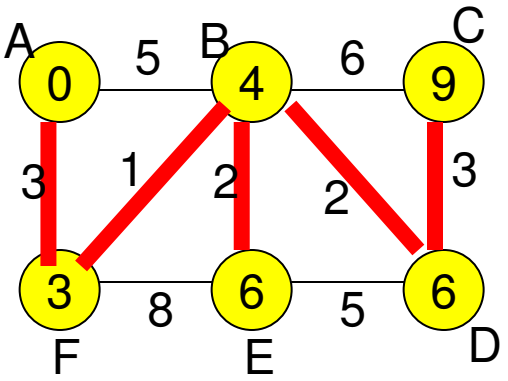
Iteration 2



Iteration 3

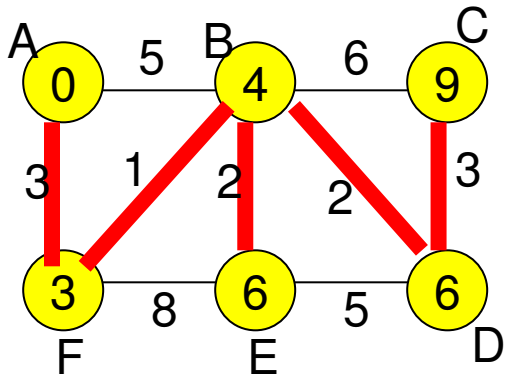


Iteration 4



Iteration 5

Dijkstra Algorithm Example 3



Shortest Path Tree

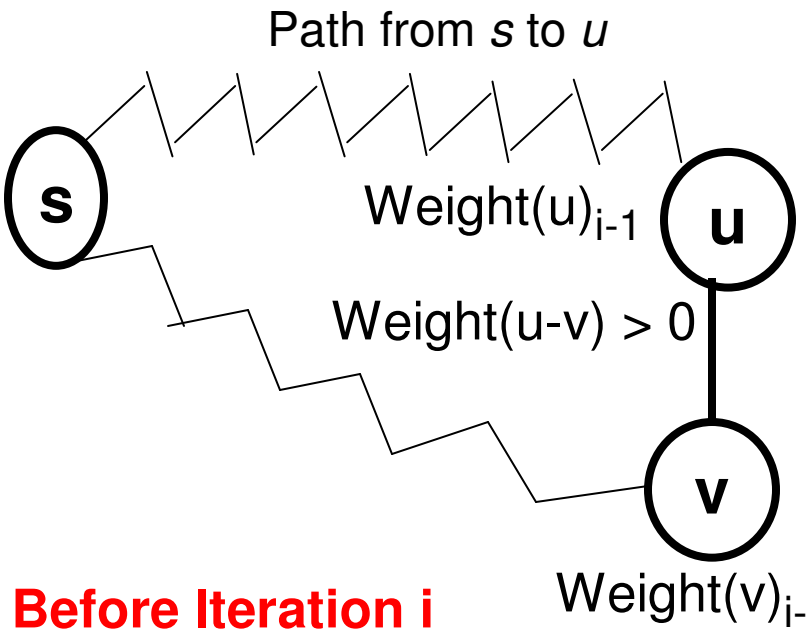
Theorems on Shortest Paths and Dijsktra Algorithm

- **Theorem 1:** Sub path of a shortest path is also shortest.
- **Proof:** Lets say there is a shortest path from s to d through the vertices s – a – b – c – d.
- Then, the shortest path from a to c is also a – b – c.
- If there is a path of lower weight than the weight of the path from a – b – c, then we could have gone from s to d through this alternate path from a to c of lower weight than a – b – c.
- However, if we do that, then the weight of the path s – a – b – c – d is not the lowest and there exists an alternate path of lower weight.
- This contradicts our assumption that s – a – b – c – d is the shortest (lowest weight) path.

Theorems on Shortest Paths and Dijsktra Algorithm

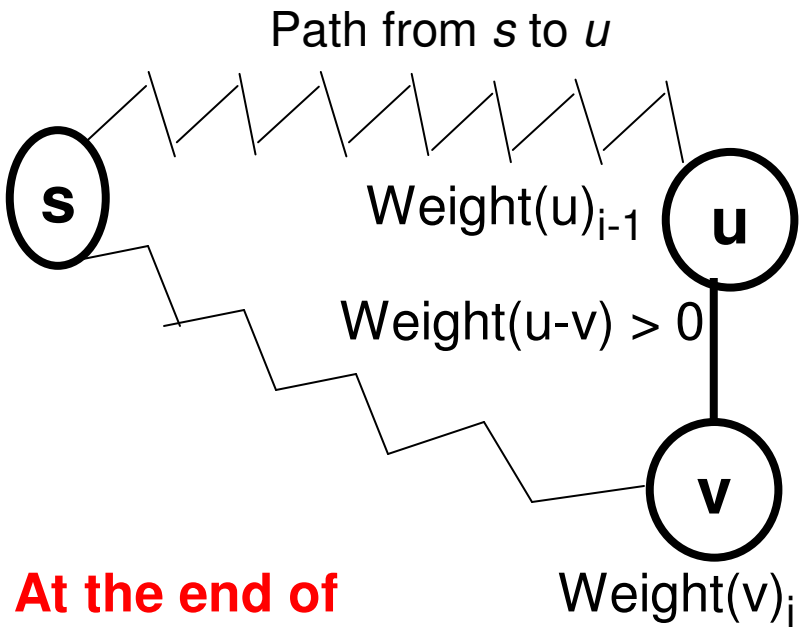
- **Theorem 2:** The weights of the vertices that are optimized are in the non-decreasing (i.e., typically increasing) order.
- **Proof:** We want to prove that if a vertex u is optimized in an earlier iteration (say iteration i), then the weight of the vertex v optimized at a later iteration (say iteration j; i < j) is always greater than or equal to that of vertex u.
- Vertex v could be either a neighbor of vertex u or not. In either case, the $weight(v)_{i-1} \geq weight(u)_{i-1}$ during the beginning of iteration i as vertex u was considered to have been optimized instead of vertex v during this iteration.
- During iteration i: we relax the neighbors of vertex u
  - If vertex v is a neighbor of vertex u, $weight(v)_i$ could have become less than $weight(v)_{i-1}$, but $weight(v)_i$ could never become $weight(u)_{i-1}$ as all edge weights are positive (including the weight of the edge u-v). Hence, $weight(v)_i$ could have become $weight(u)_{i-1}$ + weight(u-v), but it will still be only less than $weight(u)_{i-1}$, as weight(u-v) > 0.
- If vertex v is not a neighbor of vertex u, then vertex v should ultimately get optimized through some neighbor x (that is not u). But all such neighbors x should have $weight(x)_{i-1} \geq weight(u)_{i-1}$, as x was not picked for optimization in iteration i. Hence, by going through such neighbors x, the weight(v) during iterations i or later, could never become still less than $weight(u)_{i-1}$, as all the edge weights w(x-v) are greater than 0.

Proof for Theorem 2
Scenario: Vertex v is a neighbor of Vertex u



Before Iteration i (at the end of Iteration i-1)

Weight(v)$_{i-1}$ ≥ Weight(u)$_{i-1}$

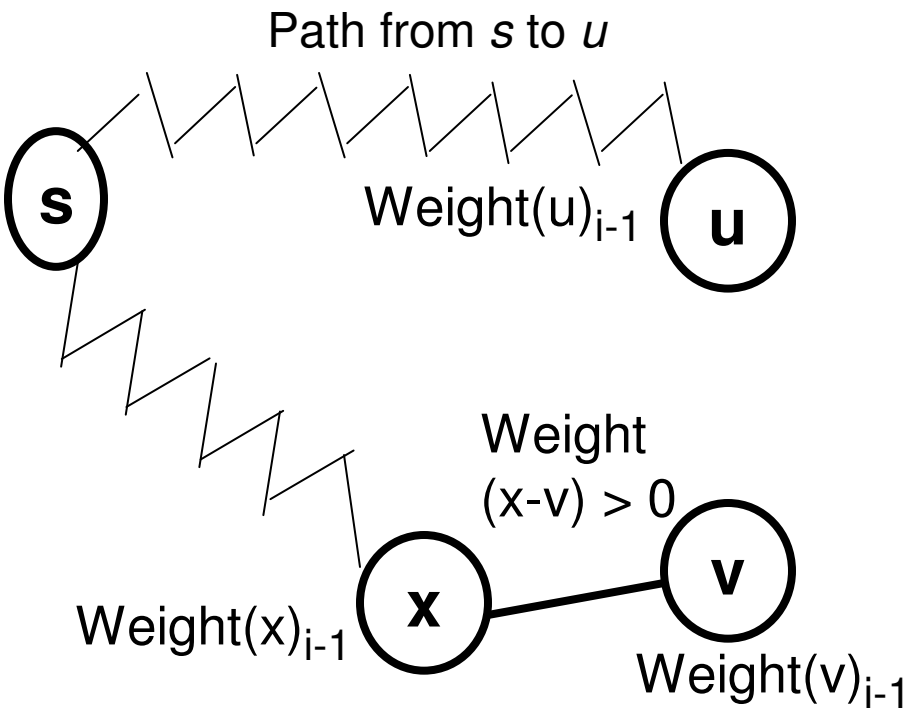


At the end of Iteration i

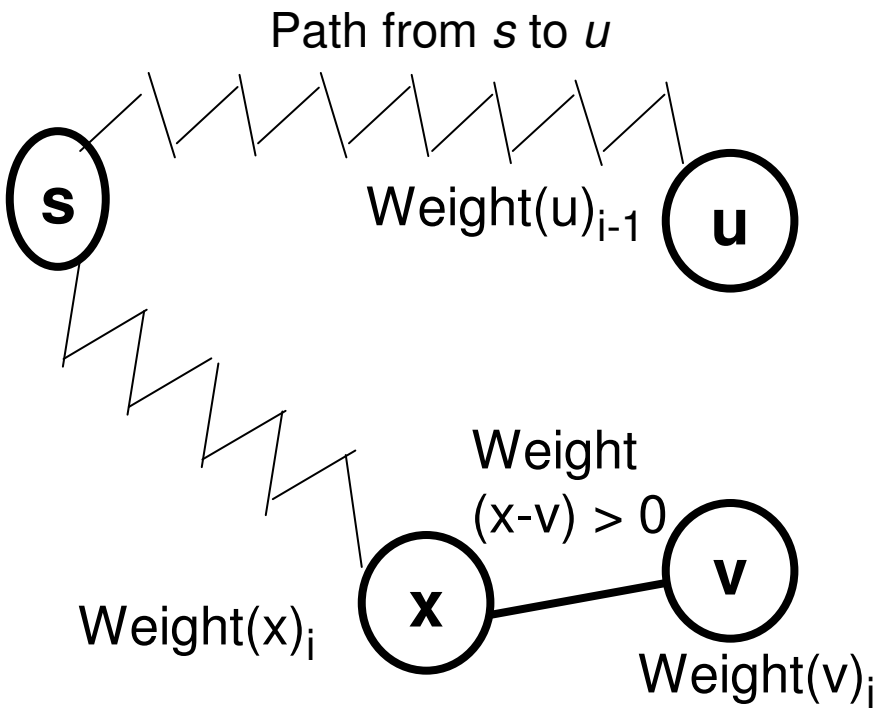
Weight(v)$_{i}$ ≥ Weight(u)$_{i-1}$

Proof for Theorem 2

Scenario: Vertex v is NOT a neighbor of Vertex u, but a neighbor of some other vertex x



Before Iteration i (at the end of Iteration i-1)



At the end of Iteration i

Weight(x)$_{i-1}$ ≥ Weight(u)$_{i-1}$

Weight(v)$_{i-1}$ ≥ Weight(u)$_{i-1}$

Weight(x)$_{i}$ ≥ Weight(u)$_{i-1}$

Weight(v)$_{i}$ ≥ Weight(u)$_{i-1}$

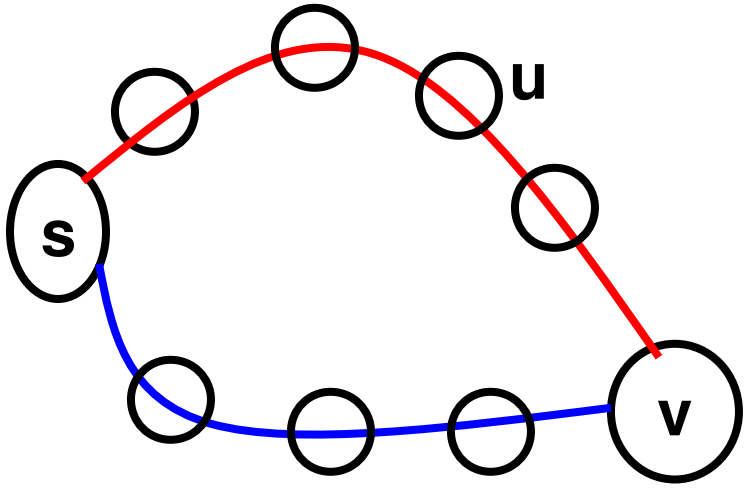
Theorems on Shortest Paths and Dijsktra Algorithm

- **Theorem 3:** When a vertex v is picked for relaxation/optimization, every intermediate vertex on the s…v shortest path is already optimized.
- **Proof:** Let there be a path from s to v that includes a vertex *x* (i.e., s...x...v) for which we have not yet found the shortest path.
- From Theorem 1, shortest path weight(s...x) < shortest path weight(s...v).
- From Theorem 2, vertices are optimized in the non-decreasing order of shortest path weights.
- So, if vertex v is picked for optimization based on the path s…x…v, then the intermediate vertex x should have been already picked (before v) for optimization. A contradiction.

- **Theorem 4:** When a vertex v is picked for relaxation, we have optimized the vertex (i.e., found the shortest path for the vertex from a source vertex s).
- **Proof:** Let P be the path from source s to vertex v based on whose weight we decide to relax the vertex. We want to prove P is the optimal path of minimum weight from s to v. We will prove this by contradiction.
- Let P' be a hypothetical shortest path from s to v such that w(P') < w(P)
- If all the intermediate vertices from s to v on the path P' are already optimized, we would have indeed found the shortest path from s to v of weight w(P').
- If P' is not chosen and P is chosen by Dijkstra algorithm for optimizing vertex v, then there should be at least one intermediate vertex (say vertex 'u') on the path P' from s to v that is not yet optimized (and because of this we were not able to optimize v from s on path P').
- From the earlier Theorems, the weight(s…u in P') ≥ weight(s…v in P) because the algorithm picks vertices for optimization in the non-decreasing (i.e., increasing) order of shortest path weights.
  - So, even if vertex u on path P' is chosen for optimization after vertex v on path P, the weight of the s…u…v path (P') would be only larger than that of the s…v path (P). Hence, a contradiction.
- Thus, the path P found by Dijkstra algorithm is the shortest path from the source s to a vertex v.

Proof for Theorem 4 (by Contradiction)

Hypothetical Path P' that We assume:
Weight(s…v)$_{P'}$ < Weight(s…v)$_{P}$



Path P found by Dijkstra algorithm

From Theorem 3,
If P' is an optimal path from s to v, then all the intermediate vertices on the path should have been already optimized, and as a result of the accompanying relaxations, we would have traced the path P' from s to v as the optimal path instead of the path P. Hence, if the algorithm did not pick P' as the optimal path, there should be some intermediate vertex u on the path P' that is not yet optimized and all the subsequent vertices on the path P' are not optimized either.

From Theorem 2,
Weight(s…u)$_{P'}$ ≥ Weight(s…v)$_{P}$

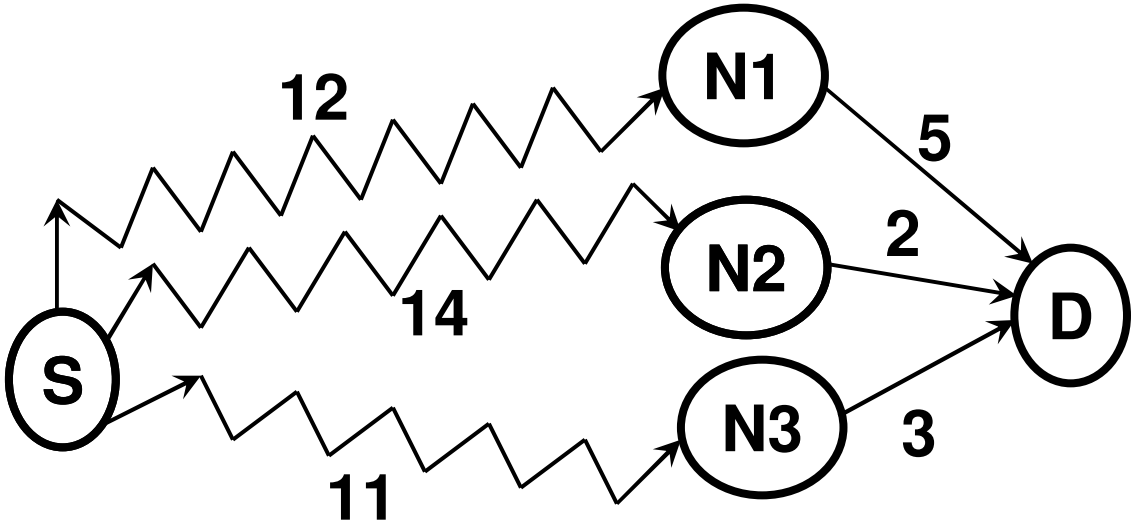
From Theorem 1,
Weight(s…u…v)$_{P'}$ > Weight(s…u)$_{P'}$

Hence:
Weight(s…u…v)$_{P'}$ > Weight(s…v)$_{P}$

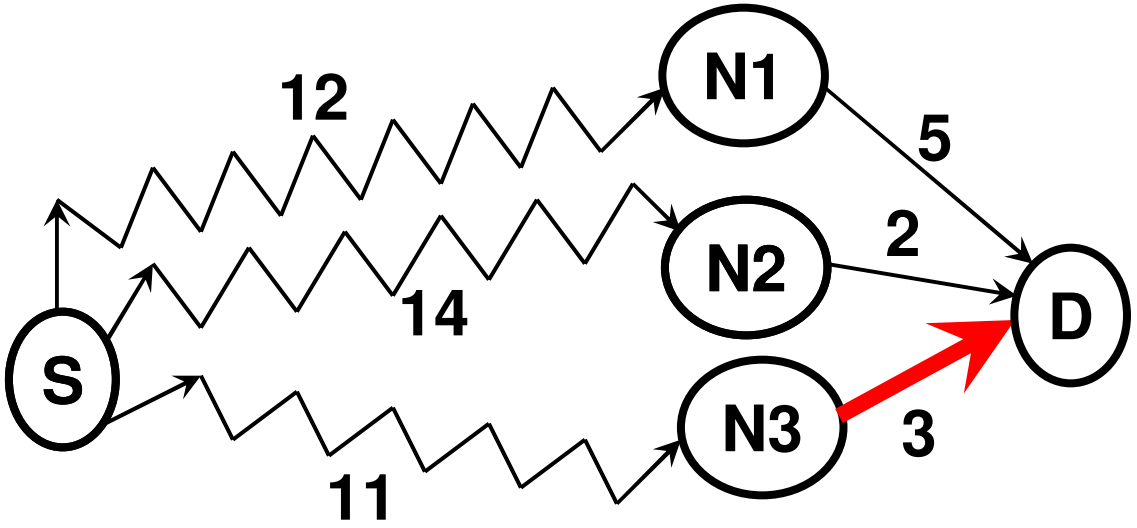
Bellman-Ford Algorithm

- The Bellman-Ford algorithm is a single source shortest path algorithm that can be run for weighted directed graphs with positive and/or negative edge weights.
  - Note that the Dijkstra algorithm will work only for graphs with positive edge weights, and is typically applied for undirected graphs.
- The Bellman-Ford algorithm maintains an estimate of the shortest path distance from the source to every vertex (including itself) and tries to reduce the estimate as much as possible by a going through series of iterations.
  - In each iteration, we try to reduce the estimate of the shortest path distance for a node on the basis of the estimate of the shortest path distance for its INCOMING neighbors (calculated in the previous iteration).
    - The incoming neighbor node that gives the smallest value for the estimate is chosen/updated as the predecessor.
  - We go through a series of V-1 iterations for a graph of V vertices.
  - Optimization: If the estimates for the shortest path distances do not change for any vertex during an iteration, stop the algorithm.

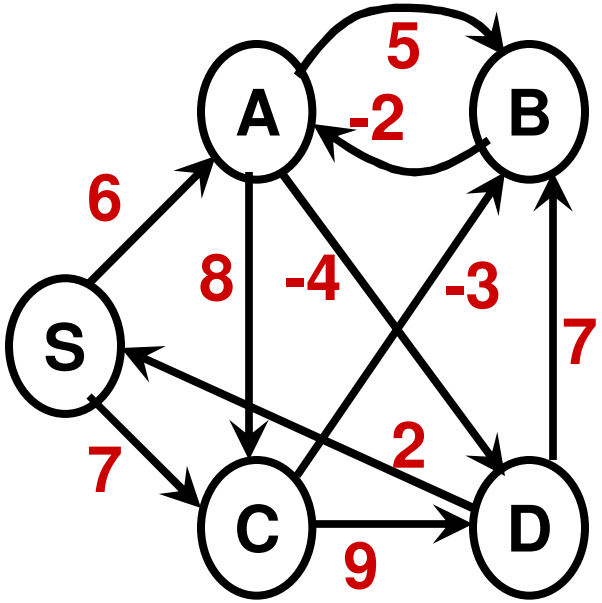
Bellman-Ford Algorithm
Operating Principle

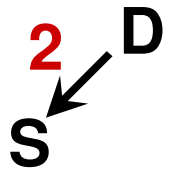


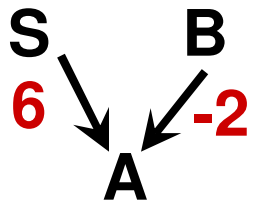
11 + 3 = 14 is lower than 12 + 5 and 14 + 2. So, N3 is chosen as the Predecessor for D

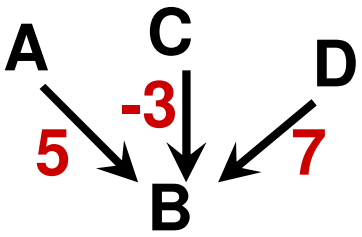


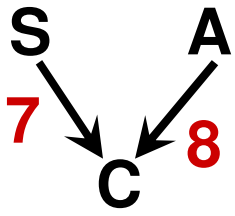
Bellman-Ford Algorithm: Example 1

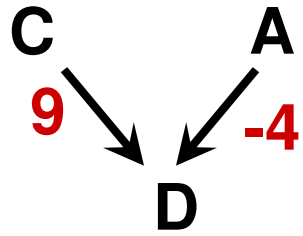






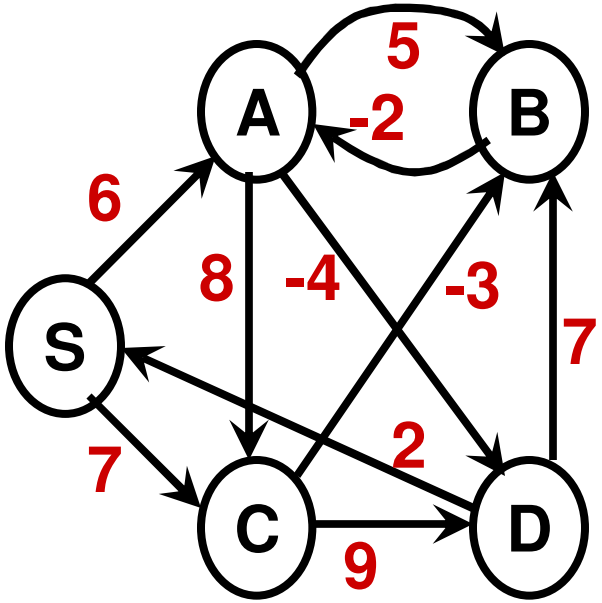


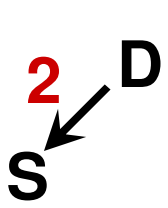


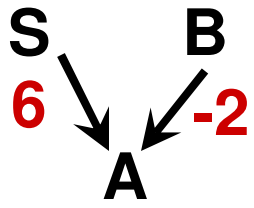


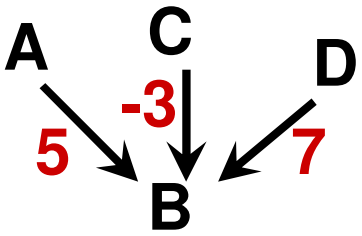
	Initial	
	Est.	Pred
S	0	-
A	inf	-
B	inf	-
C	inf	-
D	inf	-

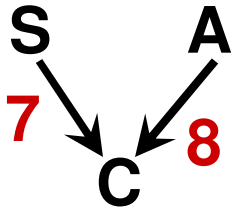
Bellman-Ford Algorithm: Example 1

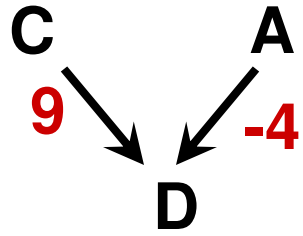






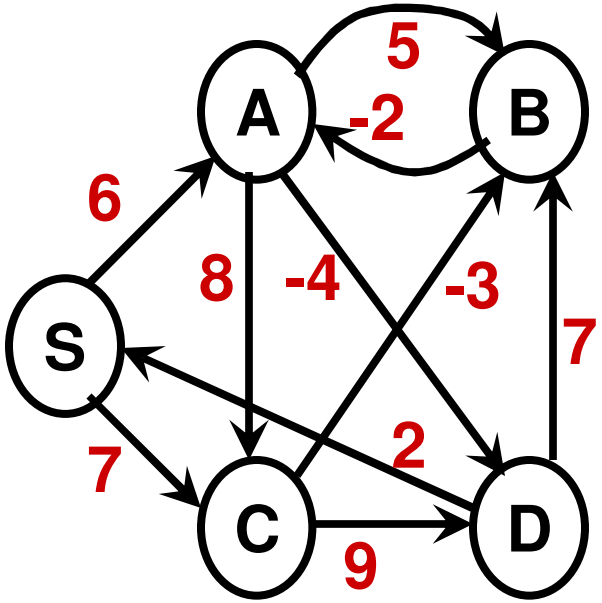


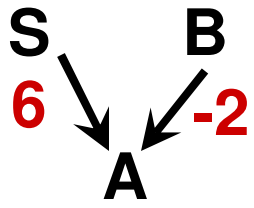


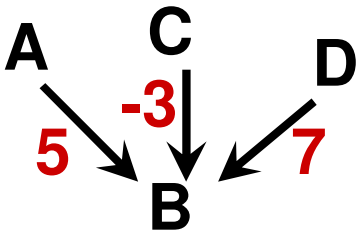


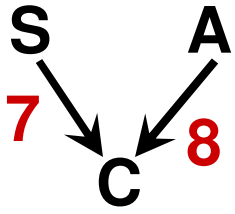
	Initial		Iteration 1	
	Est.	Pred	Est.	Pred
S	0	-	0	-
A	inf	-	6	S
B	inf	-	inf	-
C	inf	-	7	S
D	inf	-	inf	-

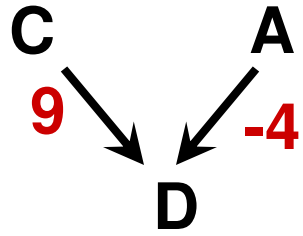
Bellman-Ford Algorithm: Example 1





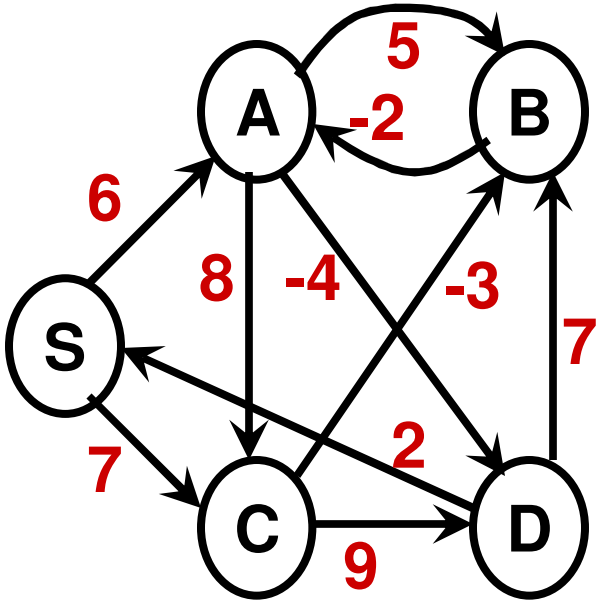


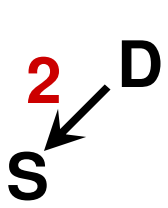


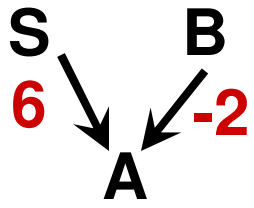


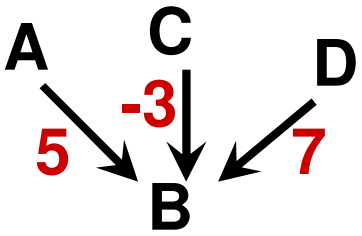
	Iteration 1		Iteration 2	
	Est.	Pred	Est.	Pred
S	0	-	0	-
A	6	S	6	S
B	inf	-	4	C
C	7	S	7	S
D	inf	-	2	A

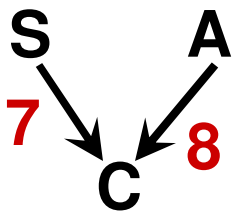
Bellman-Ford Algorithm: Example 1

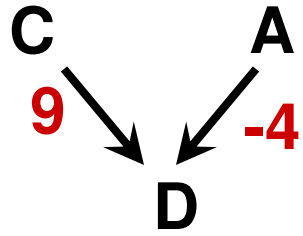






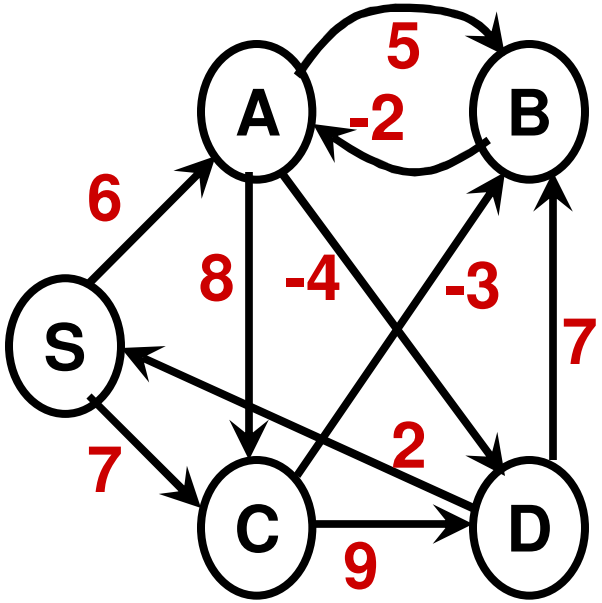


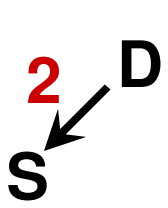


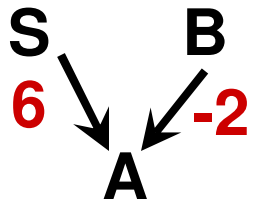


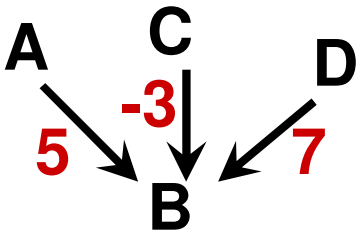
	Iteration 2		Iteration 3	
	Est.	Pred	Est.	Pred
S	0	-	0	-
A	6	S	2	B
B	4	C	4	C
C	7	S	7	S
D	2	A	2	A

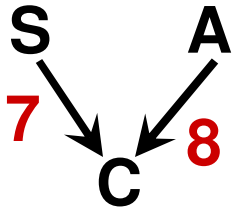
Bellman-Ford Algorithm: Example 1

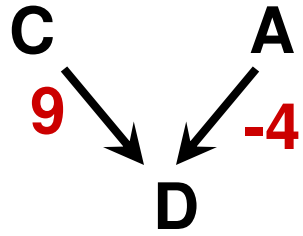






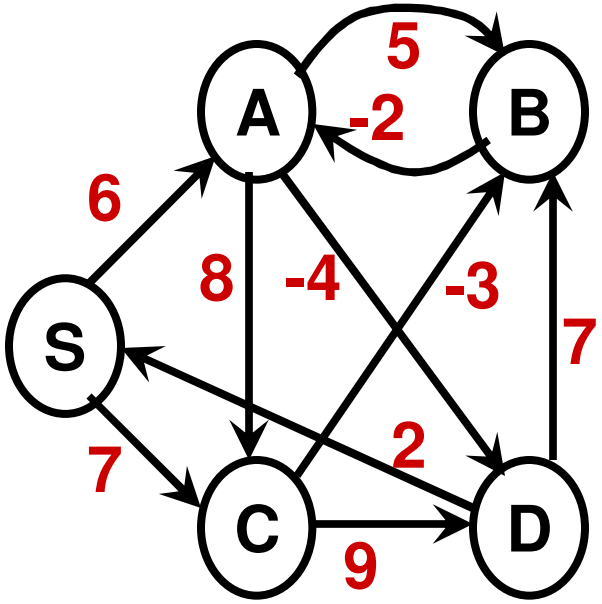






		Iteration 3		Iteration 4	
		Est.	Pred	Est.	Pred
S		0	-	0	-
A		2	B	2	B
B		4	C	4	C
C		7	S	7	S
D		2	A	-2	A

Bellman-Ford Algorithm: Example 1



Sample Shortest Path (S…D)

S ..... A → D

S ..... B → A → D

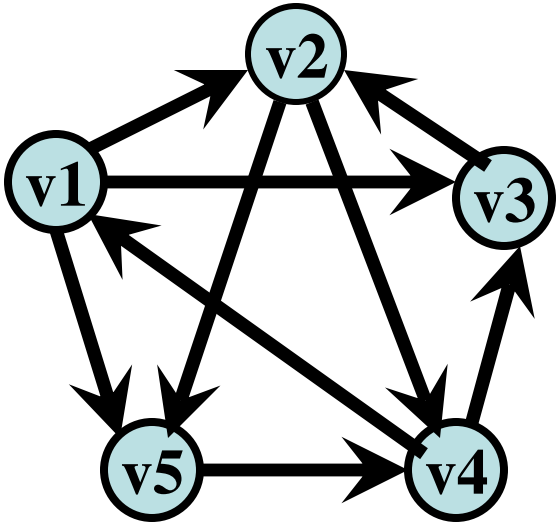
S .... C → B → A → D

S → C → B → A → D

Note that the property "sub path of a shortest path is also a shortest path" is still satisfied.

	Initial		Iteration 1		Iteration 2		Iteration 3		Iteration 4	
	Est.	Pred	Est.	Pred	Est.	Pred	Est.	Pred	Est.	Pred
S	0	-	0	-	0	-	0	-	0	-
A	inf	-	6	S	6	S	2	B	2	B
B	inf	-	inf	-	4	C	4	C	4	C
C	inf	-	7	S	7	S	7	S	7	S
D	inf	-	inf	-	2	A	2	A	-2	A

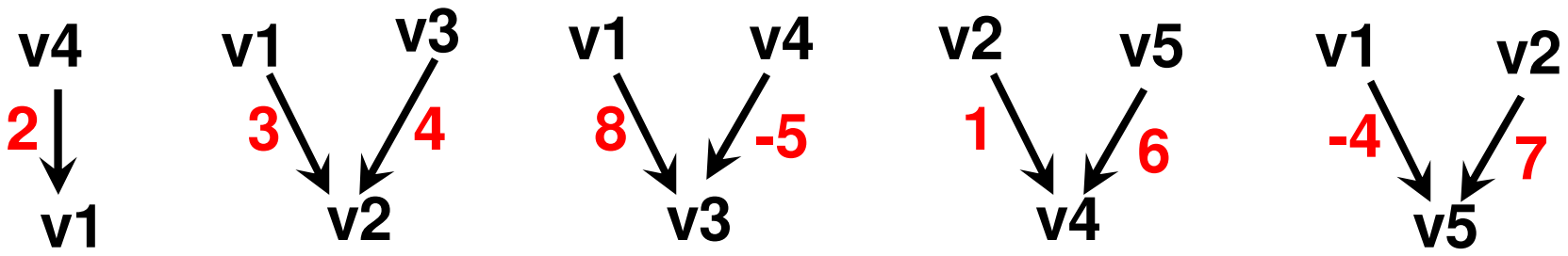
Bellman-Ford Algorithm: Example 2



		Weight Matrix			
	v1	v2	v3	v4	v5
v1	0	3	8	∞	-4
v2	∞	0	∞	1	7
v3	∞	4	0	∞	∞
v4	2	∞	-5	0	∞
v5	∞	∞	∞	6	0

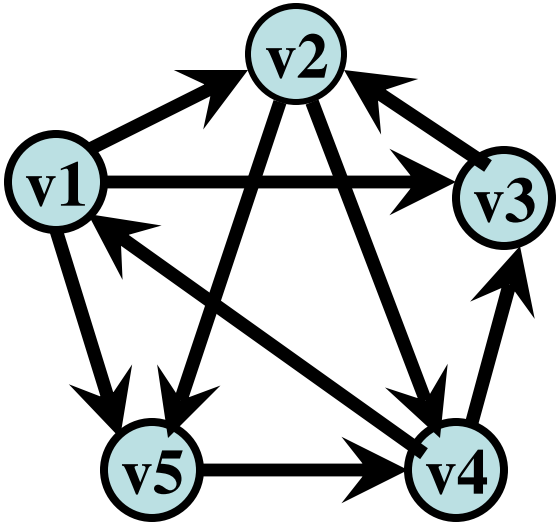
Note: An entry in the cell (i, j) indicates the weight of the edge i → j (i.e., row i, column j).

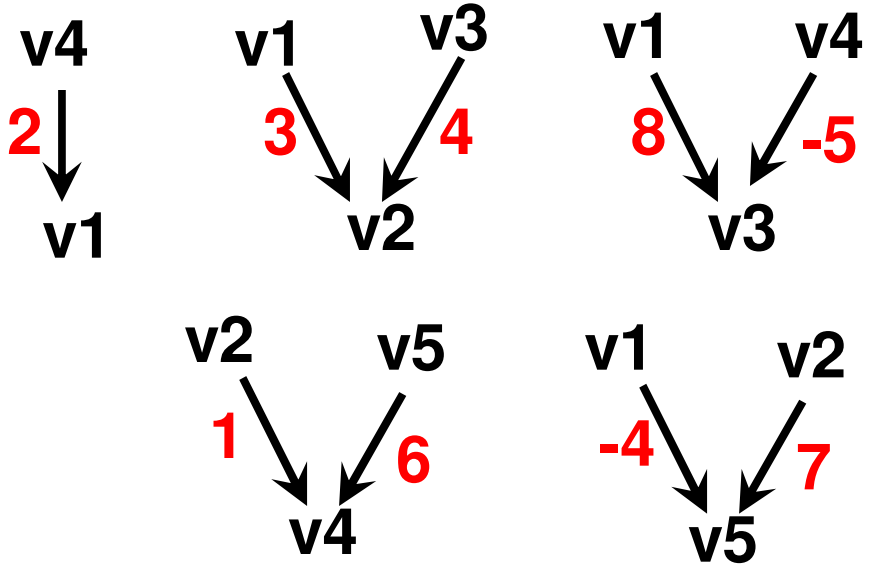
The entries in the column j indicate the weights of the incoming edges to vertex v-j.



Let v1 be the source

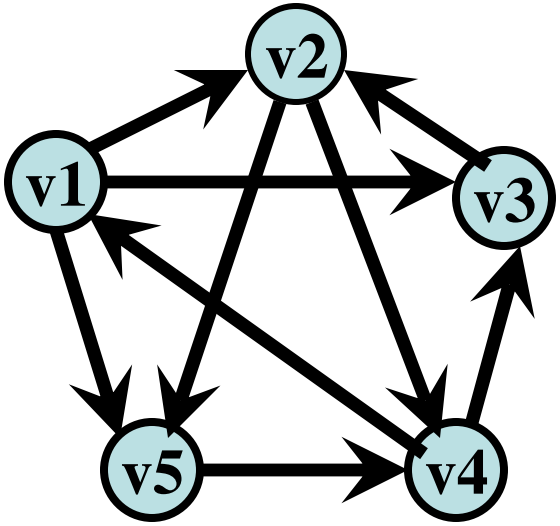
Bellman-Ford Algorithm: Example 2

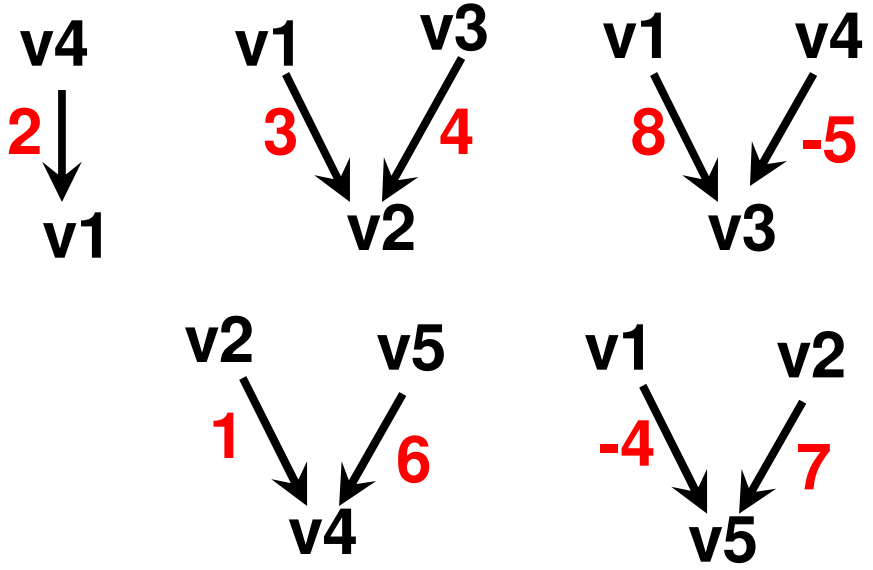




	Initial	
	Est.	Pred
v1	0	-
v2	inf	-
v3	inf	-
v4	inf	-
v5	inf	-

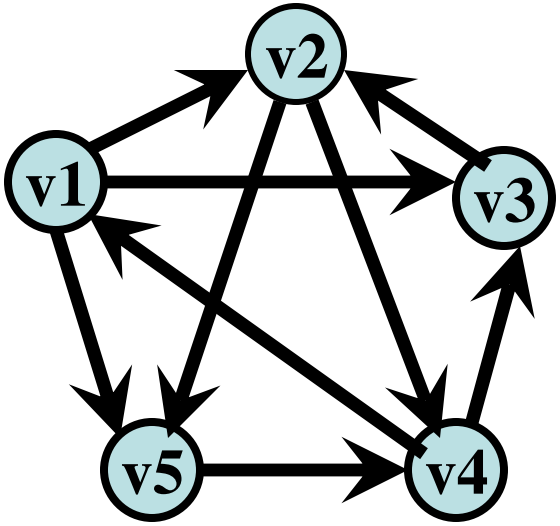
Bellman-Ford Algorithm: Example 2

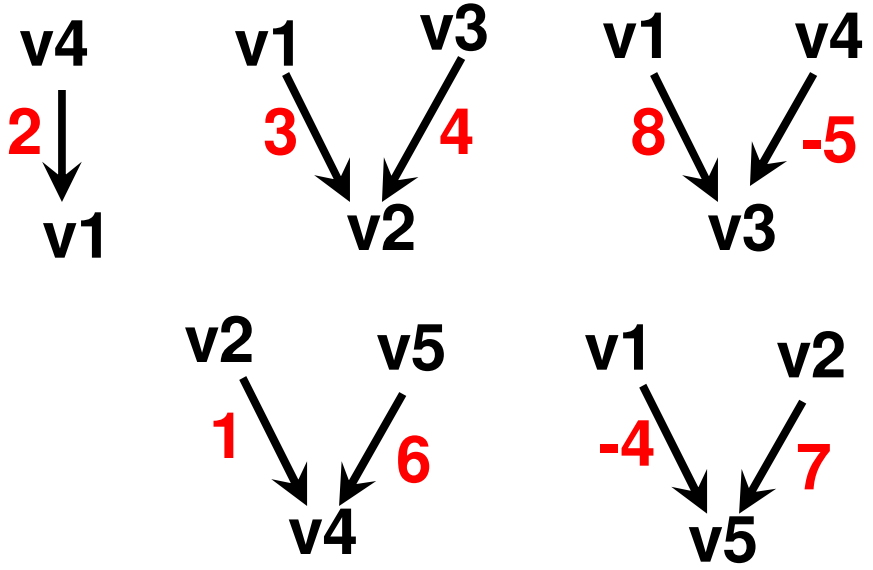




	Initial		Iteration 1	
	Est.	Pred	Est.	Pred
v1	0	-	0	-
v2	inf	-	3	v1
v3	inf	-	8	v1
v4	inf	-	inf	-
v5	inf	-	-4	v1

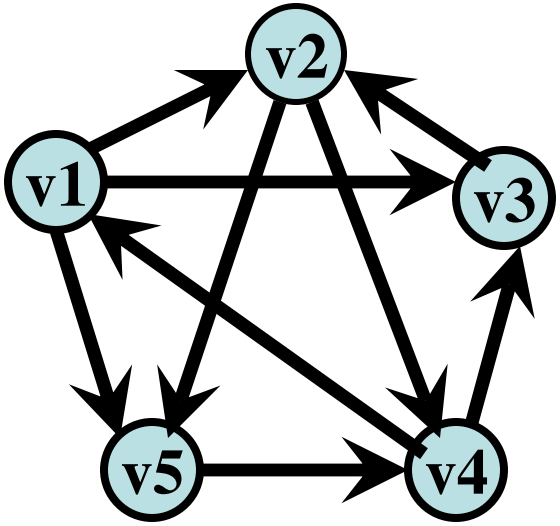
Bellman-Ford Algorithm: Example 2

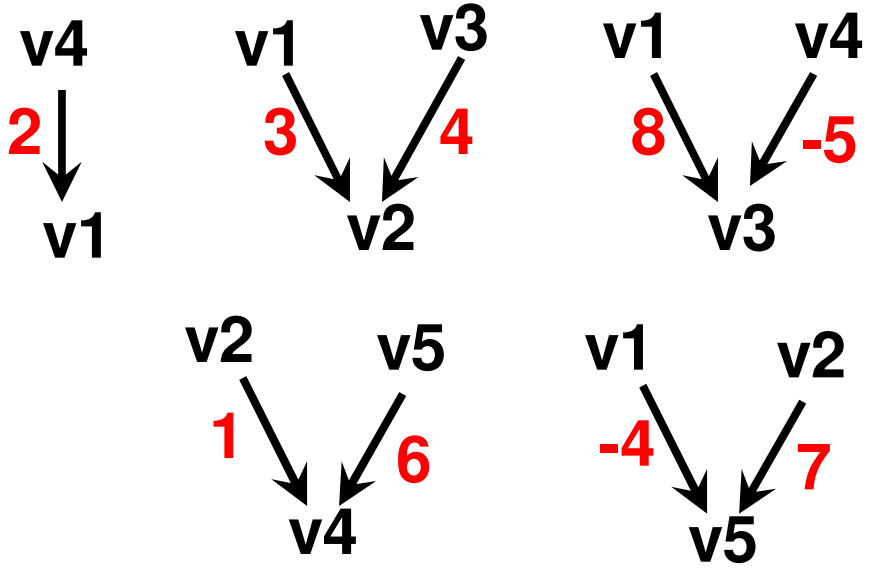




		Iteration 1		Iteration 2		
		Est.	Pred	Est.	Pred	
v1		0	-	0	-	
v2		3	v1	3	v1	
v3		8	v1	8	v1	
v4		inf	-	2	v5	
v5		-4	v1	-4	v1	

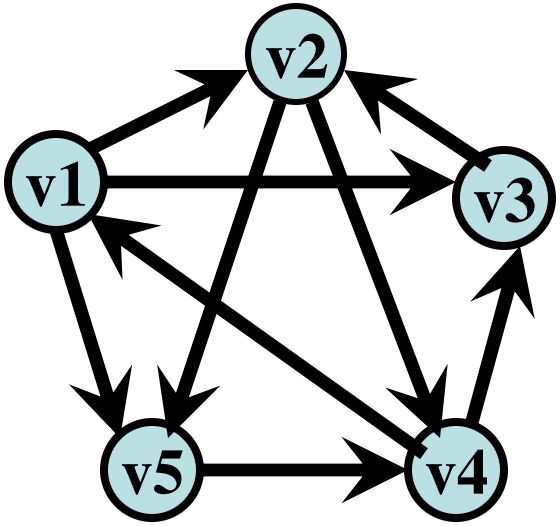
Bellman-Ford Algorithm: Example 2

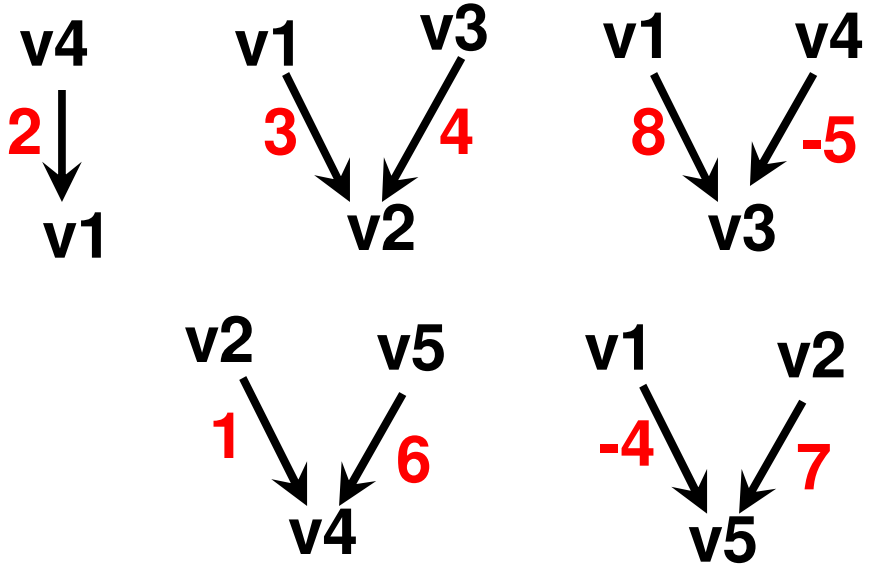




		Iteration 2		Iteration 3		
		Est.	Pred	Est.	Pred	
v1		0	-	0	-	
v2		3	v1	3	v1	
v3		8	v1	-3	v4	
v4		2	v5	2	v5	
v5		-4	v1	-4	v1	

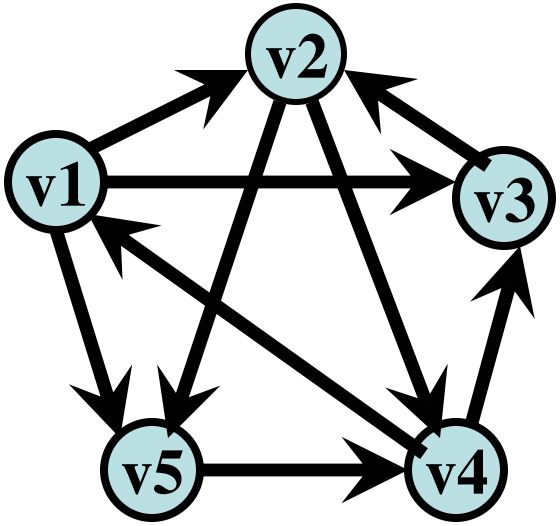
Bellman-Ford Algorithm: Example 2





		Iteration 3		Iteration 4	
		Est.	Pred	Est.	Pred
v1		0	-	0	-
v2		3	v1	1	v3
v3		-3	v4	-3	v4
v4		2	v5	2	v5
v5		-4	v1	-4	v1

Bellman-Ford Algorithm: Example 2



Sample Shortest Path (v1 … v2)

v1 …… v3 → v2

v1 ……. v4 → v3 → v2

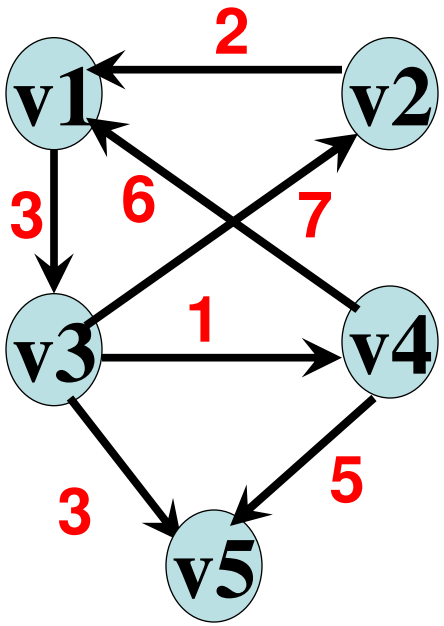
v1 …. v5 → v4 → v3 → v2

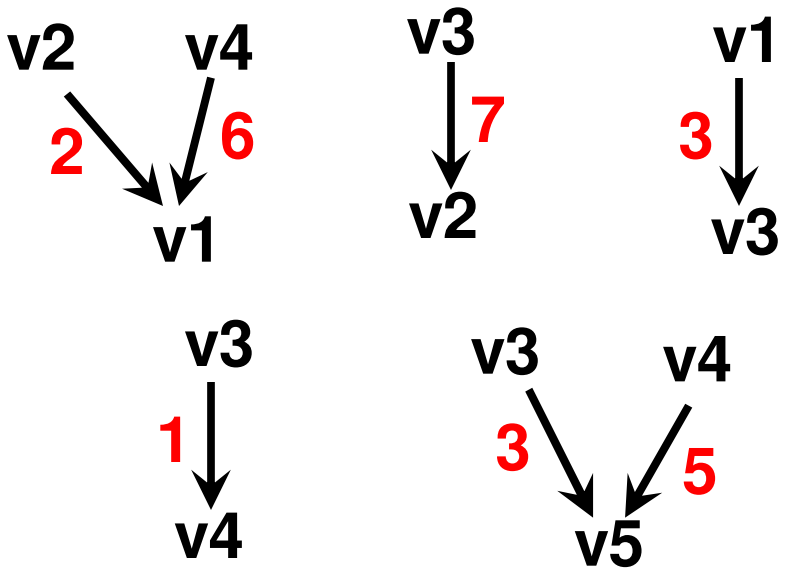
v1 → v5 → v4 → v3 → v2

-4 6 -5 4

	Initial		Iteration 1		Iteration 2		Iteration 3		Iteration 4	
	Est.	Pred	Est.	Pred	Est.	Pred	Est.	Pred	Est.	Pred
v1	0	-	0	-	0	-	0	-	0	-
v2	inf	-	3	v1	3	v1	3	v1	1	v3
v3	inf	-	8	v1	8	v1	-3	v4	-3	v4
v4	inf	-	inf	-	2	v5	2	v5	2	v5
v5	inf	-	-4	v1	-4	v1	-4	v1	-4	v1

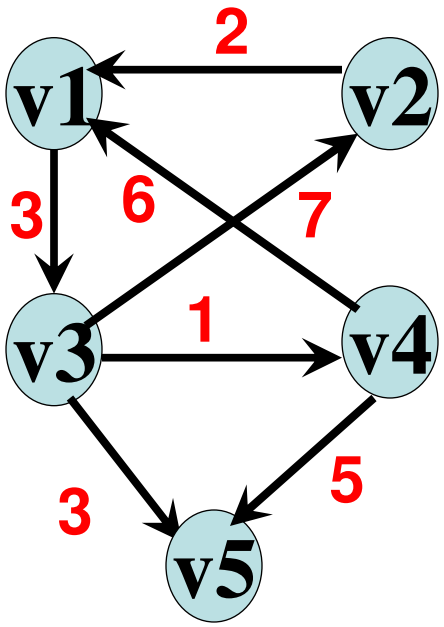
Bellman-Ford Algorithm: Example 3

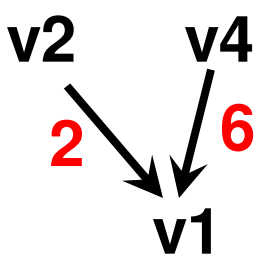


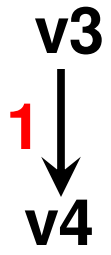


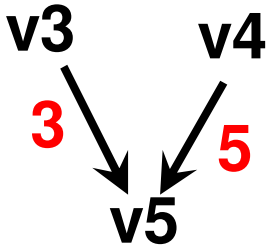
	Initial	
	Est.	Pred
v1	0	-
v2	inf	-
v3	inf	-
v4	inf	-
v5	inf	-

Bellman-Ford Algorithm: Example 3



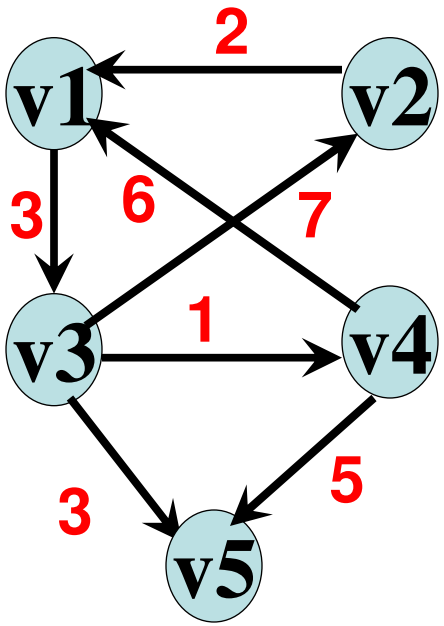


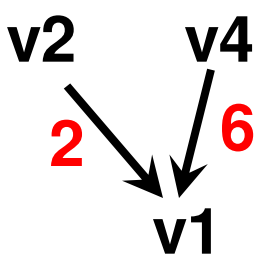


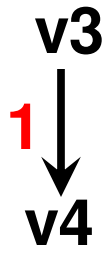


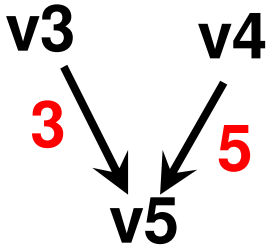
	Initial		Iteration 1	
	Est.	Pred	Est.	Pred
v1	0	-	0	-
v2	inf	-	inf	-
v3	inf	-	3	v1
v4	inf	-	inf	-
v5	inf	-	inf	-

Bellman-Ford Algorithm: Example 3



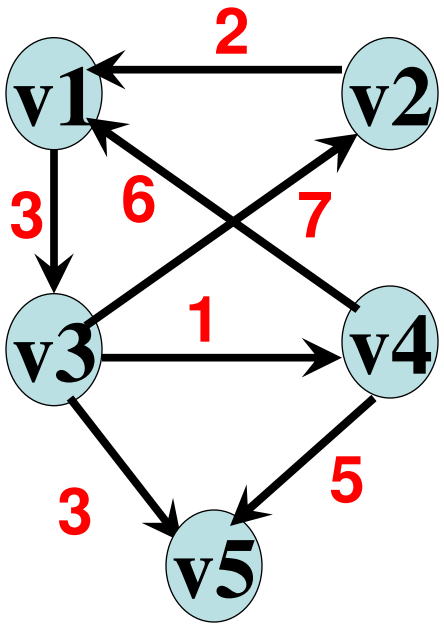


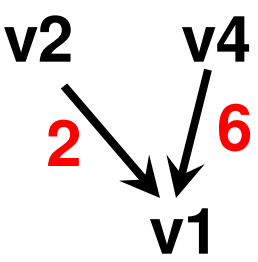




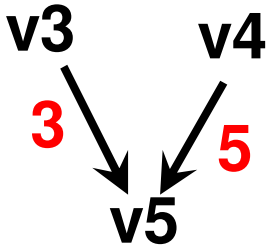
	Iteration 1		Iteration 2	
	Est.	Pred	Est.	Pred
v1	0	-	0	-
v2	inf	-	10	v3
v3	3	v1	3	v1
v4	inf	-	4	v3
v5	inf	-	6	v3

Bellman-Ford Algorithm: Example 3



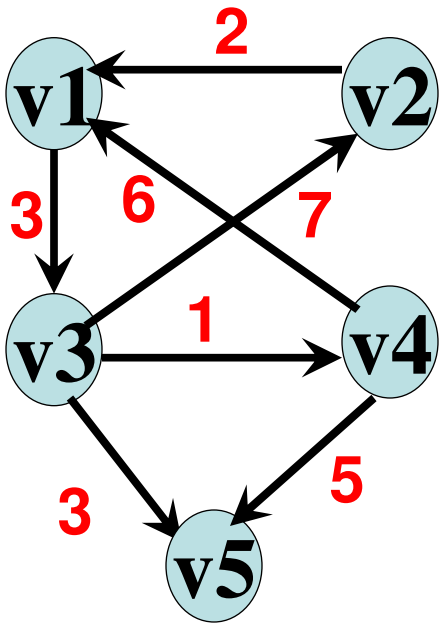


Note that the Estimates did Not change in Iterations 2 and 3. We can STOP!



	Iteration 2		Iteration 3	
	Est.	Pred	Est.	Pred
v1	0	-	0	-
v2	10	v3	10	v3
v3	3	v1	3	v1
v4	4	v3	4	v3
v5	6	v3	v6	v3

Bellman-Ford Algorithm: Example 3



V1 → V3
V1 → V3 → V2
V1 → V3 → V4
V1 → V3 → V5

Optimization Possible!!

	Initial		Iteration 1		Iteration 2		Iteration 3		Iteration 4	
	Est.	Pred	Est.	Pred	Est.	Pred	Est.	Pred	Est.	Pred
v1	0	-	0	-	0	-	0	-		
v2	inf	-	inf	-	10	v3	10	v3		
v3	inf	-	3	v1	3	v1	3	v1		
v4	inf	-	inf	-	4	v3	4	v3		
v5	inf	-	inf	-	6	v3	v6	v3		

NOT NEEDED

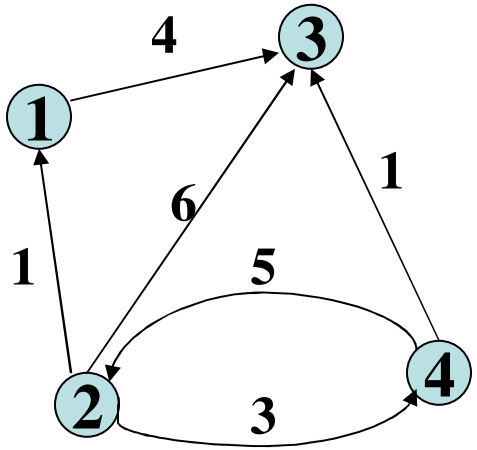
All Pairs Shortest Paths Problem

Dynamic Programming Algorithm for All Pairs Shortest Paths

Problem: **In a weighted (di)graph, find shortest paths between every pair of vertices**

idea: construct solution through series of matrices $D^{(0)}$, …, $D^{(n)}$ using increasing subsets of the vertices allowed as intermediate

Example:



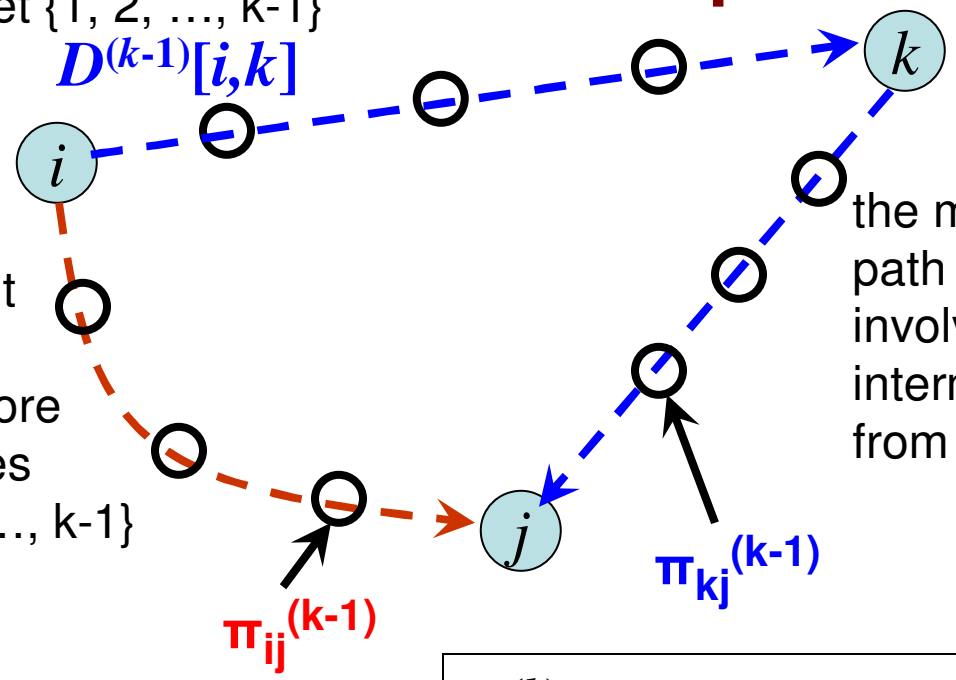
The algorithm we are going to see was developed by two people Floyd and Warshall. We will shortly refer to the algorithm as the FW algorithm

FW Algorithm: Operating Principle

- **Operating Principle:** The vertices are numbered from 1 to n. There are 'n' iterations. In the kth iteration, the candidate set of vertices available to choose from as intermediate vertices are {1, 2, 3, …, k}.
- **Initialization:** No vertex is a candidate intermediate vertex. There is a path between two vertices only if there is a direct edge between them (i.e., i → j); otherwise, not.
- **Iteration 1:** Candidate intermediate vertex {1}. Hence, the candidate paths to choose from are (depending on the graph, the following two may be true):
  i → j (or) i → 1 → j
- **Iteration 2:** Candidate intermediate vertices {1, 2}. Hence, the candidate paths to choose from are (depending on the graph; the following in an exhaustive list for a complete graph in case of a brute force approach):
  i –> j (or) i → 1 → j (or) i → 2 → j (or) i → 1 → 2 → j (or) i → 2 → 1 → j
- **Iteration 3:** Candidate intermediate vertices {1, 2, 3}. Hence, the candidate paths to choose from are (depending on the graph; the following in an exhaustive list for a complete graph in case of a brute force approach):
  i –> j (or) i → 1 → j (or) i → 2 → j (or) i → 3 → j (or) i → 1 → 2 → j (or) i → 2 → 1 → j (or) i → 1 → 3 → j (or) i → 3 → 1 → j (or) i → 2 → 3 → j (or) i → 3 → 2 → j (or) i → 1 → 2 → 3 → j (or) i → 3 → 2 → 1 → j (or) i → 1 → 3 → 2 → j (or) i → 3 → 1 → 2 → j (or) i → 2 → 3 → 1 → j (or) i → 2 → 1 → 3 → j

FW Algorithm: Operating Principle

the minimum weight path from i to k involving zero or more intermediate vertices from the set {1, 2, …, k-1}



the minimum weight path from i to j involving zero or more intermediate vertices from the set {1, 2, …, k-1}

$D^{(k-1)}[i,j]$

the minimum weight path from k to j involving zero or more intermediate vertices from the set {1, 2, …, k-1}

$D^{(k-1)}[k,j]$

$$D^{(0)}[i,j] = w_{ij} \quad \text{if } i \to j \in E$$
$$D^{(0)}[i,j] = \infty \quad \text{if } i \to j \notin E$$

$$D^{(k)}[i,j] = \min\{ D^{(k-1)}[i,j],\ D^{(k-1)}[i,k] + D^{(k-1)}[k,j] \}$$

$$\pi_{ij}^{(0)} = i \quad \text{if } i \to j \in E$$
$$\pi_{ij}^{(0)} = N/A \quad \text{if } i \to j \notin E$$

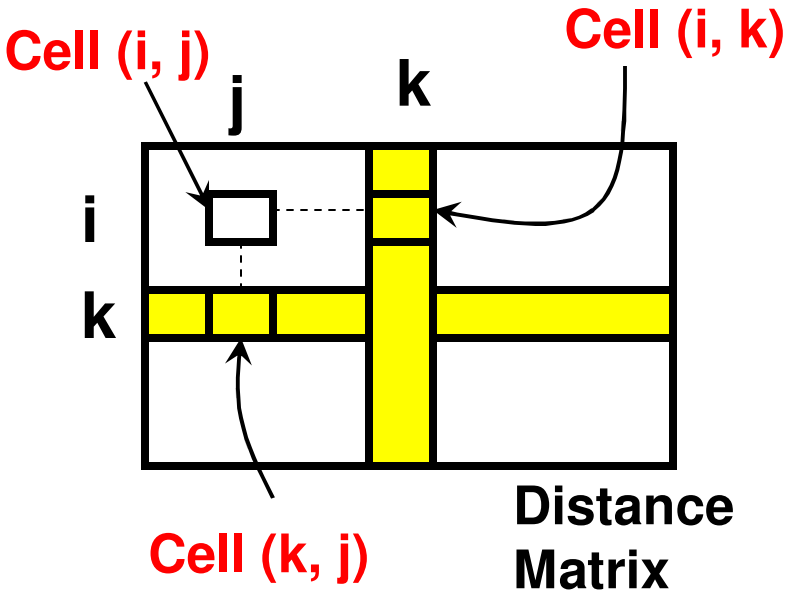
$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} & \text{if } D_{ij}^{(k-1)} \le D_{ik}^{(k-1)} + D_{kj}^{(k-1)} \\ \pi_{kj}^{(k-1)} & \text{if } D_{ij}^{(k-1)} > D_{ik}^{(k-1)} + D_{kj}^{(k-1)} \end{cases}$$

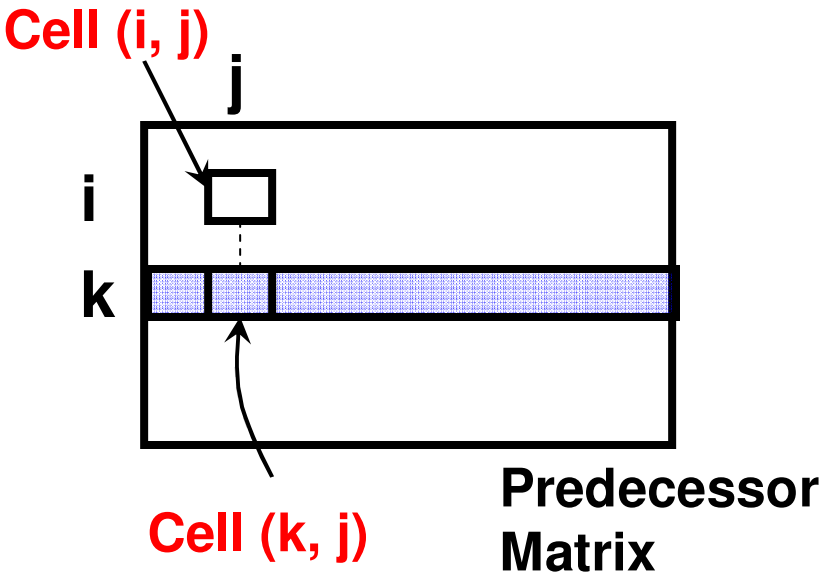
FW Algorithm: Working Principle

- In iteration k, we highlight the row and column corresponding to vertex k, and check whether the values for each of the other cells could be reduced from what they were prior to that iteration. We do not change the values for the cells in the row and column corresponding to vertex k.

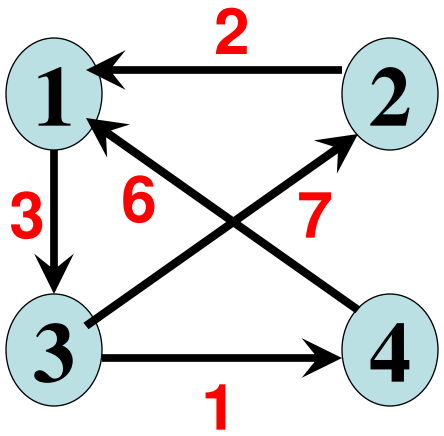
We update a cell (i, j) if the value in the cell is greater than the sum of the Values of the cells (i, k) and (k, j)

If we update cell (i, j), we also update the predecessor for (i, j) to be the value corresponding to the predecessor for (k, j) in row k.





FW Algorithm: Example 1 (1)



$D^{(0)}$

	v1	v2	v3	v4
v1	0	∞	3	∞
v2	2	0	∞	∞
v3	∞	7	0	1
v4	6	∞	∞	0

$\Pi^{(0)}$

	v1	v2	v3	v4
v1	N/A	N/A	v1	N/A
v2	v2	N/A	N/A	N/A
v3	N/A	v3	N/A	v3
v4	v4	N/A	N/A	N/A

Iteration 1

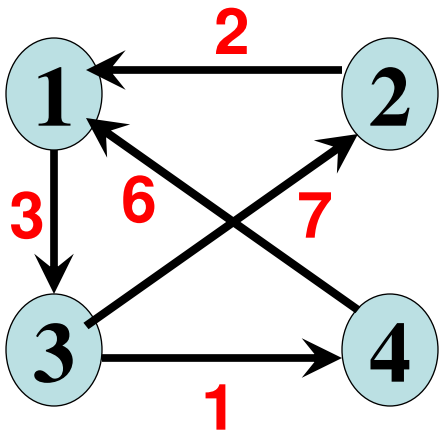
$D^{(1)}$

	v1	v2	v3	v4
v1	0	∞	3	∞
v2	2			
v3	∞			
v4	6			

$\Pi^{(1)}$

	v1	v2	v3	v4
v1	N/A	N/A	v1	N/A
v2	v2			
v3	N/A			
v4	v4			

FW Algorithm: Example 1 (1)



Iteration 1

$D^{(0)}$

	v1	v2	v3	v4
v1	0	∞	3	∞
v2	2	0	∞	∞
v3	∞	7	0	1
v4	6	∞	∞	0

$\Pi^{(0)}$

	v1	v2	v3	v4
v1	N/A	N/A	v1	N/A
v2	v2	N/A	N/A	N/A
v3	N/A	v3	N/A	v3
v4	v4	N/A	N/A	N/A

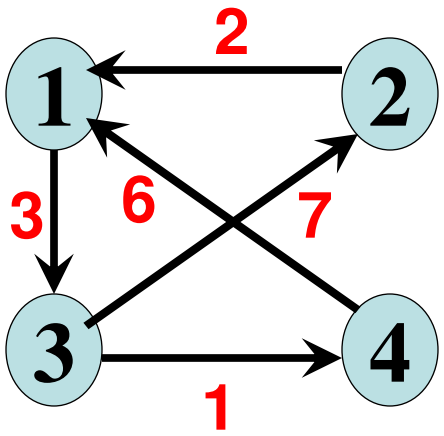
$D^{(1)}$

	v1	v2	v3	v4
v1	0	∞	3	∞
v2	2	0	5	∞
v3	∞	7	0	1
v4	6	∞	9	0

$\Pi^{(1)}$

	v1	v2	v3	v4
v1	N/A	N/A	v1	N/A
v2	v2	N/A	v1	N/A
v3	N/A	v3	N/A	v3
v4	v4	N/A	v1	N/A

FW Algorithm: Example 1 (2)



$D^{(1)}$

	v1	v2	v3	v4
v1	0	∞	3	∞
v2	2	0	5	∞
v3	∞	7	0	1
v4	6	∞	9	0

$\Pi^{(1)}$

	v1	v2	v3	v4
v1	N/A	N/A	v1	N/A
v2	v2	N/A	v1	N/A
v3	N/A	v3	N/A	v3
v4	v4	N/A	v1	N/A

Iteration 2

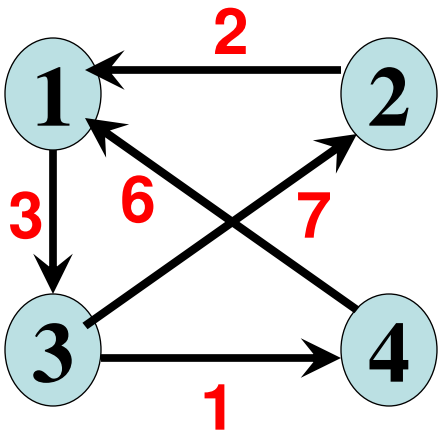
$D^{(2)}$

	v1	v2	v3	v4
v1		∞		
v2	2	0	5	∞
v3		7		
v4		∞		

$\Pi^{(2)}$

	v1	v2	v3	v4
v1		N/A		
v2	v2	N/A	v1	N/A
v3		v3		
v4		N/A		

FW Algorithm: Example 1 (2)



$D^{(1)}$

	v1	v2	v3	v4
v1	0	∞	3	∞
v2	2	0	5	∞
v3	∞	7	0	1
v4	6	∞	9	0

$\Pi^{(1)}$

	v1	v2	v3	v4
v1	N/A	N/A	v1	N/A
v2	v2	N/A	v1	N/A
v3	N/A	v3	N/A	v3
v4	v4	N/A	v1	N/A

Iteration 2

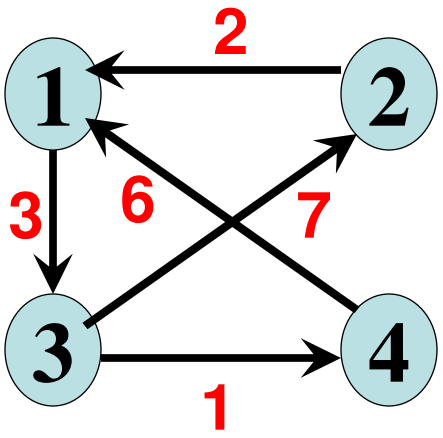
$D^{(2)}$

	v1	v2	v3	v4
v1	0	∞	3	∞
v2	2	0	5	∞
v3	9	7	0	1
v4	6	∞	9	0

$\Pi^{(2)}$

	v1	v2	v3	v4
v1	N/A	N/A	v1	N/A
v2	v2	N/A	v1	N/A
v3	v2	v3	N/A	v3
v4	v4	N/A	v1	N/A

FW Algorithm: Example 1 (3)



Iteration 3

$D^{(2)}$

	v1	v2	v3	v4
v1	0	∞	3	∞
v2	2	0	5	∞
v3	9	7	0	1
v4	6	∞	9	0

$\Pi^{(2)}$

	v1	v2	v3	v4
v1	N/A	N/A	v1	N/A
v2	v2	N/A	v1	N/A
v3	v2	v3	N/A	v3
v4	v4	N/A	v1	N/A

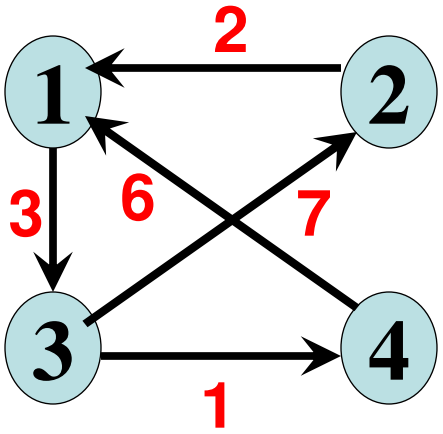
$D^{(3)}$

	v1	v2	v3	v4
v1			3	
v2			5	
v3	9	7	0	1
v4			9	

$\Pi^{(3)}$

	v1	v2	v3	v4
v1			v1	
v2			v1	
v3	v2	v3	N/A	v3
v4			v1	

FW Algorithm: Example 1 (3)



Iteration 3

$D^{(2)}$

	v1	v2	v3	v4
v1	0	∞	3	∞
v2	2	0	5	∞
v3	9	7	0	1
v4	6	∞	9	0

$\Pi^{(2)}$

	v1	v2	v3	v4
v1	N/A	N/A	v1	N/A
v2	v2	N/A	v1	N/A
v3	v2	v3	N/A	v3
v4	v4	N/A	v1	N/A

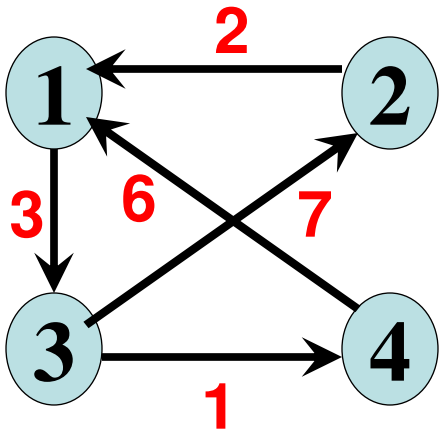
$D^{(3)}$

	v1	v2	v3	v4
v1	0	10	3	4
v2	2	0	5	6
v3	9	7	0	1
v4	6	16	9	0

$\Pi^{(3)}$

	v1	v2	v3	v4
v1	N/A	v3	v1	v3
v2	v2	N/A	v1	v3
v3	v2	v3	N/A	v3
v4	v4	v3	v1	N/A

FW Algorithm: Example 1 (4)



Iteration 4

$D^{(3)}$

	v1	v2	v3	v4
v1	0	10	3	4
v2	2	0	5	6
v3	9	7	0	1
v4	6	16	9	0

$\Pi^{(3)}$

	v1	v2	v3	v4
v1	N/A	v3	v1	v3
v2	v2	N/A	v1	v3
v3	v2	v3	N/A	v3
v4	v4	v3	v1	N/A

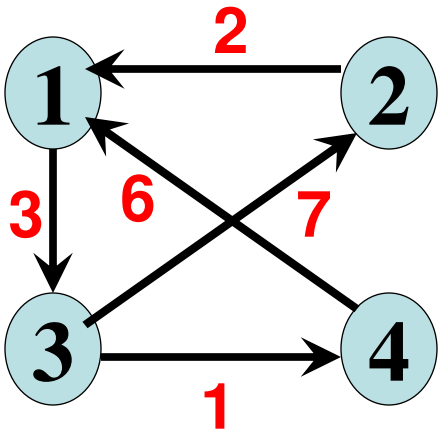
$D^{(4)}$

	v1	v2	v3	v4
v1				4
v2				6
v3				1
v4	6	16	9	0

$\Pi^{(4)}$

	v1	v2	v3	v4
v1				v3
v2				v3
v3				v3
v4	v4	v3	v1	N/A

FW Algorithm: Example 1 (4)



Iteration 4

$D^{(3)}$

	v1	v2	v3	v4
v1	0	10	3	4
v2	2	0	5	6
v3	9	7	0	1
v4	6	16	9	0

$\Pi^{(3)}$

	v1	v2	v3	v4
v1	N/A	v3	v1	v3
v2	v2	N/A	v1	v3
v3	v2	v3	N/A	v3
v4	v4	v3	v1	N/A

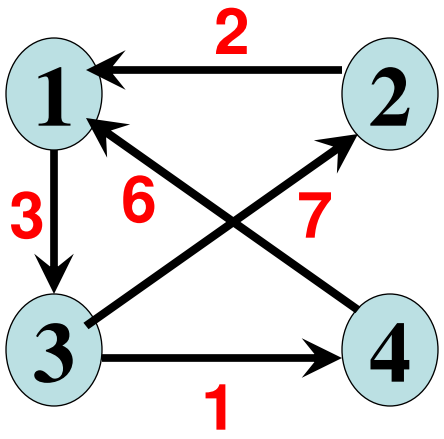
$D^{(4)}$

	v1	v2	v3	v4
v1	0	10	3	4
v2	2	0	5	6
v3	7	7	0	1
v4	6	16	9	0

$\Pi^{(4)}$

	v1	v2	v3	v4
v1	N/A	v3	v1	v3
v2	v2	N/A	v1	v3
v3	v4	v3	N/A	v3
v4	v4	v3	v1	N/A

FW Algorithm: Example 1 (5)



$D^{(4)}$

	v1	v2	v3	v4
v1	0	10	3	4
v2	2	0	5	6
v3	7	7	0	1
v4	6	16	9	0

$\Pi^{(4)}$

	v1	v2	v3	v4
v1	N/A	v3	v1	v3
v2	v2	N/A	v1	v3
v3	v4	v3	N/A	v3
v4	v4	v3	v1	N/A

Path from v2 to v4
π (v2 … v4)
= π (v2 … v3) → v3 → v4
= π (v2 … v1) → v1 → v3 → v4
= v2 → v1 → v3 → v4

Path from v4 to v2
π (v4 … v2)
= π (v4 … v3) → v3 → v2
= π (v4 … v1) → v1 → v3 → v2
= v4 → v1 → v3 → v2

FW Algorithm (pseudocode and analysis)

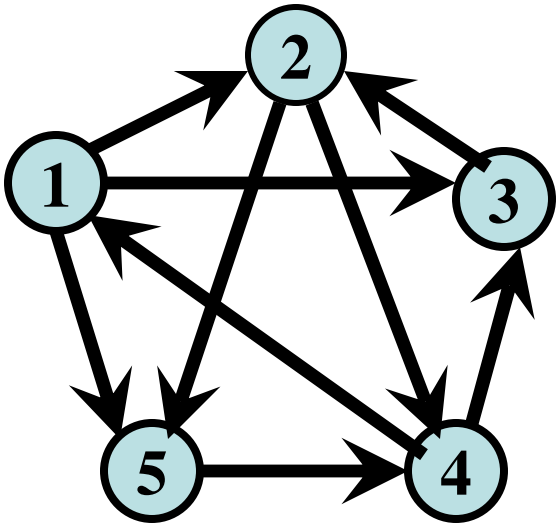
ALGORITHM *Floyd*$(W[1..n, 1..n])$

```
//Implements Floyd's algorithm for the all-pairs shortest-paths problem
//Input: The weight matrix W of a graph with no negative-length cycle
//Output: The distance matrix of the shortest paths' lengths
D ← W //is not necessary if W can be overwritten
for k ← 1 to n do
    for i ← 1 to n do
        for j ← 1 to n do
            D[i, j] ← min{D[i, j], D[i, k] + D[k, j]}
return D
```

Time efficiency: $\Theta(n^3)$

Space efficiency: $\Theta(n^2)$

FW Algorithm: Example 2(1)



	Weight Matrix				
	v1	v2	v3	v4	v5
v1	0	3	8	∞	-4
v2	∞	0	∞	1	7
v3	∞	4	0	∞	∞
v4	2	∞	-5	0	∞
v5	∞	∞	∞	6	0

$D^{(0)}$

	v1	v2	v3	v4	v5
v1	0	3	8	∞	-4
v2	∞	0	∞	1	7
v3	∞	4	0	∞	∞
v4	2	∞	-5	0	∞
v5	∞	∞	∞	6	0

$\Pi^{(0)}$

	v1	v2	v3	v4	v5
v1	N/A	v1	v1	N/A	v1
v2	N/A	N/A	N/A	v2	v2
v3	N/A	v3	N/A	N/A	N/A
v4	v4	N/A	v4	N/A	N/A
v5	N/A	N/A	N/A	v5	N/A

$D^{(1)}$

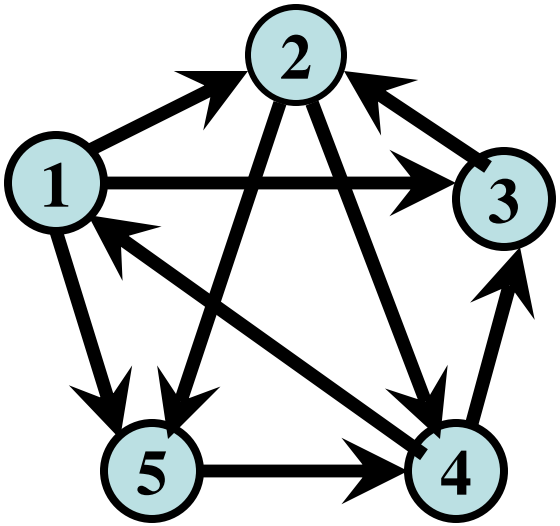
	v1	v2	v3	v4	v5
v1	0	3	8	∞	-4
v2	∞				
v3	∞				
v4	2				
v5	∞				

$\Pi^{(1)}$

	v1	v2	v3	v4	v5
v1	N/A	v1	v1	N/A	v1
v2	N/A				
v3	N/A				
v4	v4				
v5	N/A				

Iteration 1

FW Algorithm: Example 2(1)



	Weight Matrix				
	v1	v2	v3	v4	v5
v1	0	3	8	∞	-4
v2	∞	0	∞	1	7
v3	∞	4	0	∞	∞
v4	2	∞	-5	0	∞
v5	∞	∞	∞	6	0

$D^{(0)}$

	v1	v2	v3	v4	v5
v1	0	3	8	∞	-4
v2	∞	0	∞	1	7
v3	∞	4	0	∞	∞
v4	2	∞	-5	0	∞
v5	∞	∞	∞	6	0

$\Pi^{(0)}$

	v1	v2	v3	v4	v5
v1	N/A	v1	v1	N/A	v1
v2	N/A	N/A	N/A	v2	v2
v3	N/A	v3	N/A	N/A	N/A
v4	v4	N/A	v4	N/A	N/A
v5	N/A	N/A	N/A	v5	N/A

$D^{(1)}$

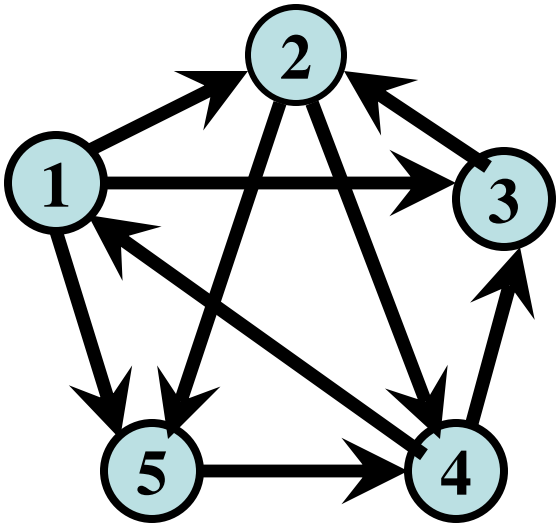
	v1	v2	v3	v4	v5
v1	0	3	8	∞	-4
v2	∞	0	∞	1	7
v3	∞	4	0	∞	∞
v4	2	5	-5	0	-2
v5	∞	∞	∞	6	0

$\Pi^{(1)}$

	v1	v2	v3	v4	v5
v1	N/A	v1	v1	N/A	v1
v2	N/A	N/A	N/A	v2	v2
v3	N/A	v3	N/A	N/A	N/A
v4	v4	v1	v4	N/A	v1
v5	N/A	N/A	N/A	v5	N/A

Iteration 1

FW Algorithm: Example 2(2)



	Weight Matrix				
	v1	v2	v3	v4	v5
v1	0	3	8	∞	-4
v2	∞	0	∞	1	7
v3	∞	4	0	∞	∞
v4	2	∞	-5	0	∞
v5	∞	∞	∞	6	0

$D^{(1)}$

	v1	v2	v3	v4	v5
v1	0	3	8	∞	-4
v2	∞	0	∞	1	7
v3	∞	4	0	∞	∞
v4	2	5	-5	0	-2
v5	∞	∞	∞	6	0

$\Pi^{(1)}$

	v1	v2	v3	v4	v5
v1	N/A	v1	v1	N/A	v1
v2	N/A	N/A	N/A	v2	v2
v3	N/A	v3	N/A	N/A	N/A
v4	v4	v1	v4	N/A	v1
v5	N/A	N/A	N/A	v5	N/A

$D^{(2)}$

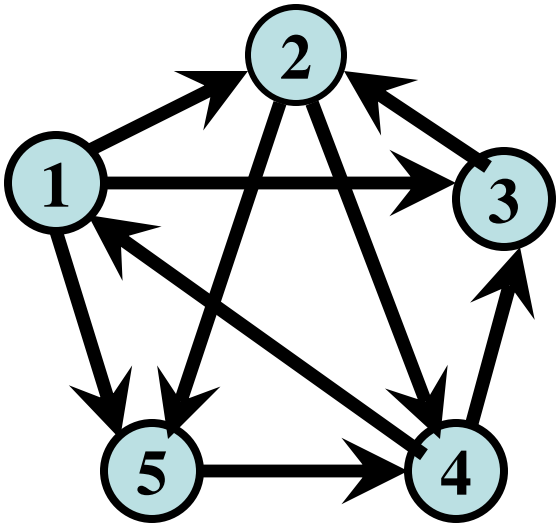
	v1	v2	v3	v4	v5
v1		3			
v2	∞	0	∞	1	7
v3		4			
v4		5			
v5		∞			

$\Pi^{(2)}$

	v1	v2	v3	v4	v5
v1		v1			
v2	N/A	N/A	N/A	v2	v2
v3		v3			
v4		v1			
v5		N/A			

Iteration 2

FW Algorithm: Example 2(2)



	Weight Matrix				
	v1	v2	v3	v4	v5
v1	0	3	8	∞	-4
v2	∞	0	∞	1	7
v3	∞	4	0	∞	∞
v4	2	∞	-5	0	∞
v5	∞	∞	∞	6	0

$D^{(1)}$

	v1	v2	v3	v4	v5
v1	0	3	8	∞	-4
v2	∞	0	∞	1	7
v3	∞	4	0	∞	∞
v4	2	5	-5	0	-2
v5	∞	∞	∞	6	0

$\Pi^{(1)}$

	v1	v2	v3	v4	v5
v1	N/A	v1	v1	N/A	v1
v2	N/A	N/A	N/A	v2	v2
v3	N/A	v3	N/A	N/A	N/A
v4	v4	v1	v4	N/A	v1
v5	N/A	N/A	N/A	v5	N/A

$D^{(2)}$

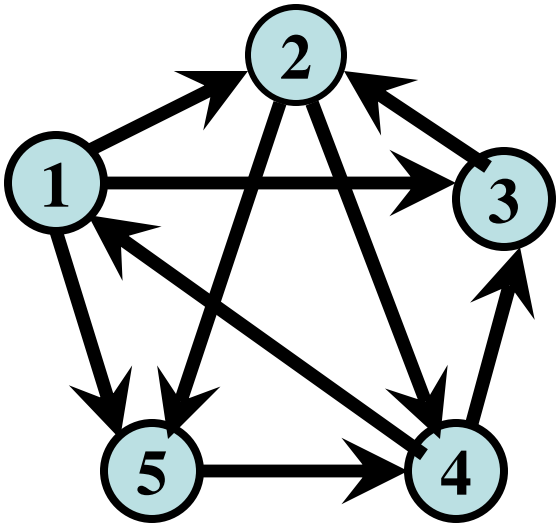
	v1	v2	v3	v4	v5
v1	0	3	8	4	-4
v2	∞	0	∞	1	7
v3	∞	4	0	5	11
v4	2	5	-5	0	-2
v5	∞	∞	∞	6	0

$\Pi^{(2)}$

	v1	v2	v3	v4	v5
v1	N/A	v1	v1	v2	v1
v2	N/A	N/A	N/A	v2	v2
v3	N/A	v3	N/A	v2	v2
v4	v4	v1	v4	N/A	v1
v5	N/A	N/A	N/A	v5	N/A

Iteration 2

FW Algorithm: Example 2(3)



	Weight Matrix				
	v1	v2	v3	v4	v5
v1	0	3	8	∞	-4
v2	∞	0	∞	1	7
v3	∞	4	0	∞	∞
v4	2	∞	-5	0	∞
v5	∞	∞	∞	6	0

$D^{(2)}$

	v1	v2	v3	v4	v5
v1	0	3	8	4	-4
v2	∞	0	∞	1	7
v3	∞	4	0	5	11
v4	2	5	-5	0	-2
v5	∞	∞	∞	6	0

$\Pi^{(2)}$

	v1	v2	v3	v4	v5
v1	N/A	v1	v1	v2	v1
v2	N/A	N/A	N/A	v2	v2
v3	N/A	v3	N/A	v2	v2
v4	v4	v1	v4	N/A	v1
v5	N/A	N/A	N/A	v5	N/A

$D^{(3)}$

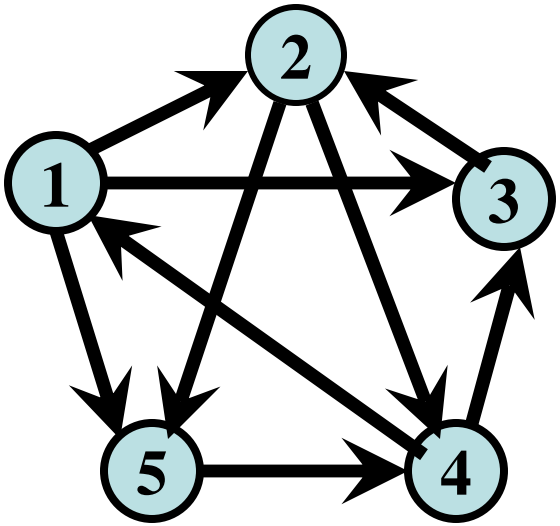
	v1	v2	v3	v4	v5
v1	0	3	8	4	-4
v2	∞	0	∞	1	7
v3	∞	4	0	5	11
v4	2	-1	-5	0	-2
v5	∞	∞	∞	6	0

$\Pi^{(3)}$

	v1	v2	v3	v4	v5
v1	N/A	v1	v1	v2	v1
v2	N/A	N/A	N/A	v2	v2
v3	N/A	v3	N/A	v2	v2
v4	v4	v3	v4	N/A	v1
v5	N/A	N/A	N/A	v5	N/A

Iteration 3

FW Algorithm: Example 2(4)



	Weight Matrix				
	v1	v2	v3	v4	v5
v1	0	3	8	∞	-4
v2	∞	0	∞	1	7
v3	∞	4	0	∞	∞
v4	2	∞	-5	0	∞
v5	∞	∞	∞	6	0

$D^{(3)}$

	v1	v2	v3	v4	v5
v1	0	3	8	4	-4
v2	∞	0	∞	1	7
v3	∞	4	0	5	11
v4	2	-1	-5	0	-2
v5	∞	∞	∞	6	0

$\Pi^{(3)}$

	v1	v2	v3	v4	v5
v1	N/A	v1	v1	v2	v1
v2	N/A	N/A	N/A	v2	v2
v3	N/A	v3	N/A	v2	v2
v4	v4	v3	v4	N/A	v1
v5	N/A	N/A	N/A	v5	N/A

$D^{(4)}$

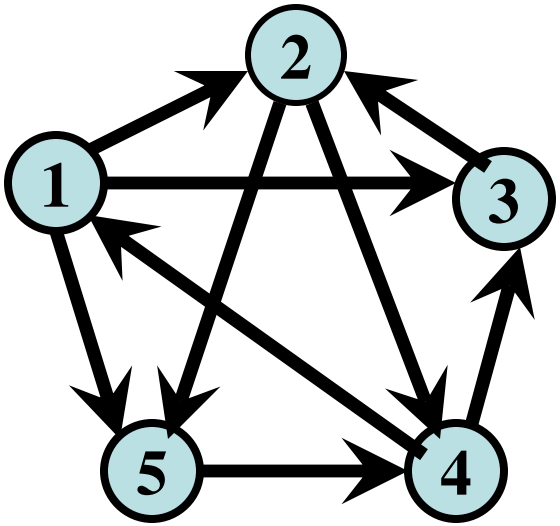
	v1	v2	v3	v4	v5
v1	0	3	-1	4	-4
v2	3	0	-4	1	-1
v3	7	4	0	5	3
v4	2	-1	-5	0	-2
v5	8	5	1	6	0

$\Pi^{(4)}$

	v1	v2	v3	v4	v5
v1	N/A	v1	v4	v2	v1
v2	v4	N/A	v4	v2	v1
v3	v4	v3	N/A	v2	v1
v4	v4	v3	v4	N/A	v1
v5	v4	v3	v4	v5	N/A

Iteration 4

FW Algorithm: Example 2(5)



	Weight Matrix				
	v1	v2	v3	v4	v5
v1	0	3	8	∞	-4
v2	∞	0	∞	1	7
v3	∞	4	0	∞	∞
v4	2	∞	-5	0	∞
v5	∞	∞	∞	6	0

$D^{(4)}$

	v1	v2	v3	v4	v5
v1	0	3	-1	4	-4
v2	3	0	-4	1	-1
v3	7	4	0	5	3
v4	2	-1	-5	0	-2
v5	8	5	1	6	0

$\Pi^{(4)}$

	v1	v2	v3	v4	v5
v1	N/A	v1	v4	v2	v1
v2	v4	N/A	v4	v2	v1
v3	v4	v3	N/A	v2	v1
v4	v4	v3	v4	N/A	v1
v5	v4	v3	v4	v5	N/A

$D^{(5)}$

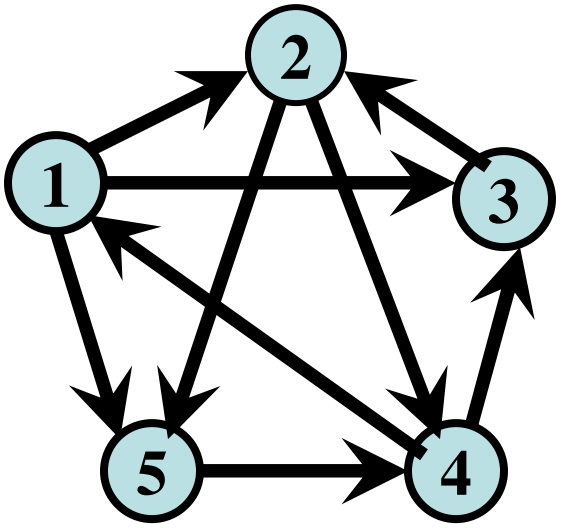
	v1	v2	v3	v4	v5
v1	0	1	-3	2	-4
v2	3	0	-4	1	-1
v3	7	4	0	5	3
v4	2	-1	-5	0	-2
v5	8	5	1	6	0

$\Pi^{(5)}$

	v1	v2	v3	v4	v5
v1	N/A	v3	v4	v5	v1
v2	v4	N/A	v4	v2	v1
v3	v4	v3	N/A	v2	v1
v4	v4	v3	v4	N/A	v1
v5	v4	v3	v4	v5	N/A

Iteration 5

FW Algorithm: Example 2(6)



$D^{(5)}$

	v1	v2	v3	v4	v5
v1	0	1	-3	2	-4
v2	3	0	-4	1	-1
v3	7	4	0	5	3
v4	2	-1	-5	0	-2
v5	8	5	1	6	0

$\Pi^{(5)}$

	v1	v2	v3	v4	v5
v1	N/A	v3	v4	v5	v1
v2	v4	N/A	v4	v2	v1
v3	v4	v3	N/A	v2	v1
v4	v4	v3	v4	N/A	v1
v5	v4	v3	v4	v5	N/A

Path from v3 to v1
π (v3 … v1)
= π (v3 … v4) → v4 → v1
= π (v3 … v2) → v2 → v4 → v1
= v3 → v2 → v4 → v1

Path from v1 to v3
π (v1 … v3)
= π (v1 … v4) → v4 → v3
= π (v1 … v5) → v5 → v4 → v3
= v1 → v5 → v4 → v3

Comparison of the Shortest Path Algorithms

	Dijkstra	**Bellman-Ford**	**Floyd-Warshall**
Type	Single source shortest path	Single source shortest path	All pairs shortest path
Typical Graphs	Undirected	Directed	Undirected and Directed
Edge Weights	Positive only	Positive and/ or Negative	Positive and/ or Negative
Time Complexity	$\Theta(E^*\log V)$	$\Theta(E^*V)$	$\Theta(V^3)$

Breadth First Search (BFS)

- BFS is a graph traversal algorithm (like DFS); but, BFS proceeds in a concentric breadth-wise manner (not depth wise) by first visiting all the vertices that are adjacent to a starting vertex, then all unvisited vertices that are two edges apart from it, and so on.
  - The above traversal strategy of BFS makes it ideal for determining minimum-edge (i.e., minimum-hop paths) on graphs.
- If the underling graph is connected, then all the vertices of the graph should have been visited when BFS is started from a randomly chosen vertex.
  - If there still remains unvisited vertices, the graph is not connected and the algorithm has to restarted on an arbitrary vertex of another connected component of the graph.
- BFS is typically implemented using a FIFO-queue (not a LIFO-stack like that of DFS).
  - The queue is initialized with the traversal's starting vertex, which is marked as visited. On each iteration, BFS identifies all unvisited vertices that are adjacent to the front vertex, marks them as visited, and adds them to the queue; after that, the front vertex is removed from the queue.
- When a vertex is visited for the first time, the corresponding edge that facilitated this visit is called the *tree edge*. When a vertex that is already visited is re-visited through a different edge, the corresponding edge is called a *cross edge*.

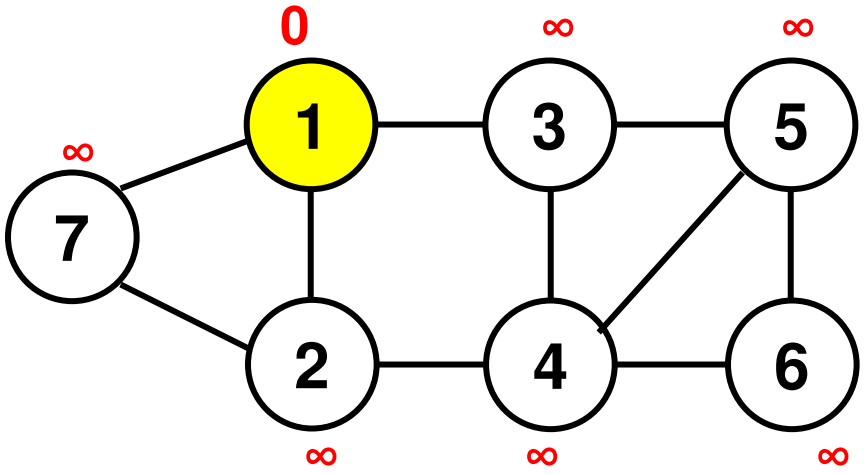
Breadth First Search (BFS)

```
BFS(G, s)
    Queue queue
    queue.enqueue(s)        // 's' is the starting vertex
    Level[s] = 0
    Level[v] = ∞;  for all vertices v other than 's'
                    // The level # is also the estimated number of edges
                    // on the minimum edge path (shortest path) from 's'
    Visited[v] = false; for all vertices v other than 's'
    while (!queue.isEmpty()) do
      u = queue.dequeue();
      for every vertex v that is a neighbor of u
          if (Visited[v] = false) then
                    Level[v] = Level[u] + 1
                    Visited[v] = true
                    Queue.enqueue(v)
                    Edge u-v is a tree edge
          end if
          else
                    Edge u-v is a cross edge
       end for
    end while
End BFS
```

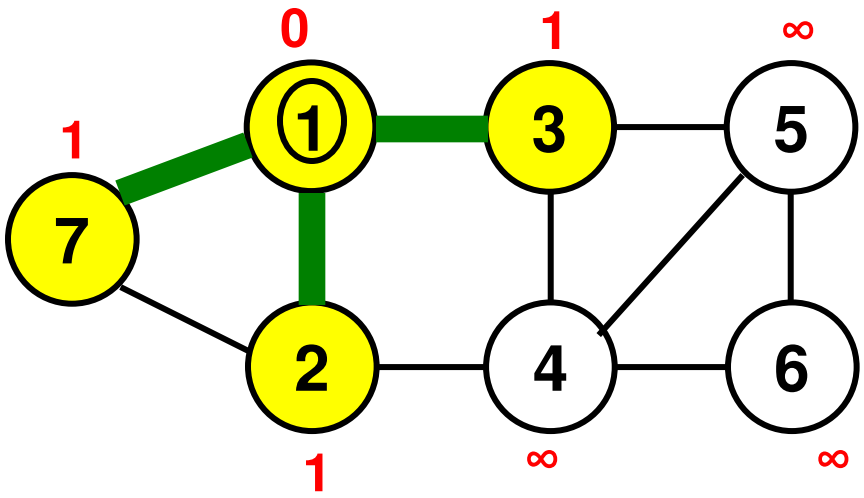
Time Complexity:

If there are 'V' vertices and 'E' edges, we traverse each edge exactly once as well as enqueue and dequeue each vertex exactly once. Hence, the time complexity of BFS is Θ(V+E) when implemented using an Adjacency list and $\Theta(V^2)$ when implemented using an Adjacency matrix.

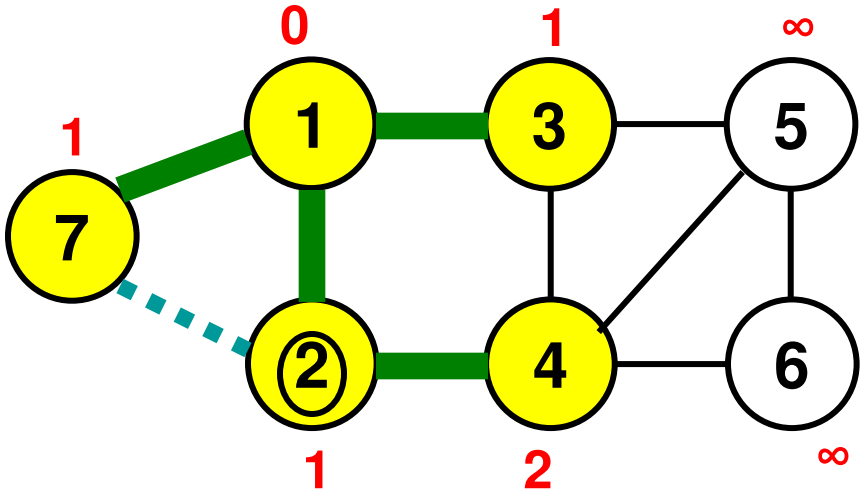
BFS: Example 1



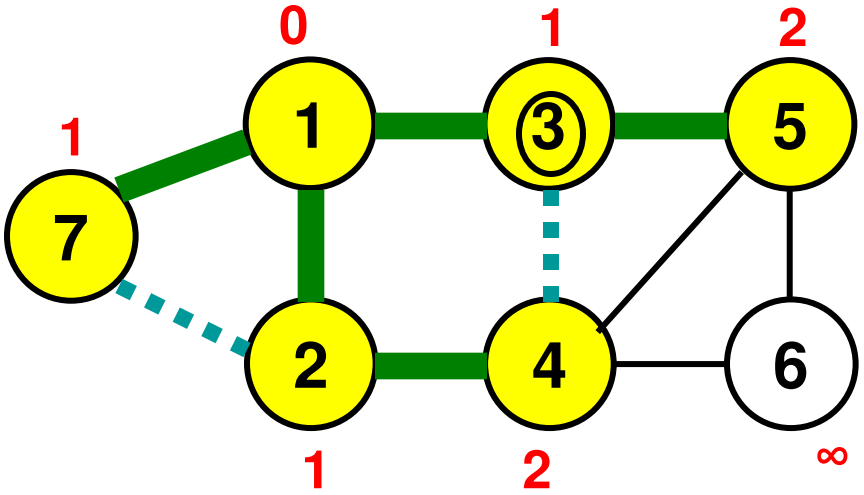
Initialization: Queue = {1}



Iteration 1: Queue = {2, 3, 7}

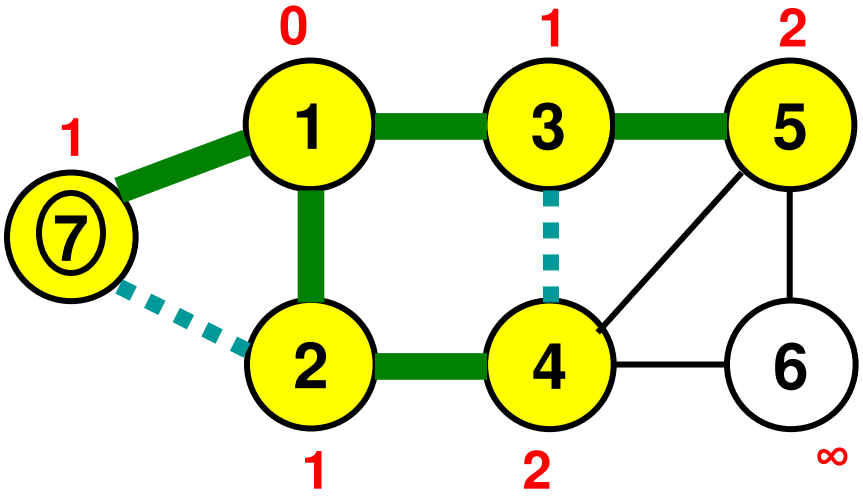


Iteration 2: Queue = {3, 7, 4}

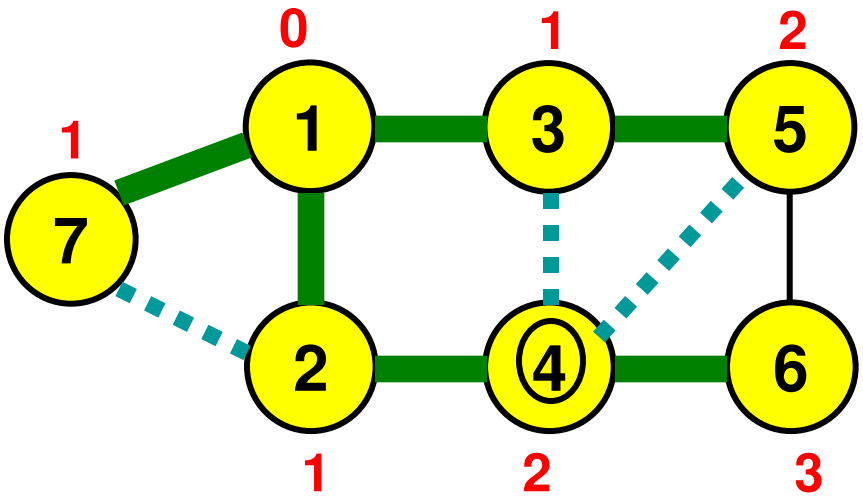


Iteration 3: Queue = {7, 4, 5}

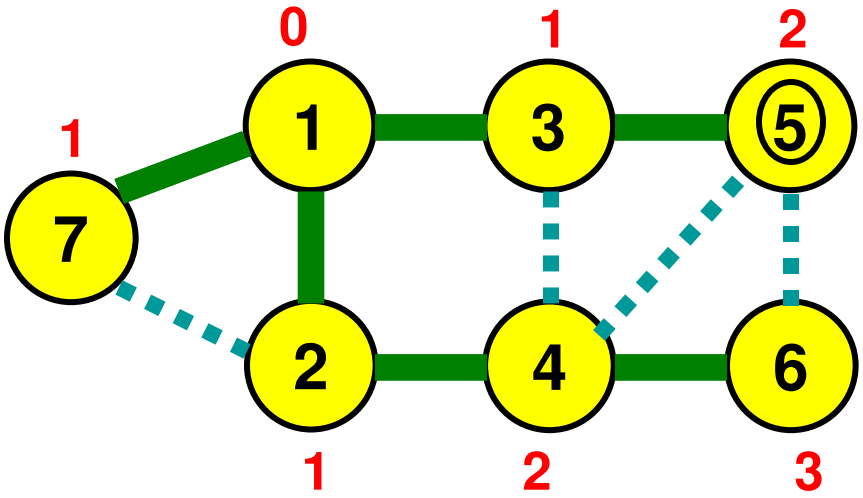
BFS: Example 1



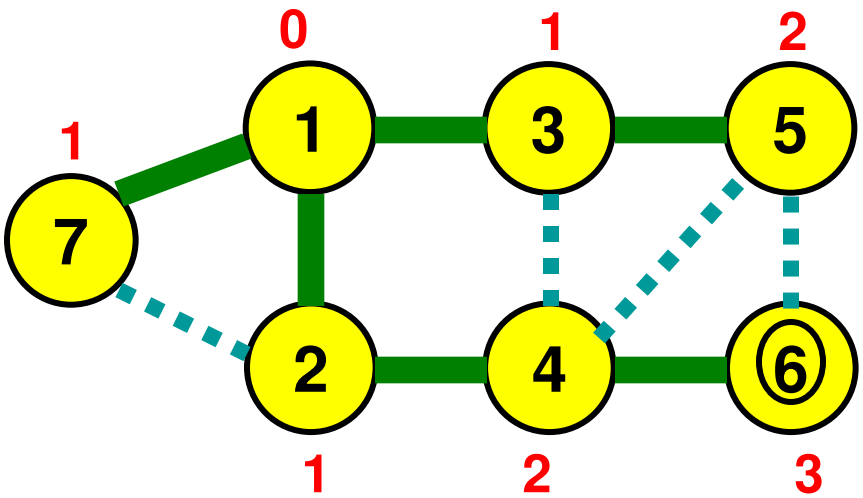
Iteration 4: Queue = {4, 5}



Iteration 5: Queue = {5, 6}



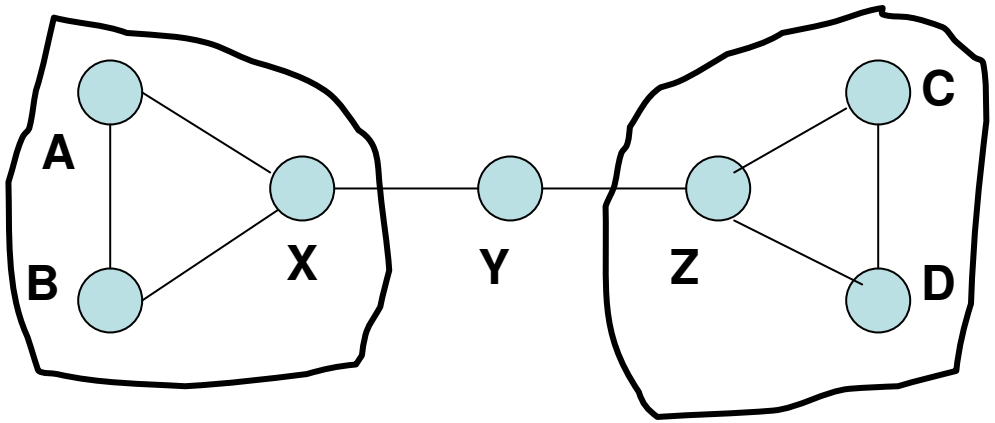
Iteration 6: Queue = {6}



Iteration 7: Queue = {}

Centrality Metrics

- Centrality metrics quantify the importance of a vertex based on its position (topological importance) in the graph
- Among the various centrality metrics that exist, we will look at one of the most important metrics called the Betweenness Centrality (BWC) of a vertex.
- The BWC of a vertex is a measure of the presence of the vertex (as an intermediate vertex) on the shortest paths between any two pairs of vertices



In the graph shown here, there exists only one shortest path between any two vertices.
Vertex Y is an intermediate vertex on all the '9' shortest paths between any vertex in the set {A, B, X} and any vertex in the set {Z, C, D}.
Vertices A, B, C and D do not lie on the shortest path for any two vertices.

Betweeness Centrality

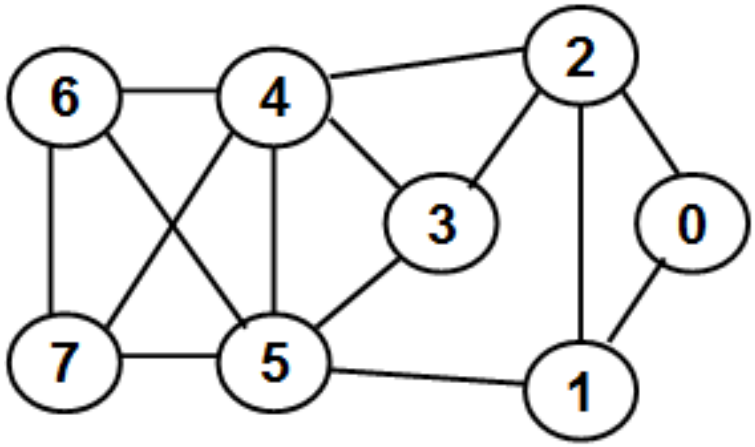
$$BWC(i) = \sum_{j \neq k \neq i} \frac{sp_{jk}(i)}{sp_{jk}}$$

(j < k for undirected graphs)

Time Complexity: Θ(VE + V^2)

- We will now discuss how to find the total number of shortest paths between any two vertices *j* and *k* as well as to find out how many of these shortest paths go through a vertex *i* ($j \neq k \neq i$).
- Use Breadth First Search (BFS) to find the shortest path tree from vertex j to every other vertex k
  - Root vertex j is at level 0
  - Vertices that are 1-hop away from j are at level 1; 2-hops away from j are at level 2, and so on.
  - The number of shortest paths from *j* to a vertex *k* at level *p* is the sum of the number of shortest paths from *j* to the neighbors of *k* in the original graph that are at level *p*-1
  - The number of shortest paths from *j* to *k* that go through vertex *i* is the maximum of the number of shortest paths from *j* to *i* and the number of shortest paths from *k* to *i*.

Betweenness Centrality Example



BWC for node 0: 0.0

BWC for node 1
Pair (0, 5): → 1 / 1
Pair (0, 6): → 1 / 2
Pair (0, 7): → 1 / 2
Pair (2, 5): → 1 / 3
BWC (1) = 2.333

BWC for node 2
Pair (0, 3): → 1 / 1
Pair (0, 4): → 1 / 1
Pair (0, 6): → 1 / 2
Pair (0, 7): → 1 / 2
Pair (1, 3): → 1 / 2
Pair (1, 4): → 1 / 2
BWC (2): 4.0

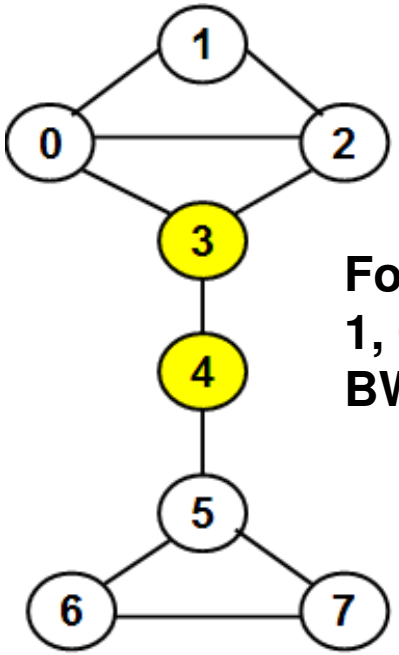
BWC for node 3
Pair (2, 5) → 1 / 3
BWC (3) = 0.333

BWC for node 4
Pair (0, 6) → 1 / 2
Pair (0, 7) → 1 / 2
Pair (2, 5) → 1 / 3
Pair (2, 6) → 1 / 1
Pair (2, 7) → 1 / 1
Pair (3, 6) → 1 / 2
Pair (3, 7) → 1 / 2
BWC (2) = 4.333

BWC for node 6: 0.0
BWC for node 7: 0.0

BWC for node 5
Pair (0, 6) → 1 / 2
Pair (0, 7) → 1 / 2
Pair (1, 3) → 1 / 2
Pair (1, 4) → 1 / 2
Pair (1, 6) → 1 / 1
Pair (1, 7) → 1 / 1
Pair (3, 6) → 1 / 2
Pair (3, 7) → 1 / 2
BWC (5) = 5.0

ID	BWC
0	0.0
1	2.333
2	4.0
3	0.333
4	4.333
5	5.0
6	0.0
7	0.0



For vertices
1, 6 and 7
BWC = 0

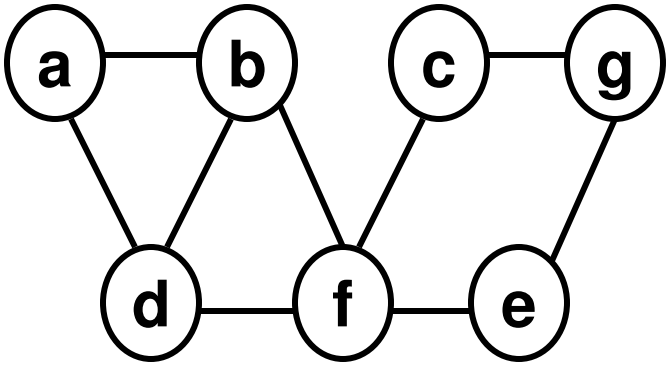
Betweeness for Vertex 0
Pair (3,1) --->1 / 2
Pair (4,1) --->1 / 2
Pair (5,1) --->1 / 2
Pair (6,1) --->1 / 2
Pair (7,1) --->1 / 2
Total of all Betweeness
(Vertex 0): 2.5

Betweeness for Vertex 2
Pair (3,1) --->1 / 2
Pair (4,1) --->1 / 2
Pair (5,1) --->1 / 2
Pair (6,1) --->1 / 2
Pair (7,1) --->1 / 2
Total of all Betweeness
(Vertex 2): 2.5

Betweeness for Vertex 5
Pair (6,0) --->1 / 1
Pair (7,0) --->1 / 1
Pair (6,1) --->2 / 2
Pair (7,1) --->2 / 2
Pair (6,2) --->1 / 1
Pair (7,2) --->1 / 1
Pair (6,3) --->1 / 1
Pair (7,3) --->1 / 1
Pair (6,4) --->1 / 1
Pair (7,4) --->1 / 1
Total of all Betweeness
(Vertex 5): 10

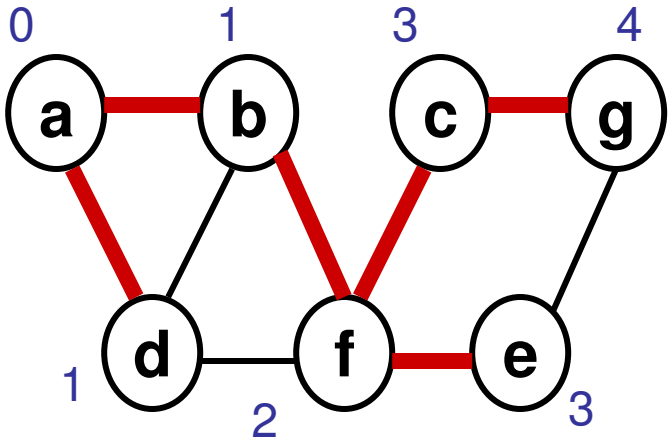
Betweeness for Vertex 3
Pair (4,0) --->1 / 1
Pair (5,0) --->1 / 1
Pair (6,0) --->1 / 1
Pair (7,0) --->1 / 1
Pair (4,1) --->2 / 2
Pair (5,1) --->2 / 2
Pair (6,1) --->2 / 2
Pair (7,1) --->2 / 2
Pair (4,2) --->1 / 1
Pair (5,2) --->1 / 1
Pair (6,2) --->1 / 1
Pair (7,2) --->1 / 1
Total of all Betweeness
(Vertex 3): 12

Betweeness for Vertex 4
Pair (5,0) --->1 / 1
Pair (6,0) --->1 / 1
Pair (7,0) --->1 / 1
Pair (5,1) --->2 / 2
Pair (6,1) --->2 / 2
Pair (7,1) --->2 / 2
Pair (5,2) --->1 / 1
Pair (6,2) --->1 / 1
Pair (7,2) --->1 / 1
Pair (5,3) --->1 / 1
Pair (6,3) --->1 / 1
Pair (7,3) --->1 / 1
Total of all Betweeness
(Vertex 4): 12

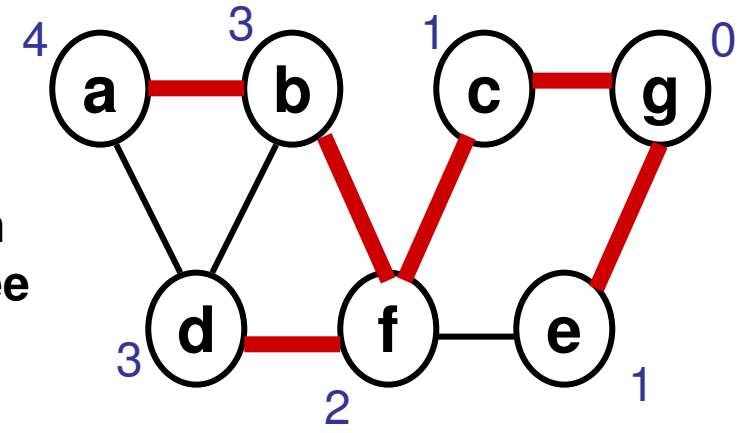


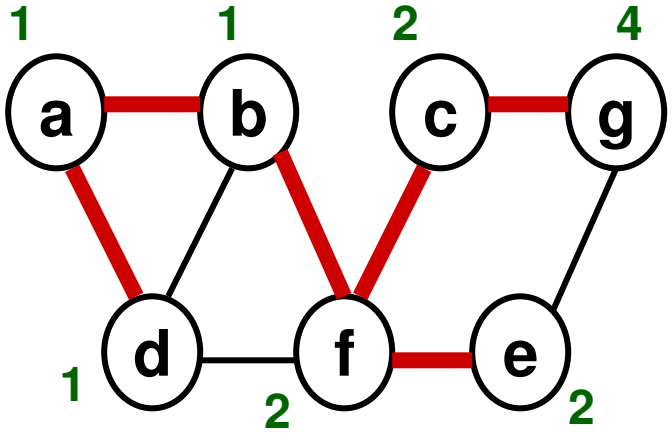
Important Note (Pre-requisite Check)

To find the BWC('x' w.r.t. the pair 'y, z'):
For a non-zero BWC fraction, the sum of the level numbers of 'x' on the BFS trees rooted at 'y' and 'z' should be the same as the level number of 'y' (or equivalently 'z') on the BFS tree rooted at 'z' (or equivalently 'y').

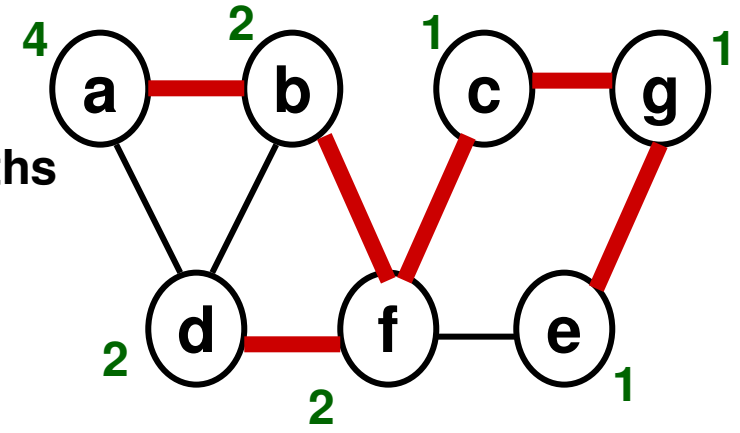


Levels of Vertices on the BFS tree





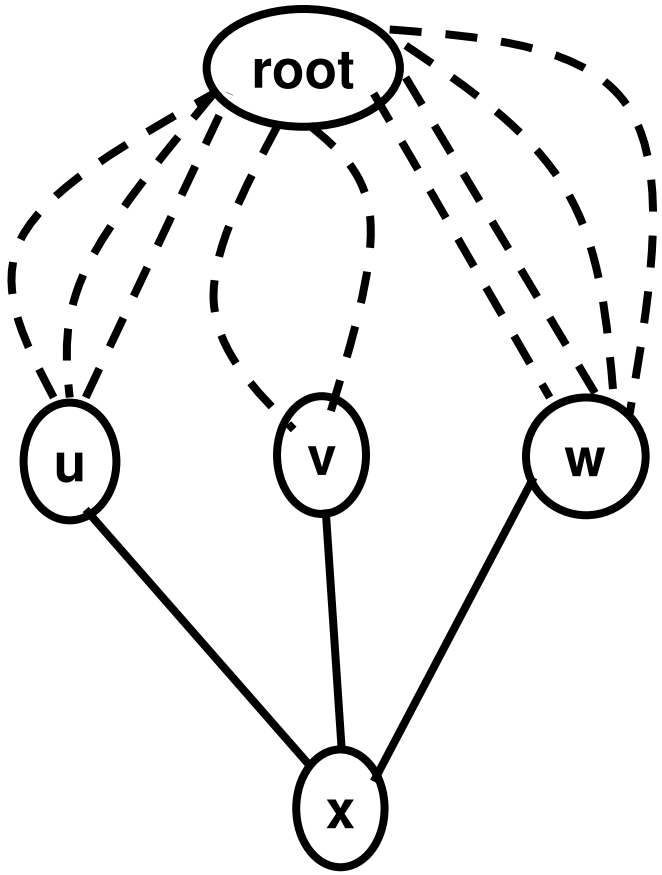
 # shortest paths from the root to the other vertices



Shortest Paths from the Root to a Vertex

- The number of shortest paths from the root of a Breadth First Search (BFS) tree to a vertex ‘x’ (at level ‘L’ in the BFS tree) is the sum of the number of shortest paths from the root to the predecessors of ‘x’ (that are at level ‘L-1’ in the BFS tree) to which ‘x’ has an edge in the graph.

- In the example shown on the right side, vertex ‘x’ (at level L) has an edge to vertices ‘u’, ‘v’ and ‘w’ (at level L-1 in the BFS tree) in the given graph. The # shortest paths from the root to ‘x’ is the sum of the # shortest paths from the root to each of ‘u’, ‘v’ and ‘w’.

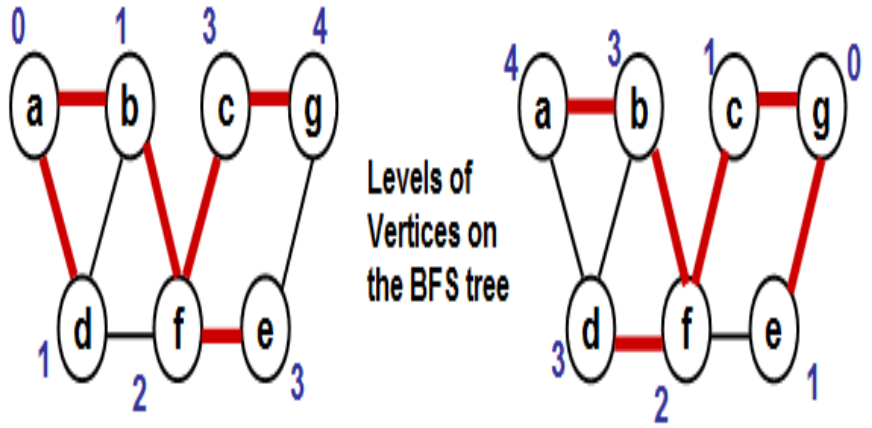


Shortest Paths from the root to ‘x’
= # Shortest Paths from the root to ‘u’ +
Shortest Paths from the root to ‘v’ +
Shortest Paths from the root to ‘w’
= 3 + 2 + 4 = 9.

To Find BWC(b: pair a, g)

Level (b; BFS tree of a) = 1
Level (b; BFS tree of g) = 3
Sum of the levels = 1 + 3 = 4
is not greater than the
Level (a; BFS tree of g) or
equivalently Level (g; BFS tree of a).

 # Shortest paths (b; BFS tree of a) = 1
 # Shortest paths (b; BFS tree of g) = 2
Total # Shortest paths from a to g = 4.

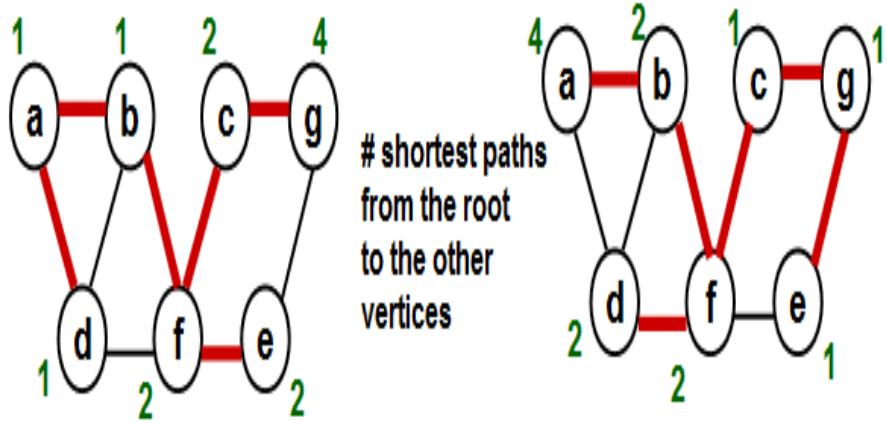


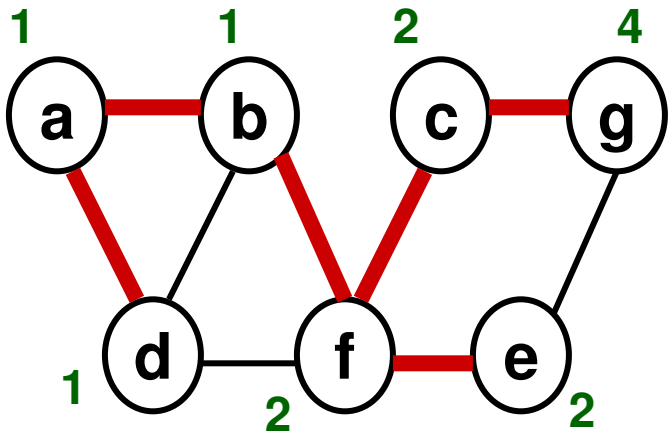
BWC(b; pair a, g)

 # Shortest paths (b; BFS tree of a) *
 # Shortest paths (b; BFS tree of g)

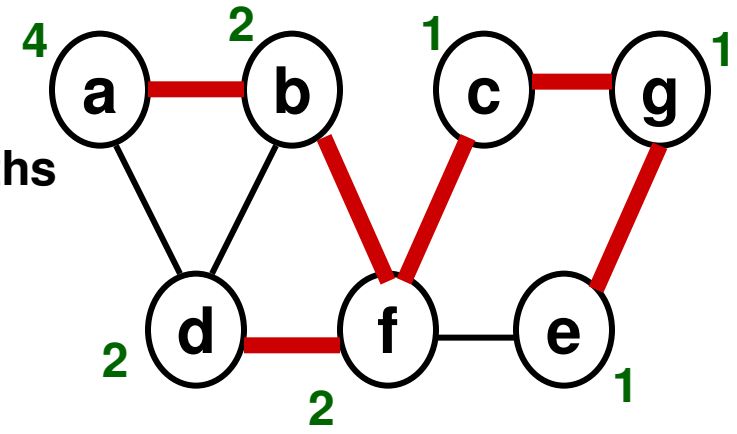
Total # Shortest paths from a to g

$$= \frac{(1 * 2)}{4} = \frac{2}{4} = \frac{1}{2}.$$





shortest paths from the root to the other vertices



To Find BWC(f: pair a, g)

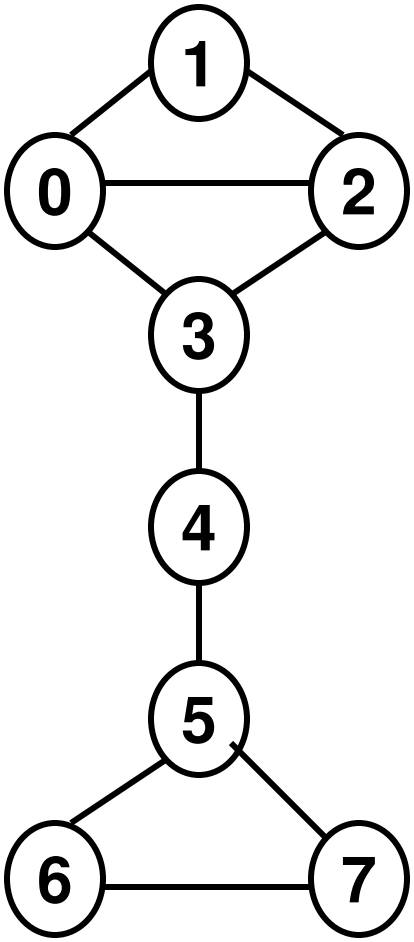
Level (f; BFS tree of a) = 2
Level (f; BFS tree of g) = 2
Sum of the levels = 2 + 2 = 4
is not greater than the
Level (a; BFS tree of g) or
equivalently Level (g; BFS tree of a).

Shortest paths (f; BFS tree of a) = 2
Shortest paths (f; BFS tree of g) = 2
Total # Shortest paths from a to g = 4.

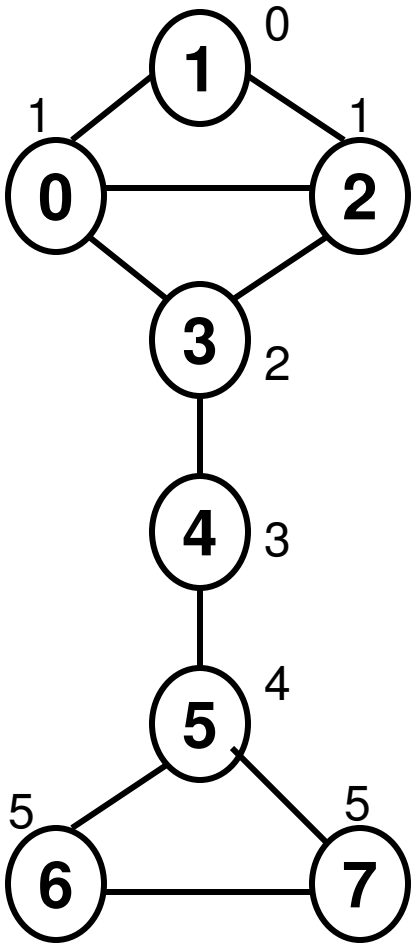
BWC(f; pair a, g) = (2*2)/4 = 1.0

Time Complexity to Determine BWC

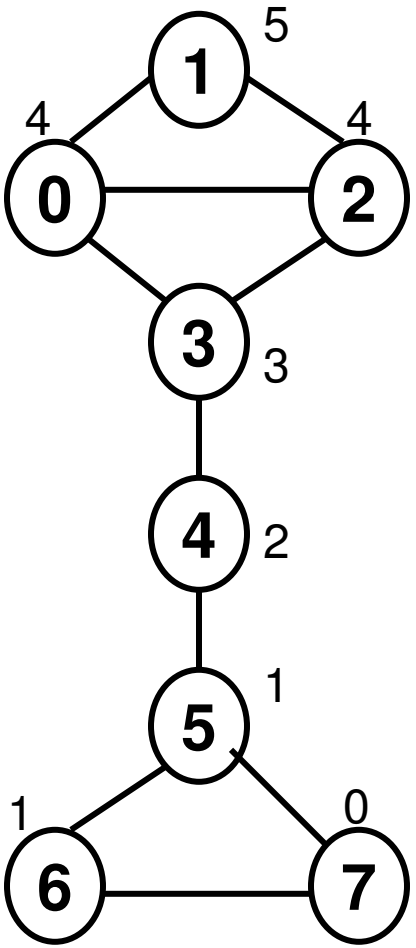
We can run the Θ(V+E)-BFS algorithm once for each vertex and
Determine the number of shortest paths from that vertex to every vertex
This would be of time complexity **Θ(V(V+E)).**



To determine how many Shortest paths from nodes 1 to 7 that go through node 4.

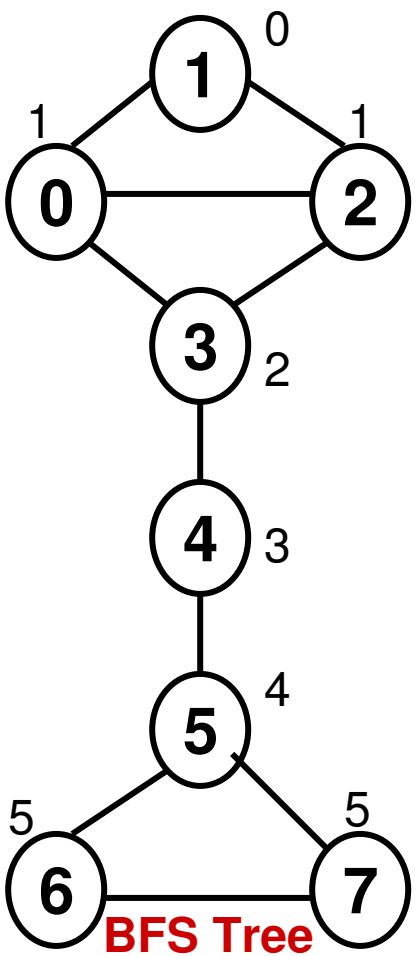


Level Numbers of the vertices starting from root ‘1’.

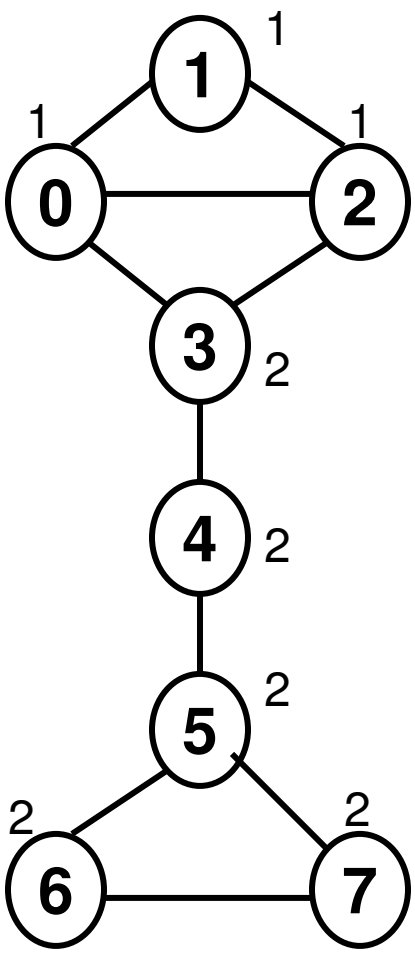


Level Numbers of the vertices starting from root ‘7’.

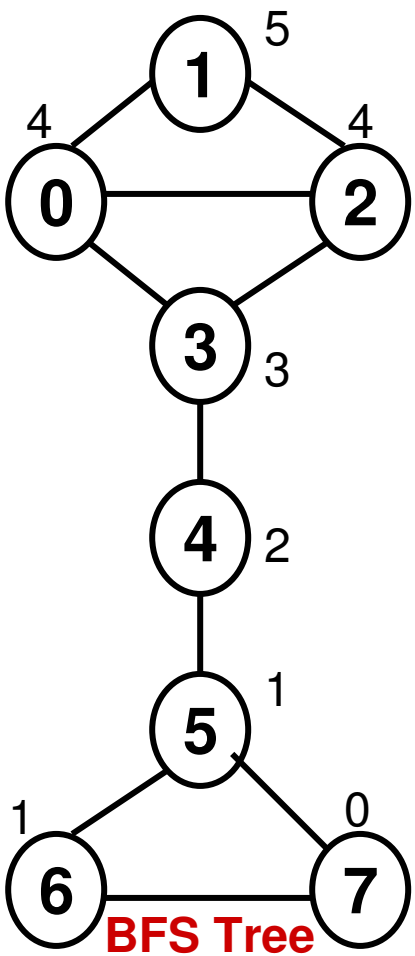
Since the sum of the level numbers from vertex 1 to vertex 6 and from vertex 7 to 6 exceeds the level number from vertex 1 to 7; BWC(6; pair 1, 7) = 0.0



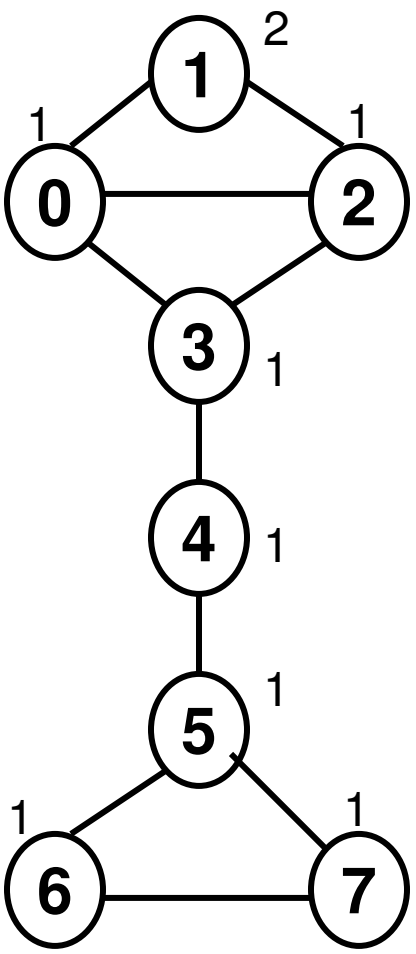
BFS Tree rooted at Vertex 1



 # shortest paths from vertex 1 to the other vertices



BFS Tree rooted at Vertex 7

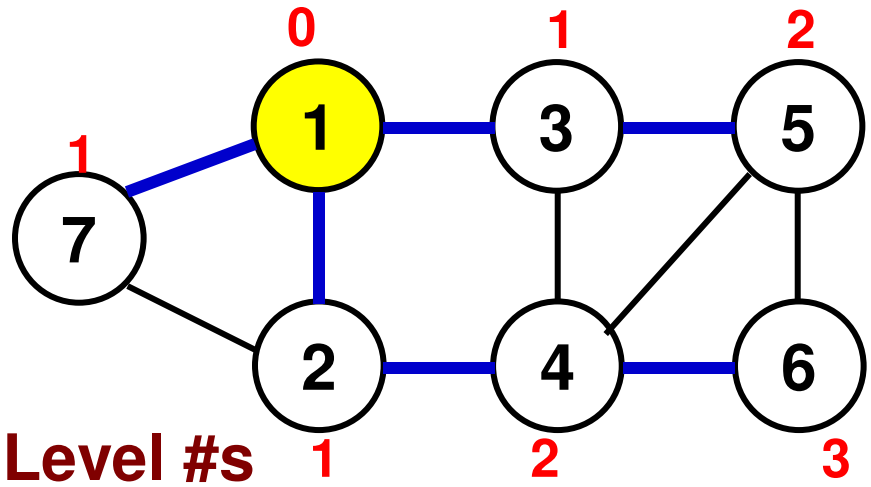


 # shortest paths from vertex 7 to the other vertices

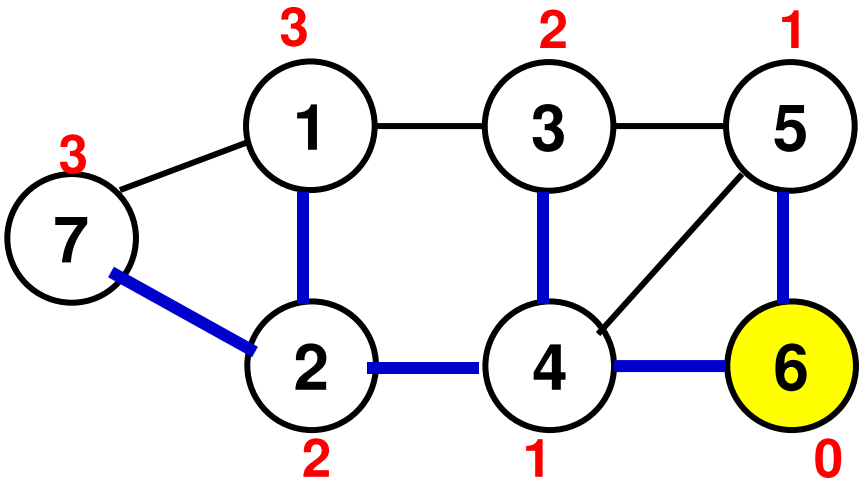
BWC(4; pair 1, 7)
Level (4; BFS tree of 1) = 3
Level (4; BFS tree of 7) = 2
Sum of the levels = 5
= Distance from 1 to 7

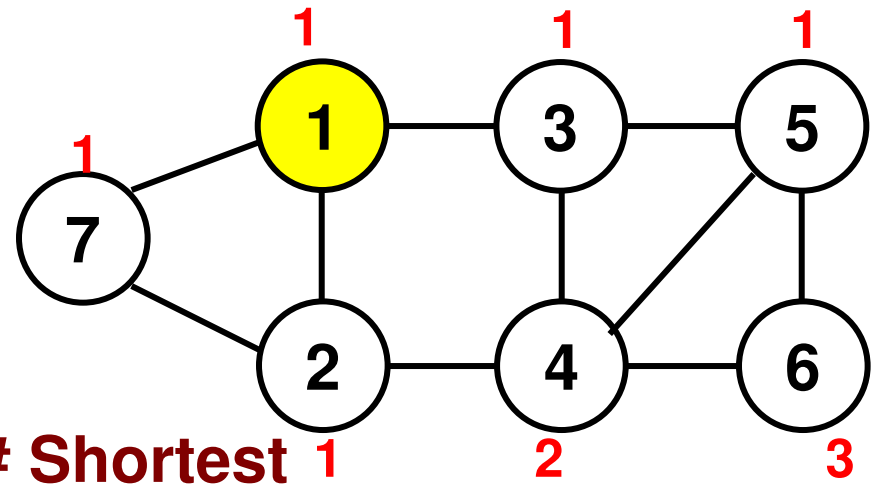
To determine how many Shortest paths from nodes 1 to 7 that go through node 4: = Product(2, 1) = 2

BWC(node 4 with respect to pair 1-7) = 2/2

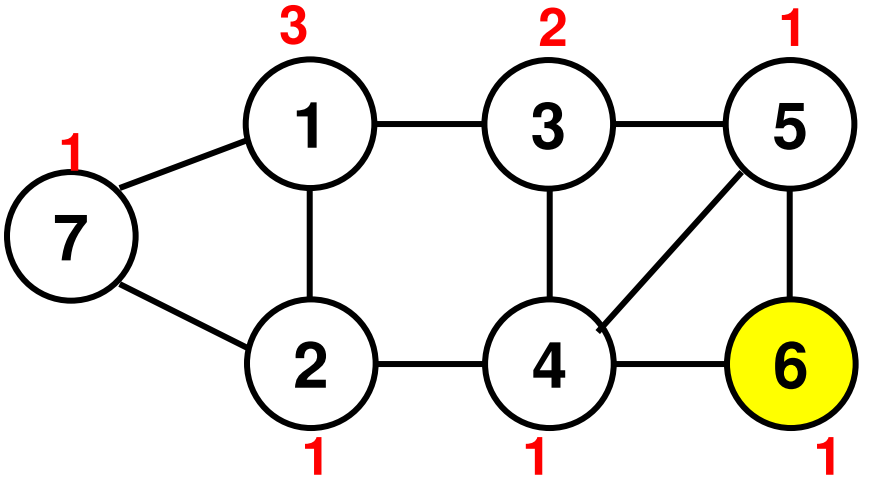


Level #s



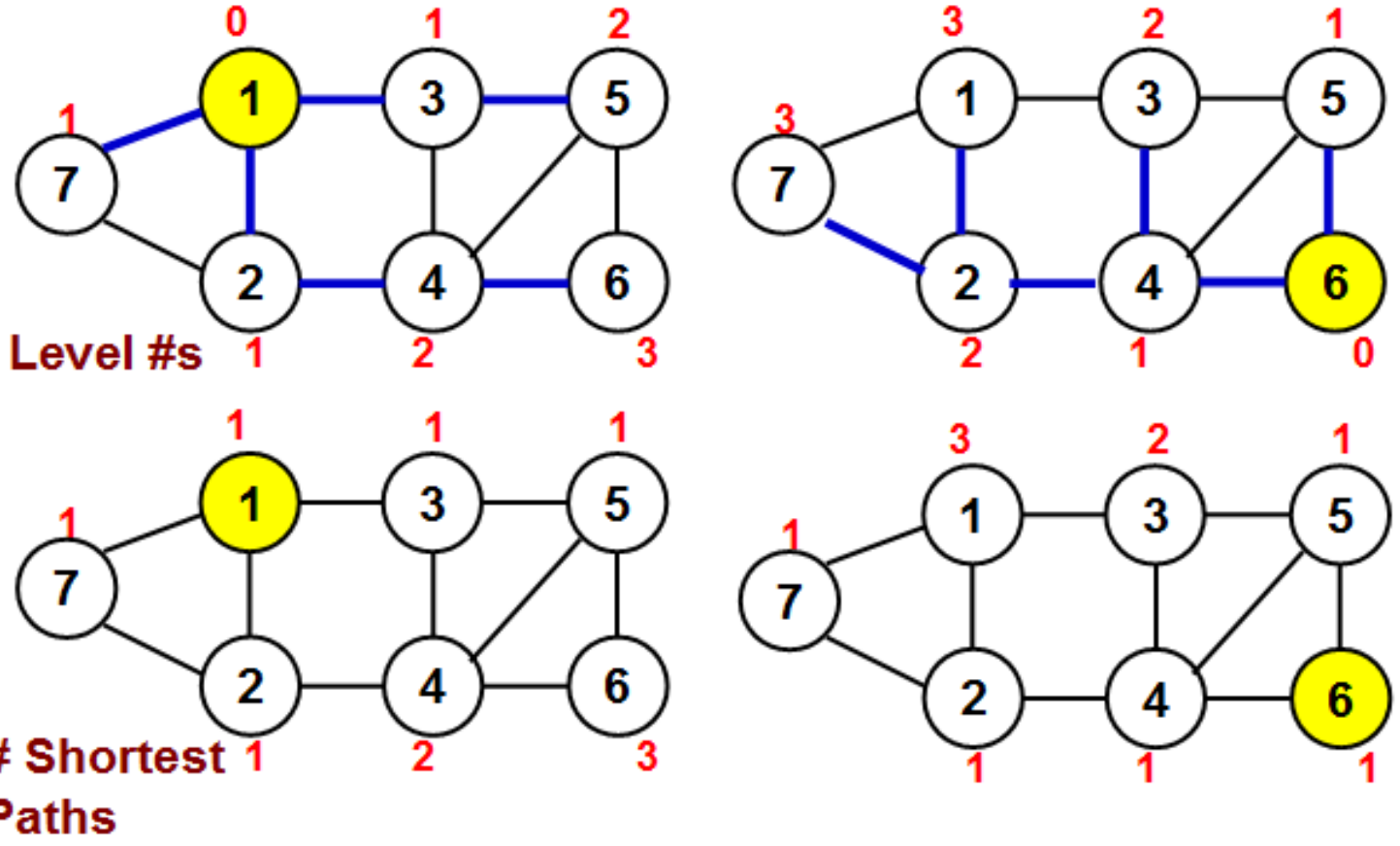


Shortest Paths



To Find BWC(7 for pair 1, 6)

Level number for vertex 7 in the BFS tree rooted at vertex 1 is: 1
Level number for vertex 7 in the BFS tree rooted at vertex 6 is: 3
Sum of the level numbers (1 + 3 = 4) is greater than the level number for vertex 6 on the BFS tree rooted at 1 (and vice-versa). Hence, BWC(7 for pair 1, 6) = 0.0



BWC (3 for pair 1, 6): sum of the level numbers (1 + 2) from vertex 1 to 3 (1) and from vertex 6 to 3 is not greater than the level number (3) from vertex 1 to 6 (or equivalently from vertex 6 to 1).

$$\text{BWC (3 for pair 1, 6)} = \frac{\text{(\# shortest paths from 1 to 3 * \# shortest paths from 6 to 3)}}{\text{(\# shortest paths from 1 to 6)}}$$

= (1 * 2) / 3 = 2/3.

Ford-Fulkerson Algorithm

- **Maximal Flow Problem:** Given a weighted graph, where the weight of an edge represents the capacity of the edge, we want to determine the maximum amount of flow that we can send from one vertex to another vertex in the graph.
- **Constraint:** The maximum flow on an edge is limited by the capacity of that edge.

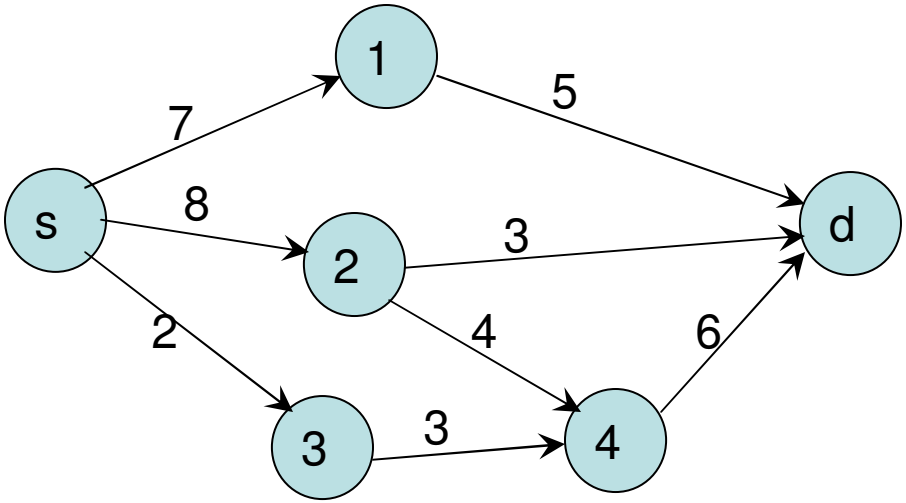
Input: Graph $G = (V, E, C)$, source node s and sink node d
Output: A maximum flow f from s to d
Auxiliary Variables: Residual Graph $G_R (V, E_R, C) = G (V, E, C)$
Initialization: $\forall$ $(u, v) \in E$, $(u, v) \in E_R$ and $(v, u) \in E_R$;
$f_R(u, v) = 0$ and $f_R(v, u) = 0$
$c_R(u, v) = c(u, v) - f_R(u, v)$
$c_R(v, u) = c(v, u) - f_R(v, u)$
Maximal flow f $\leftarrow$ 0

Begin *Ford-Fulkerson*
while there is an s-d path p in G_R, such that $c_R (u, v) > 0$ for all edges $(u, v) \in p$ **do**
$c_R(p) \leftarrow$ Minimum $\{c_R(u, v) \mid (u, v) \in p\}$
$f_R(u, v) \leftarrow f_R(u, v) + c_R(p)$
$c_R(u, v) \leftarrow c_R(u, v) - f_R(u, v)$
$f_R(v, u) \leftarrow f_R(v, u) - c_R(p)$
$c_R(v, u) \leftarrow c_R(v, u) - f_R(v, u)$
$f \leftarrow f + c_R(p)$
return f
End *Ford-Fulkerson*

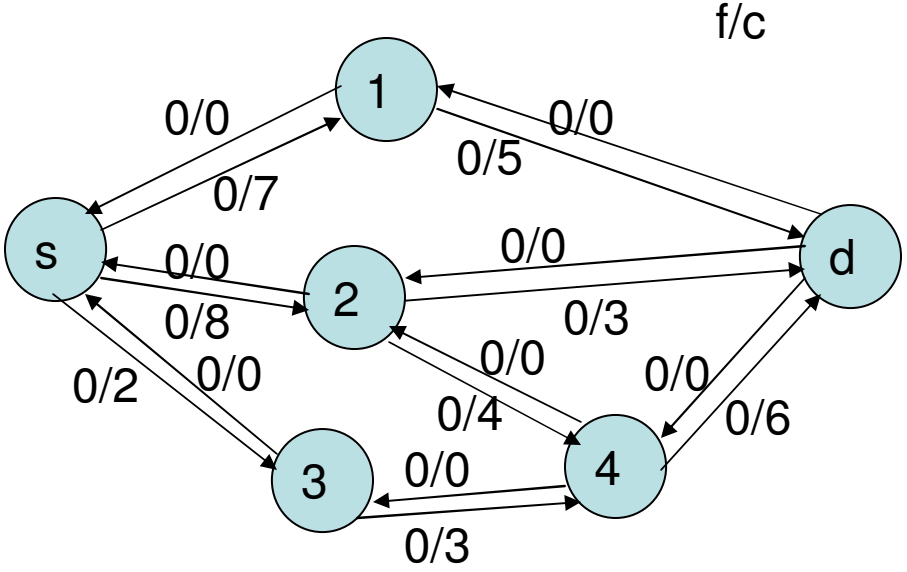
Ford-Fulkerson Algorithm

- The path determined in each iteration is called an augmenting path.
  - It takes O(E) time to determine an augmenting path – using the Breadth-First-Search or the Depth-First-Search algorithms.
- The sum of the flows of the augmenting paths is the maximum flow obtained.
  - Since an augmenting path has at least a flow of 1, the time complexity of the Ford-Fulkerson algorithm is given by O(f * E), where f is the maximum flow.

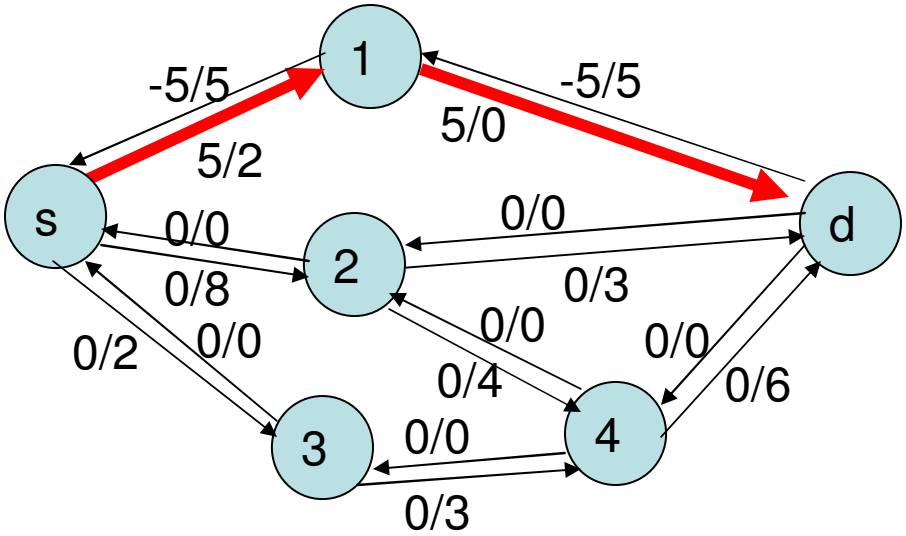
Example 1 - Ford-Fulkerson Algorithm



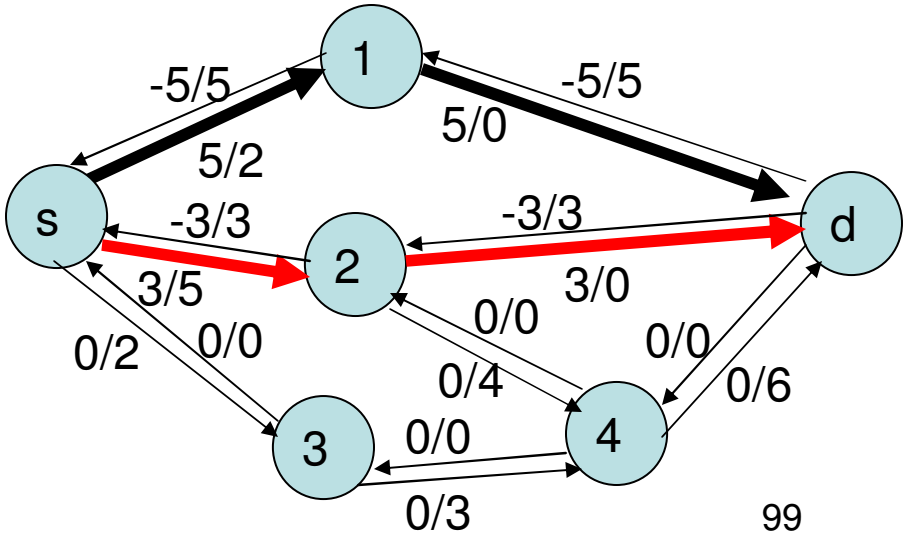
Initial Condition



Residual Graph

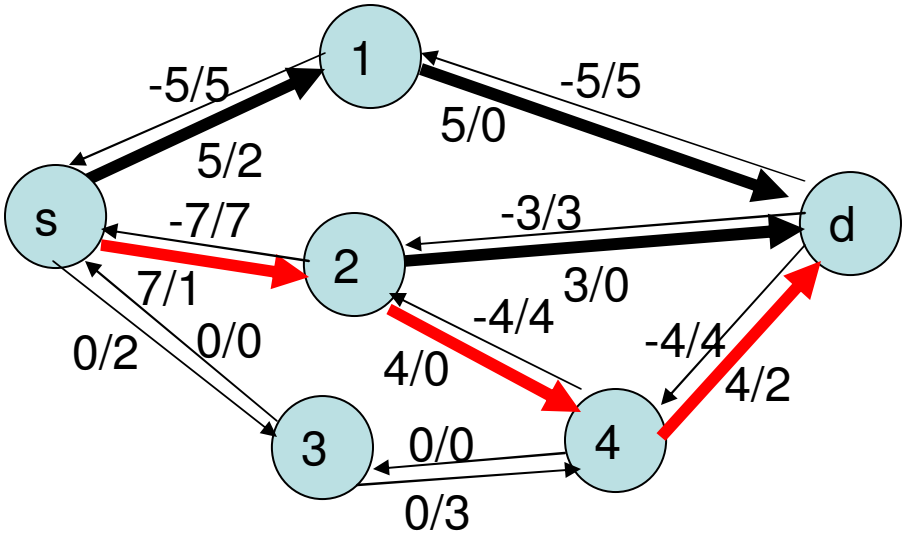


Iteration 1 (flow: 5)

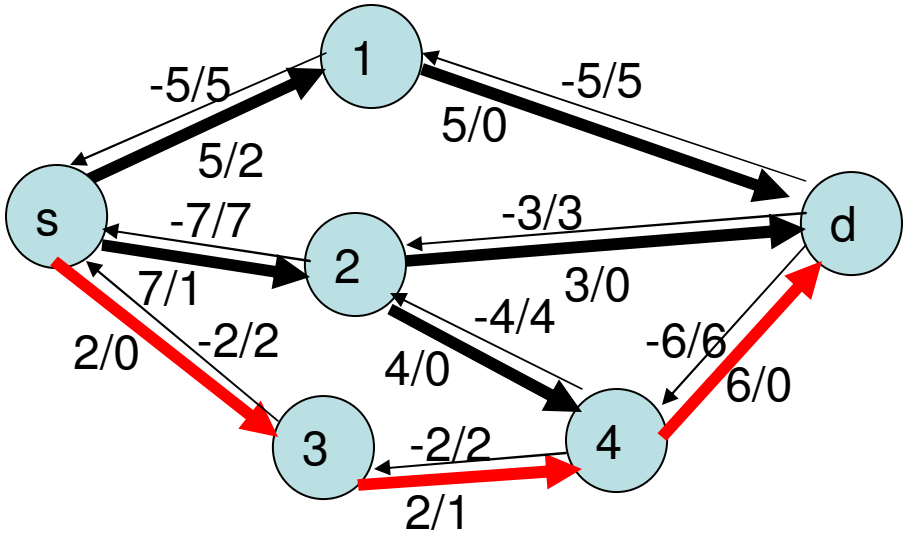


Iteration 2 (flow: 3)

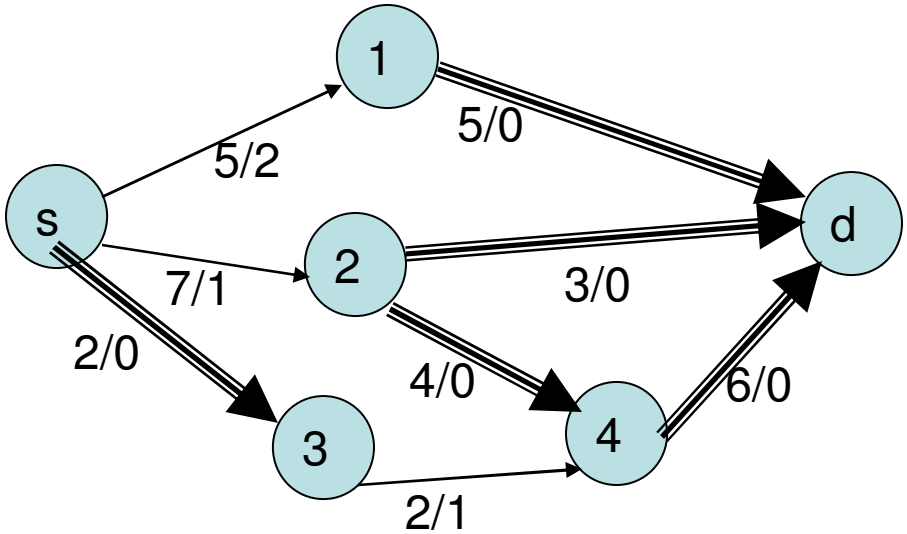
Example 1 for Ford-Fulkerson Algorithm



Iteration 3 (flow: 4)



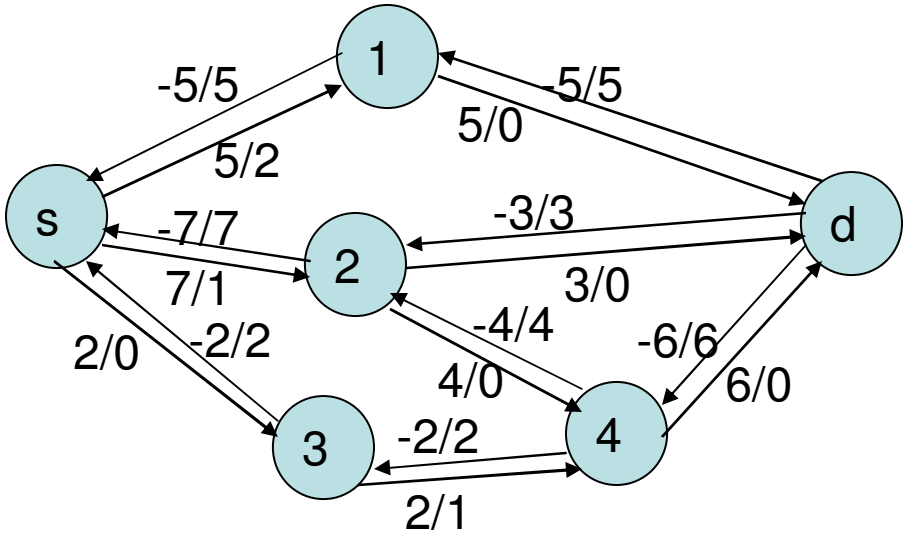
Iteration 4 (flow: 2)



Final Flow Graph

Maximum flow = 5 + 3 + 4 +2 =14

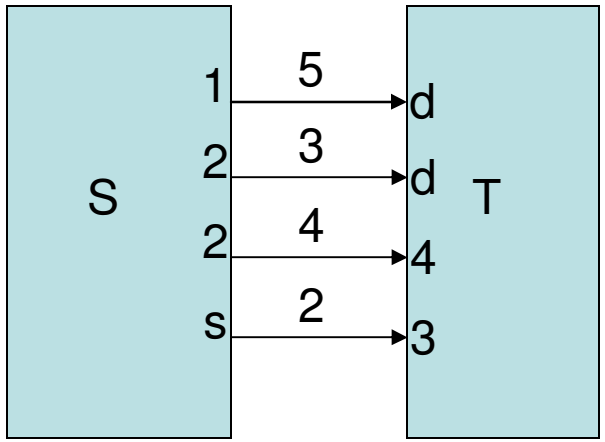
Example 1 for Ford-Fulkerson Algorithm



Final Residual Graph

In the final Residual graph, we see that Source s is reachable to nodes 1 and 2 Nodes 3, 4 and destination d are not reachable from s.

S = {s, 1, 2} and T = {3, 4, d}
Find edges between S and T – These are the edges that form the bottleneck edges and if we remove these edges s and d are disconnected.



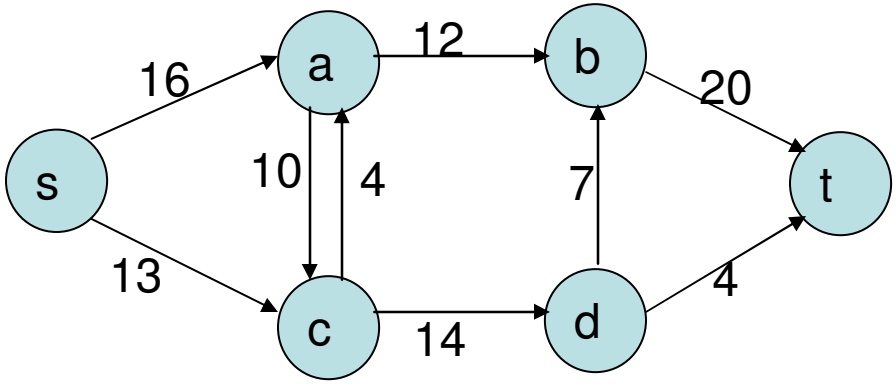
Min Cut Edges

Edges (1, d), (2, d), (2, 4) and (s, 3) are said to be the bottleneck edges in the input graph. If these edges are removed, the source s and destination d are disconnected.

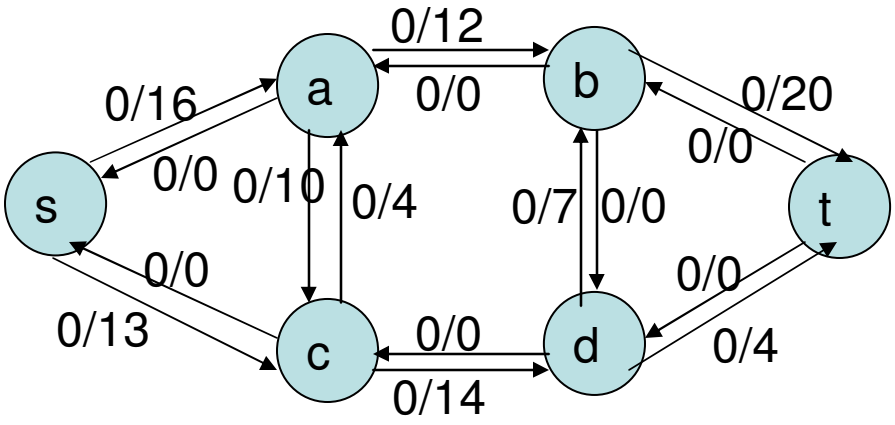
Max Flow – Min Cut Theorem

- Ford Fulkerson algorithm is said to have determined the maximum flow once the source *s* and sink *d* get disconnected in the residual graph.
- We determine the set S of vertices that are reachable from the source *s* (including *s*) and the set T of vertices that are not reachable from the source *s*.
- An s-d cut is a set of edges that if removed will disconnect the source *s* and the sink *d*.
- A Min. Cut (over all the s-d cuts) is the set of edges with minimum capacity (sum of all edge weights forming the cut) that also corresponds to the Max. Flow.
- Thus, the Ford Fulkerson algorithm is said to simultaneously determine both the Max Flow and the Min Cut.

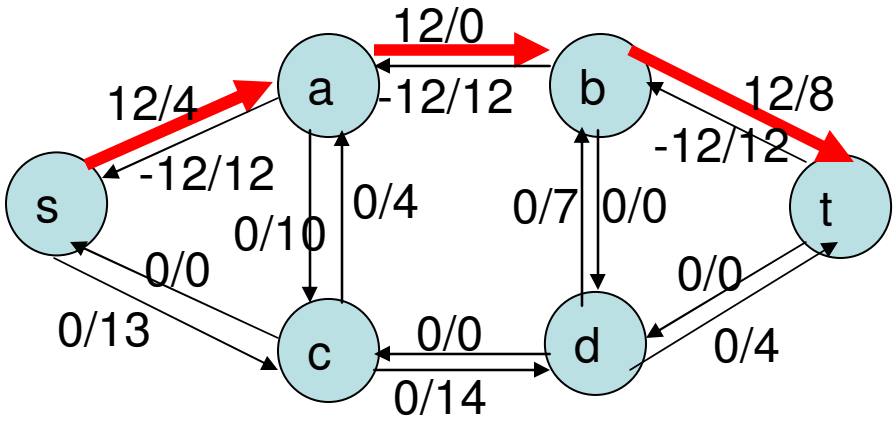
Example 2 - Ford-Fulkerson Algorithm



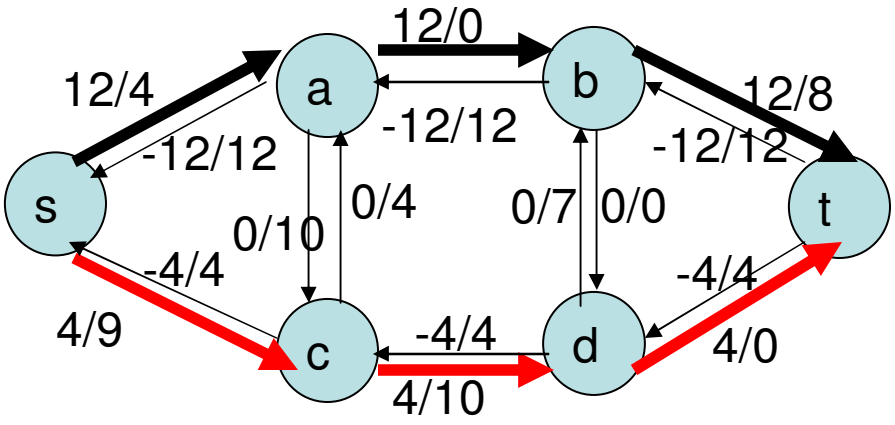
Initial Graph



Residual Graph

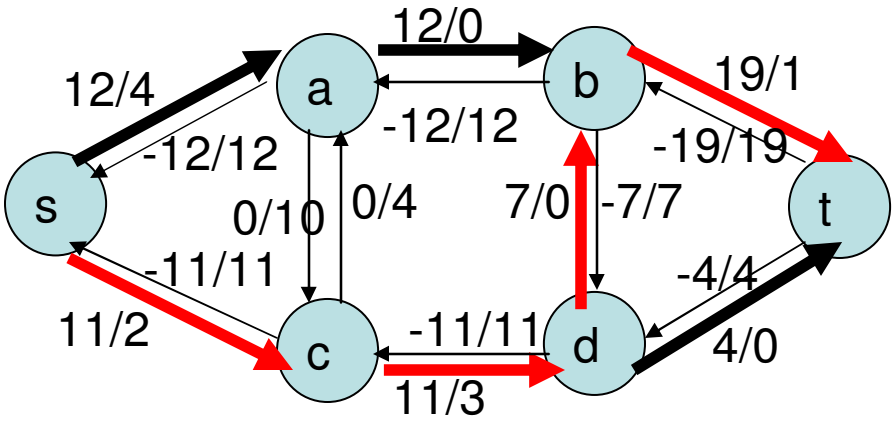


Iteration 1

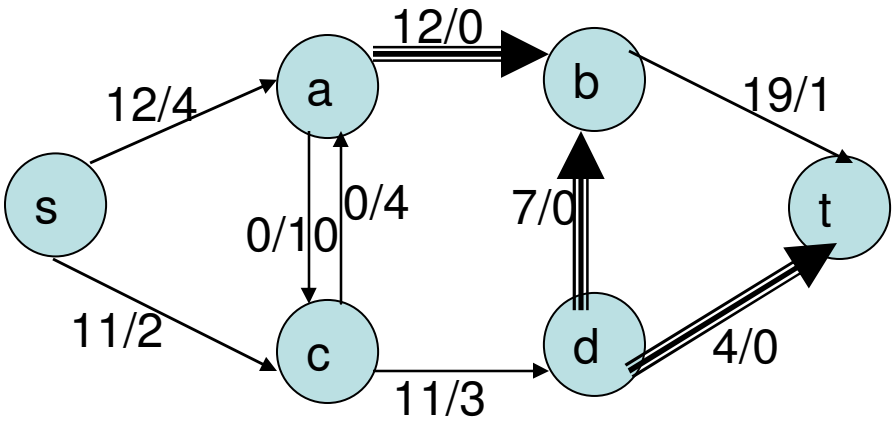


Iteration 2

Example 2 - Ford-Fulkerson Algorithm



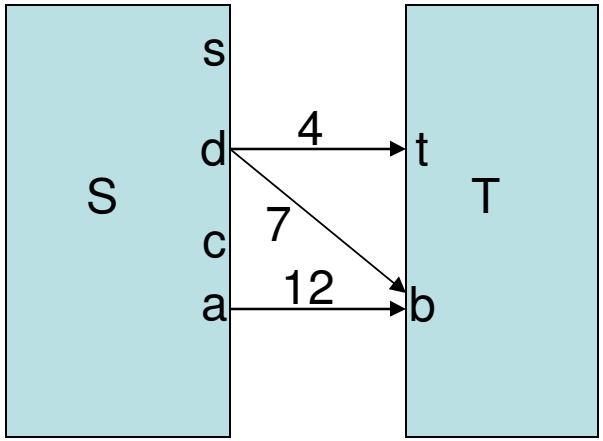
Final Residual Graph: Iteration 3



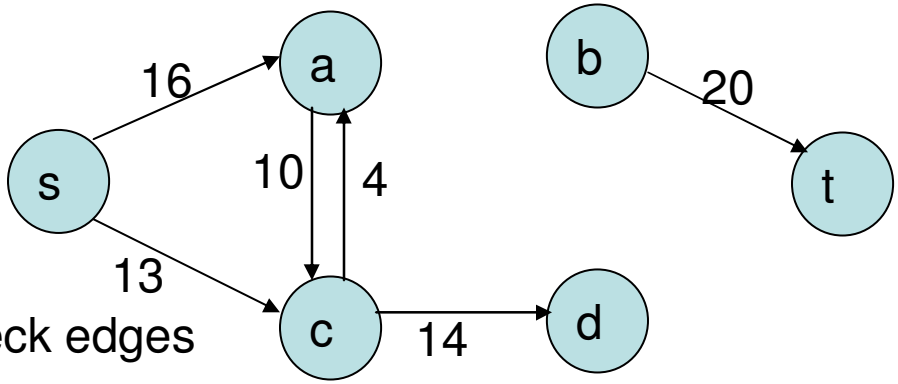
Final Flow Graph

Maximum Flow = 12 + 4 + 7 = 23

S = {s, a, d, c}
T = {b, t}

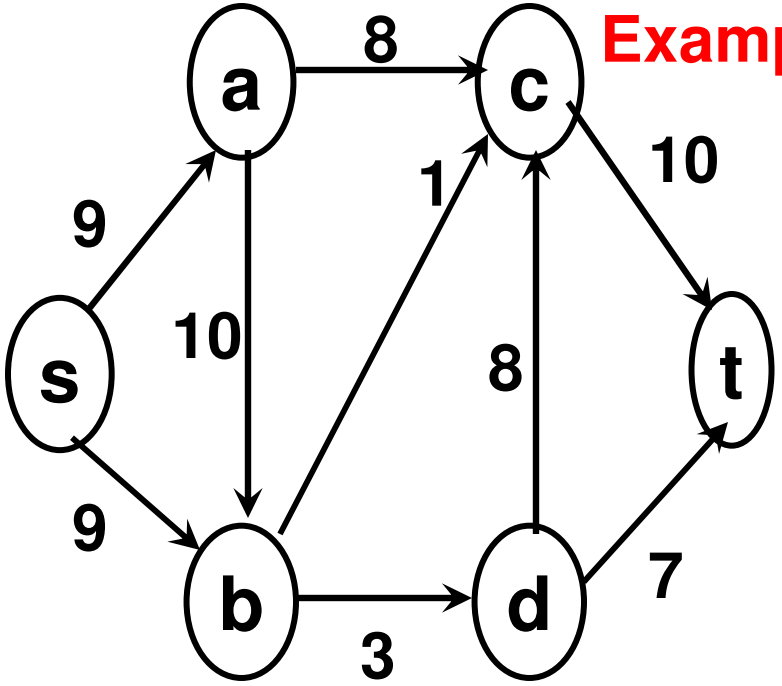


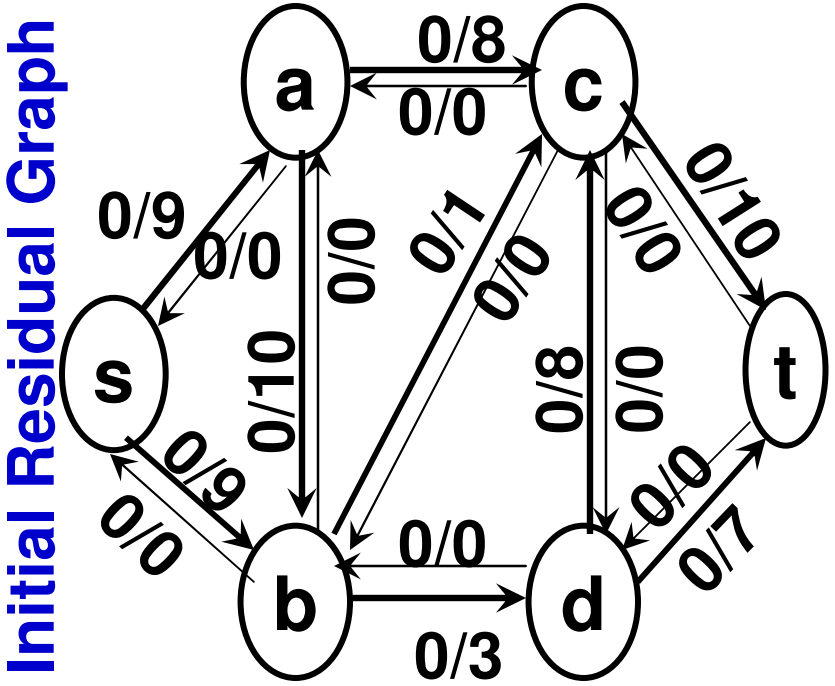
Min Cut Edges

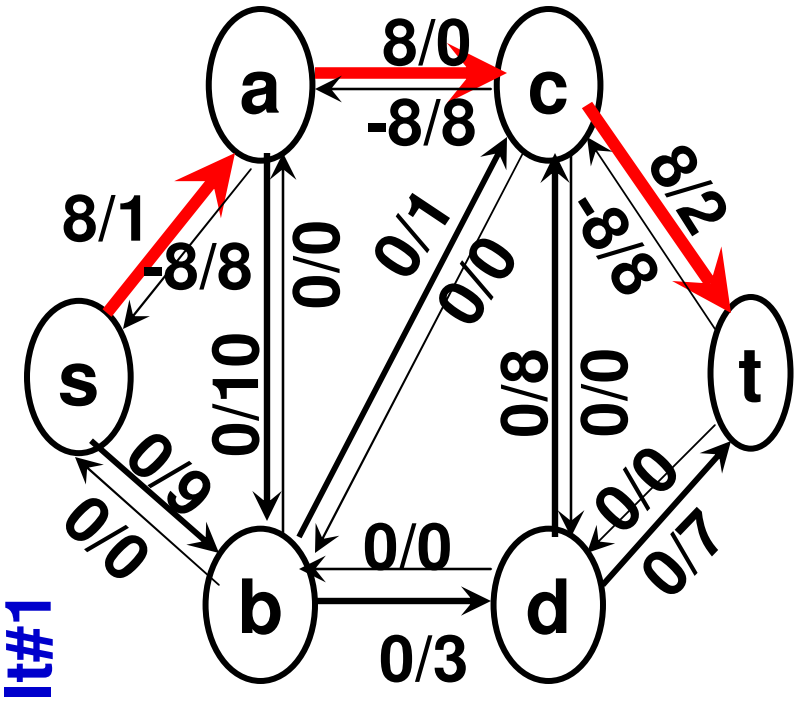


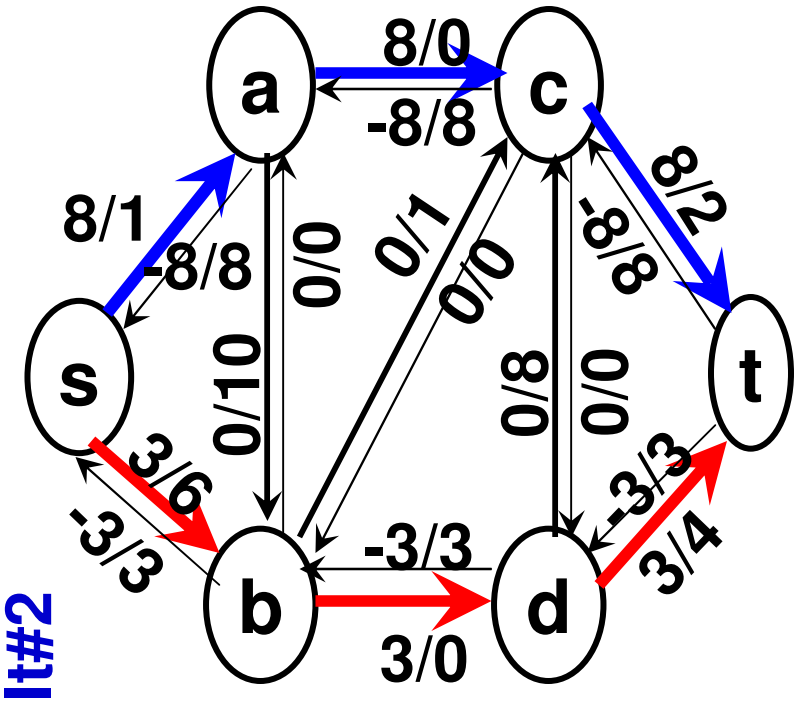
Initial Graph
With bottleneck edges removed

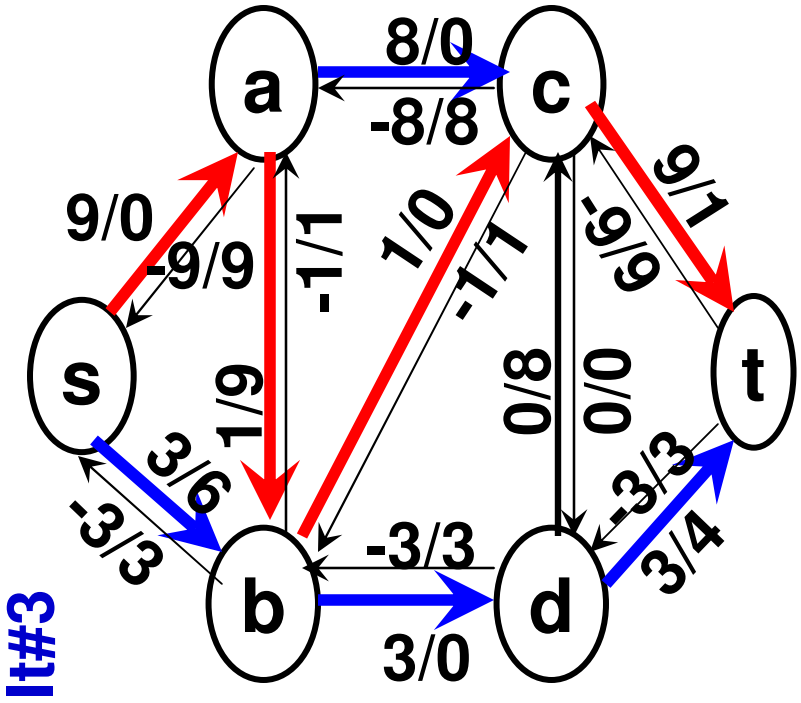
Edges (d, t), (d, b) and (a, b) are the bottleneck edges in the input graph. If these three edges are removed, then the source *s* and the sink *t* are disconnected

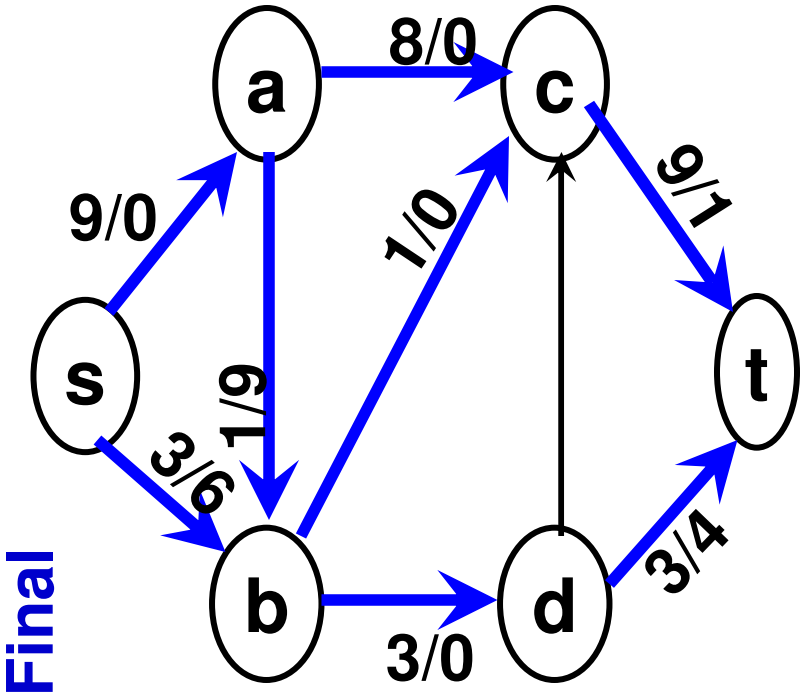

Example 3
a
c
s
t
b
d
8
10
1
9
10
8
9
7
3

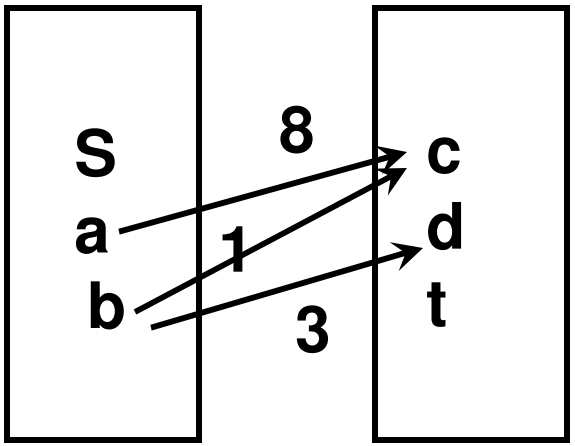

Initial Residual Graph
a
c
s
t
b
d
0/8
0/0
0/9
0/0
0/0
0/1
0/0
0/0
0/10
0/10
0/8
0/8
0/0
0/0
0/9
0/0
0/0
0/7
0/0
0/3


It#1
a
c
s
t
b
d
8/0
-8/8
8/1
-8/8
0/0
0/1
0/0
-8/8
8/2
0/10
0/8
0/0
0/9
0/0
0/0
0/0
0/7
0/3

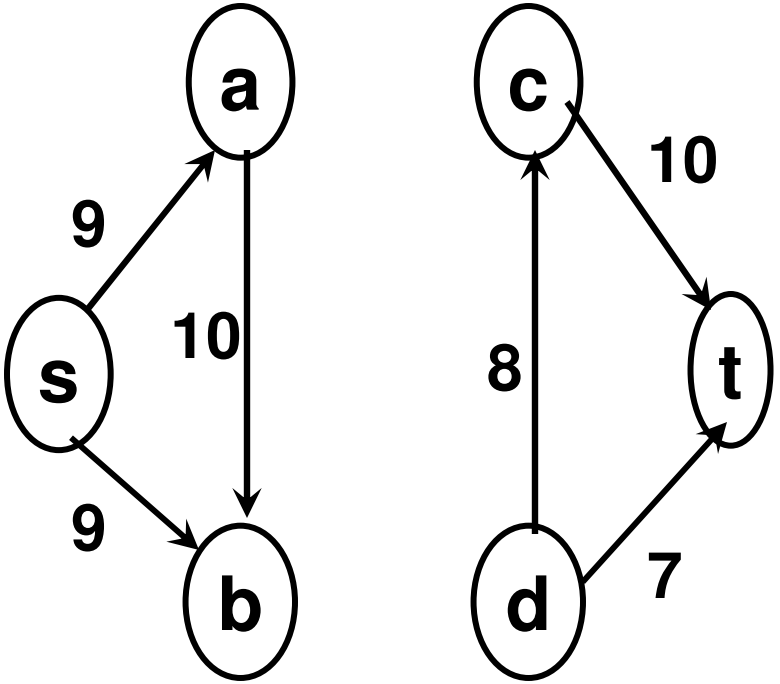

It#2
a
c
s
t
b
d
8/0
-8/8
8/1
-8/8
0/0
0/1
0/0
-8/8
8/2
0/10
0/8
0/0
3/6
-3/3
-3/3
-3/3
3/4
3/0





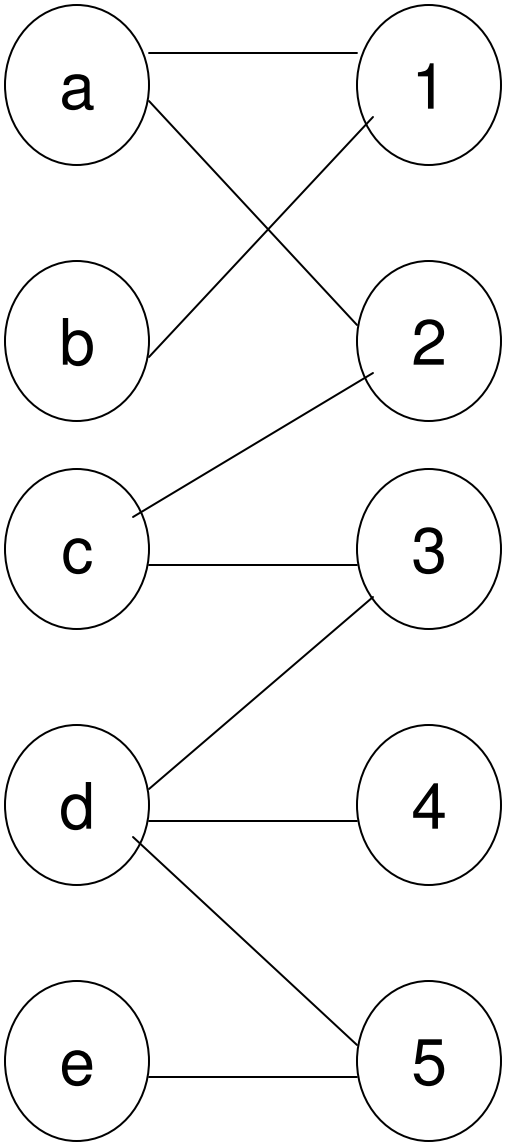


Min Cut Edges

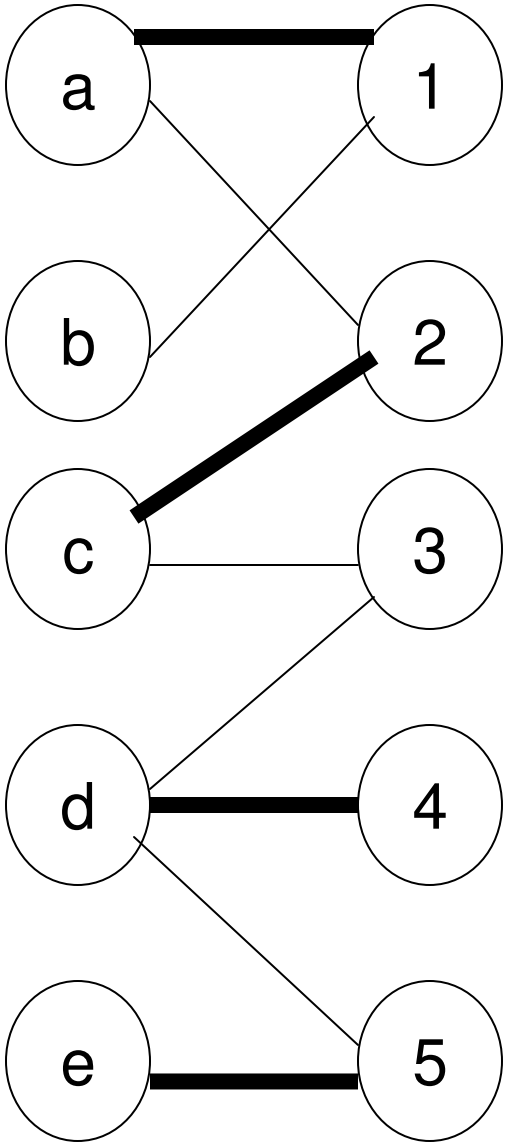


Bipartite Graphs

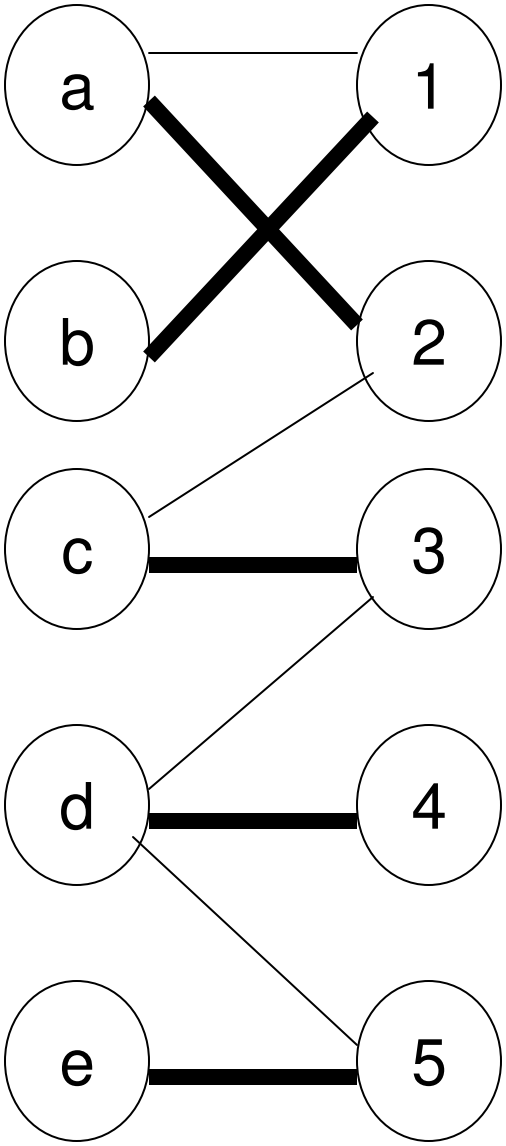
- A bipartite graph is a graph with two disjoint sets of vertices such that every edge in the graph connects a vertex in one set to a vertex in another set.
- Maximum bipartite matching problem: Given a bipartite graph, we want to find a largest subset of the edges of the graph such that each vertex in the two disjoint sets is incident on at most one edge in this subset.
- Solution Strategy: Transform the bipartite graph to a network flow graph and run the Ford-Fulkerson algorithm to find the max. flow – the edges on the bipartite graph that have a unit/positive flow are the edges constituting the maximum bipartite matching.
- Problem Reduction:
  - Given a bipartite graph G (A U B, E), direct the edges from A to B
  - Add new vertices s and t
  - Add an edge from s to every vertex in A
  - Add an edge from every vertex in B to t
  - Make all the edge capacities to 1.
  - Solve the maximum flow network problem on the new network graph
  - *Note that since all edges have unit capacity, there can be at most only one edge leaving a vertex in A, as part of the edges constituting the max. flow.*



Bipartite Graph

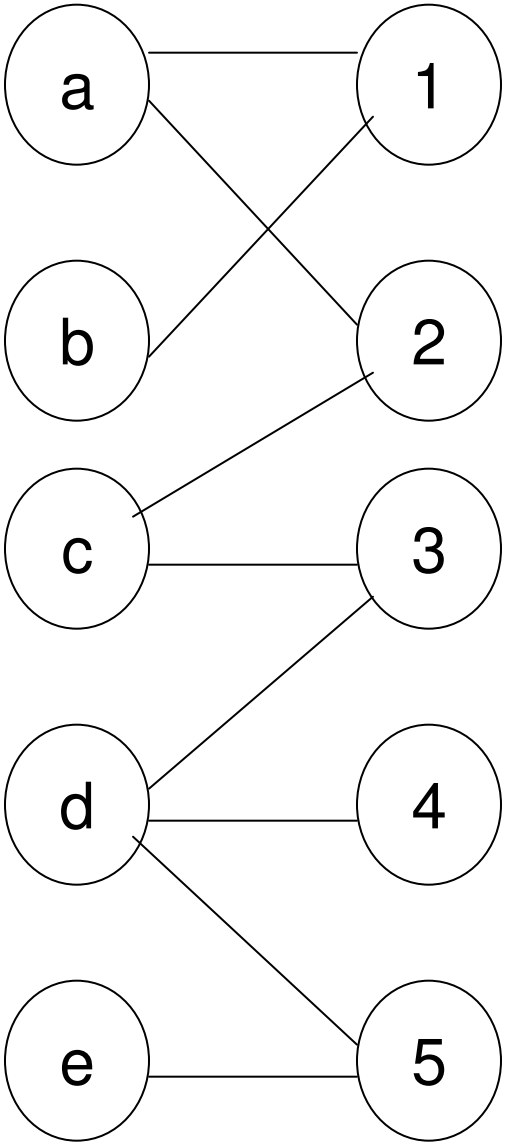


A Matching on the Bipartite Graph (4 edges)

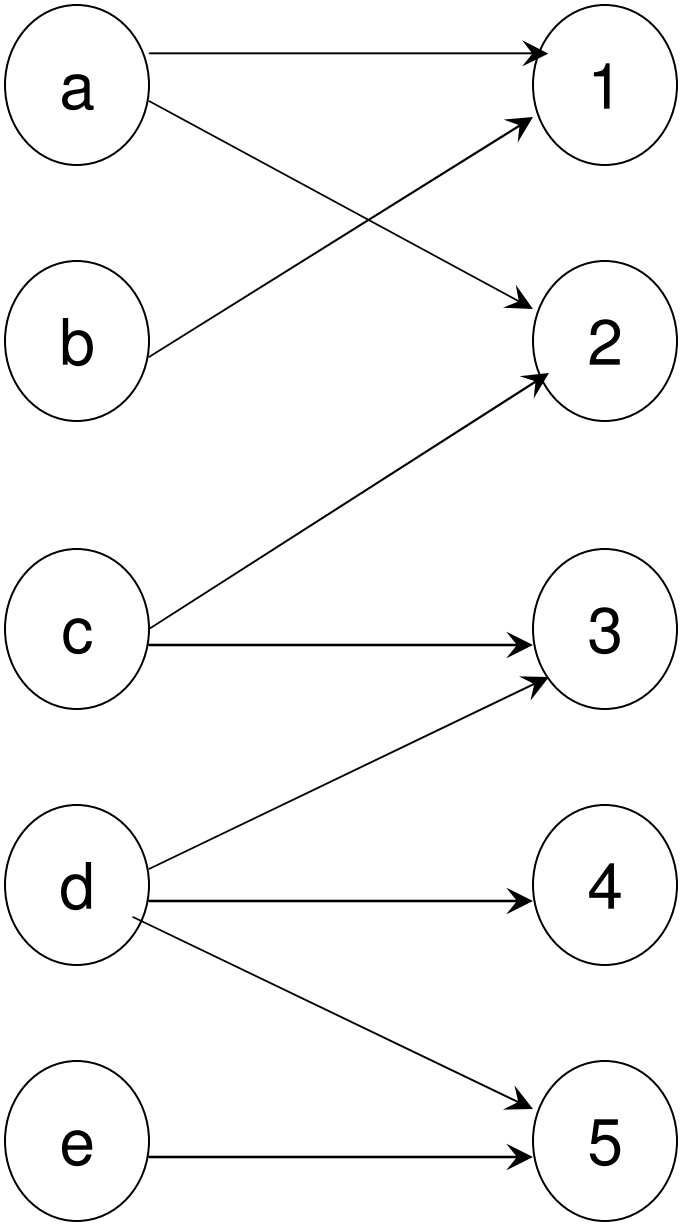


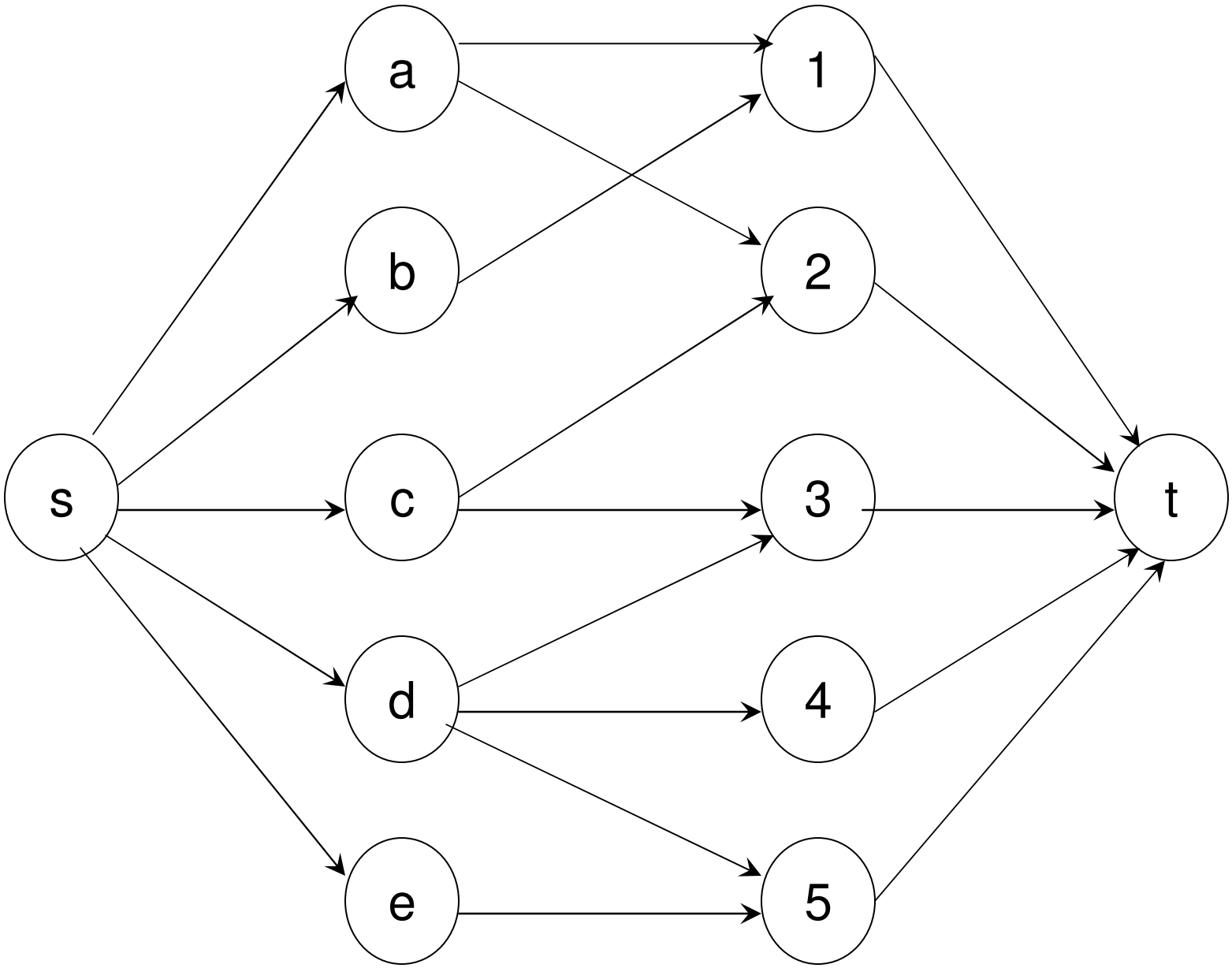
Maximum Matching on the Bipartite Graph (5 edges)

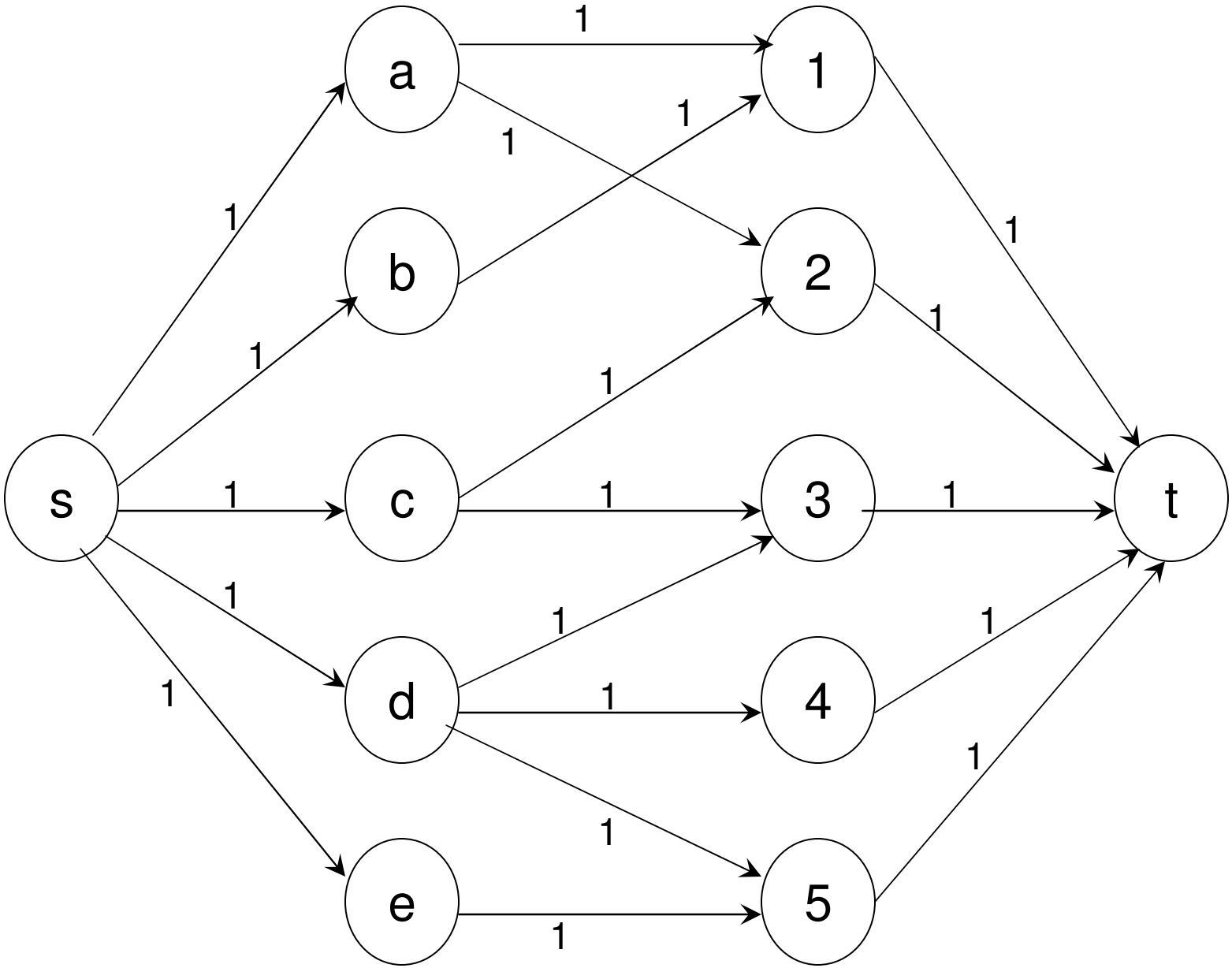
EXAMPLE 1

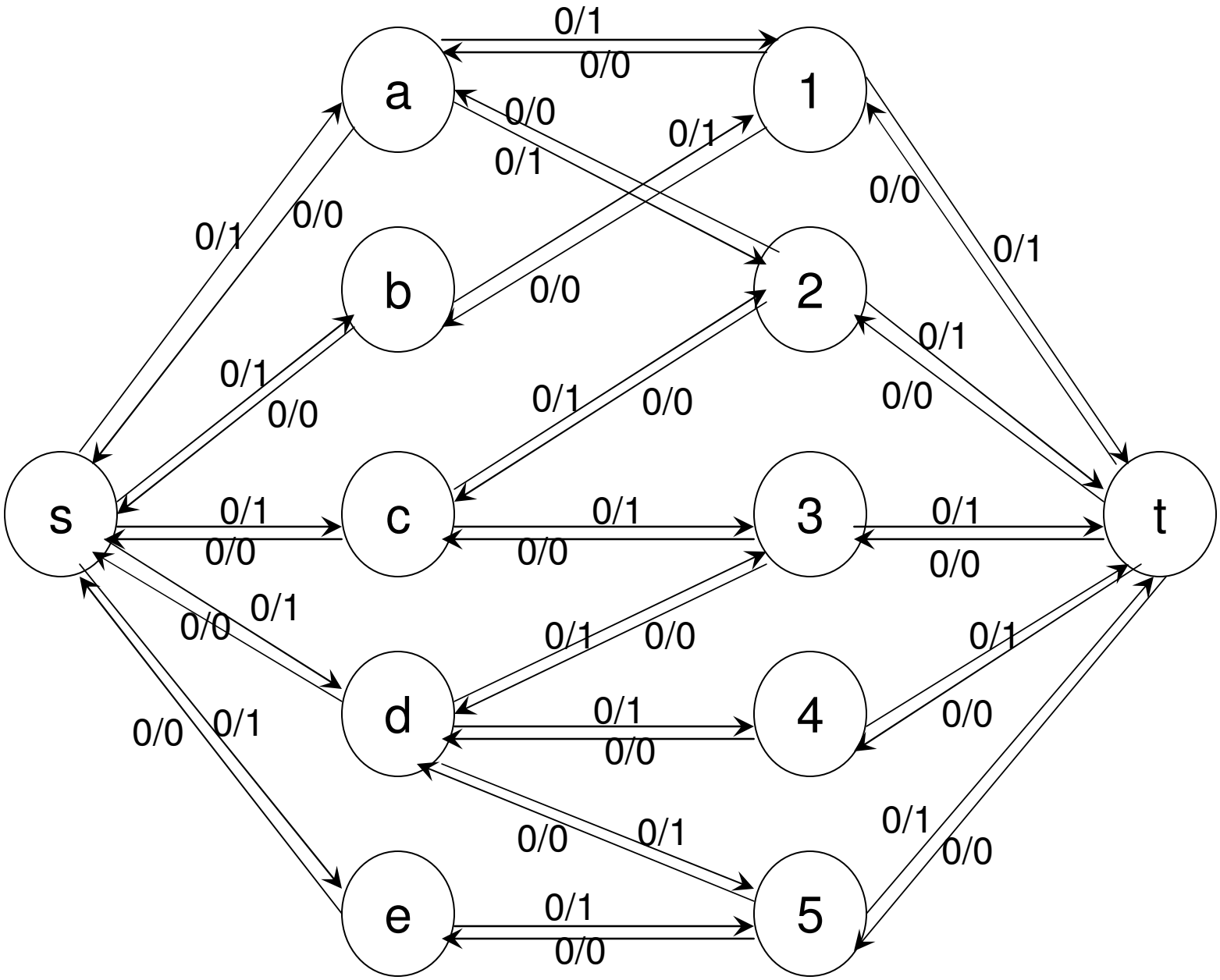


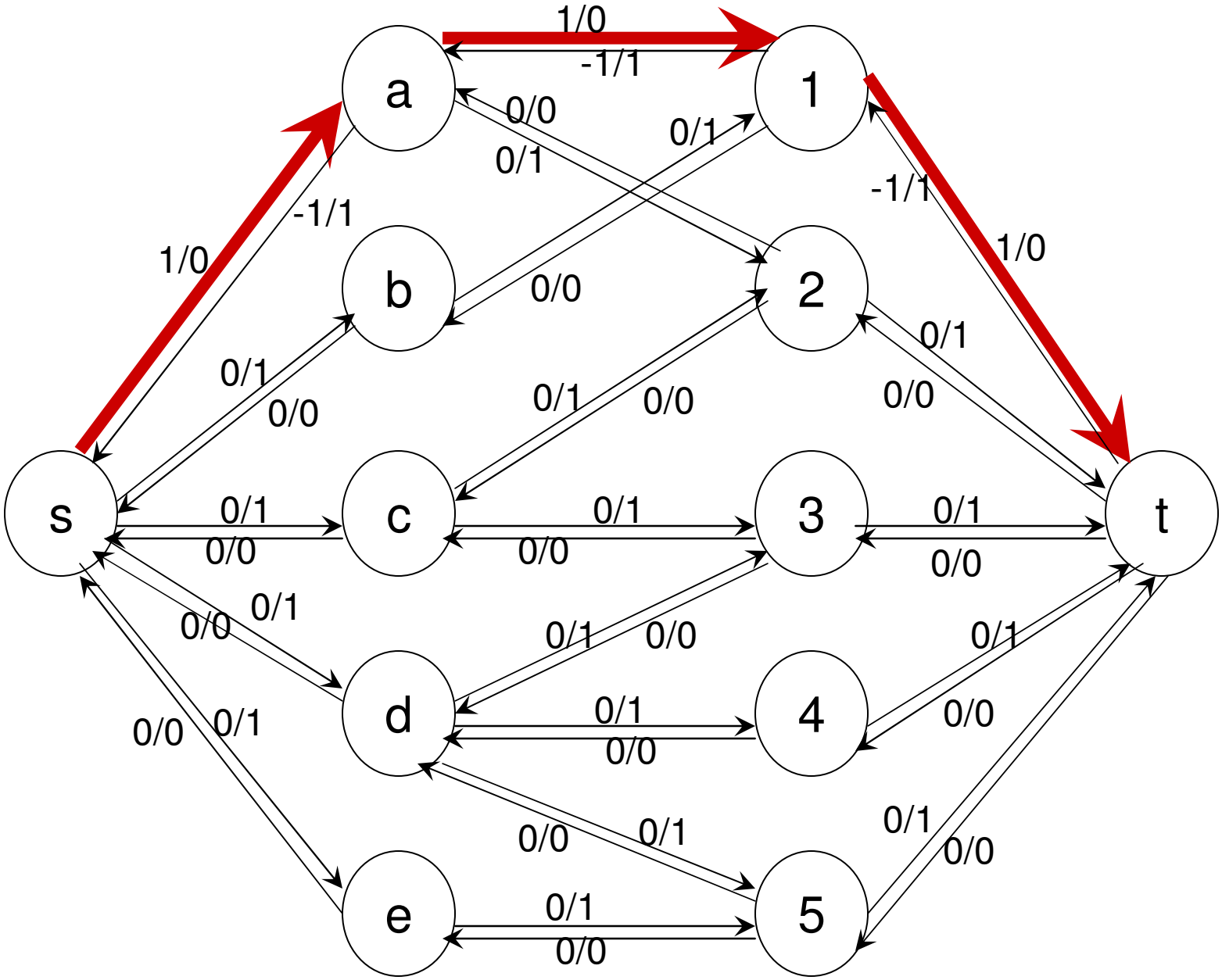
Bipartite Graph




a
1
b
2
s
c
3
t
d
4
e
5


s
a
b
c
d
e
1
2
3
4
5
t
1


a
1
b
2
s
c
3
t
d
4
e
5
0/1
0/0


a
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b
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c
3
t
d
4
e
5
1/0
-1/1
0/0
0/1
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1/0
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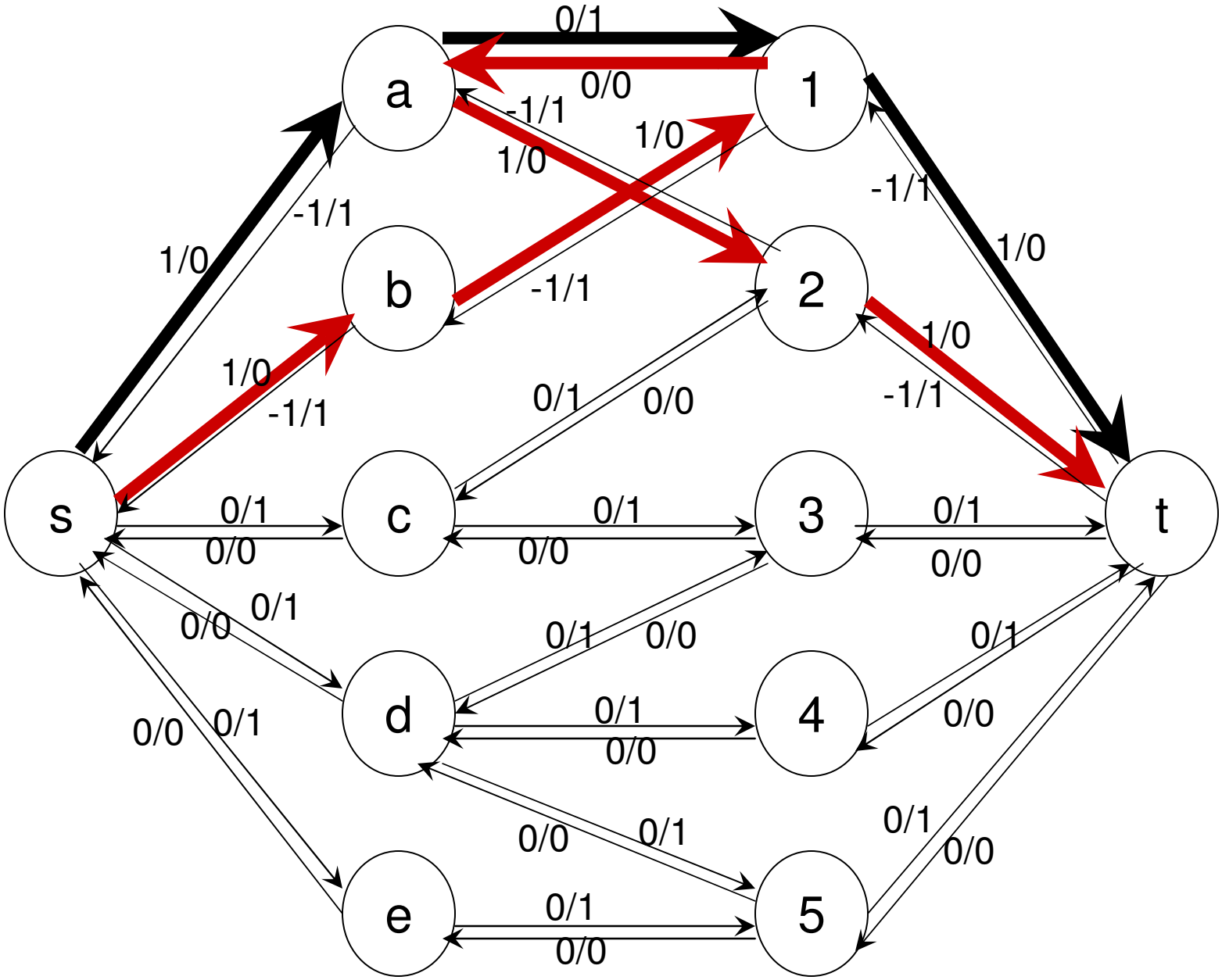
a
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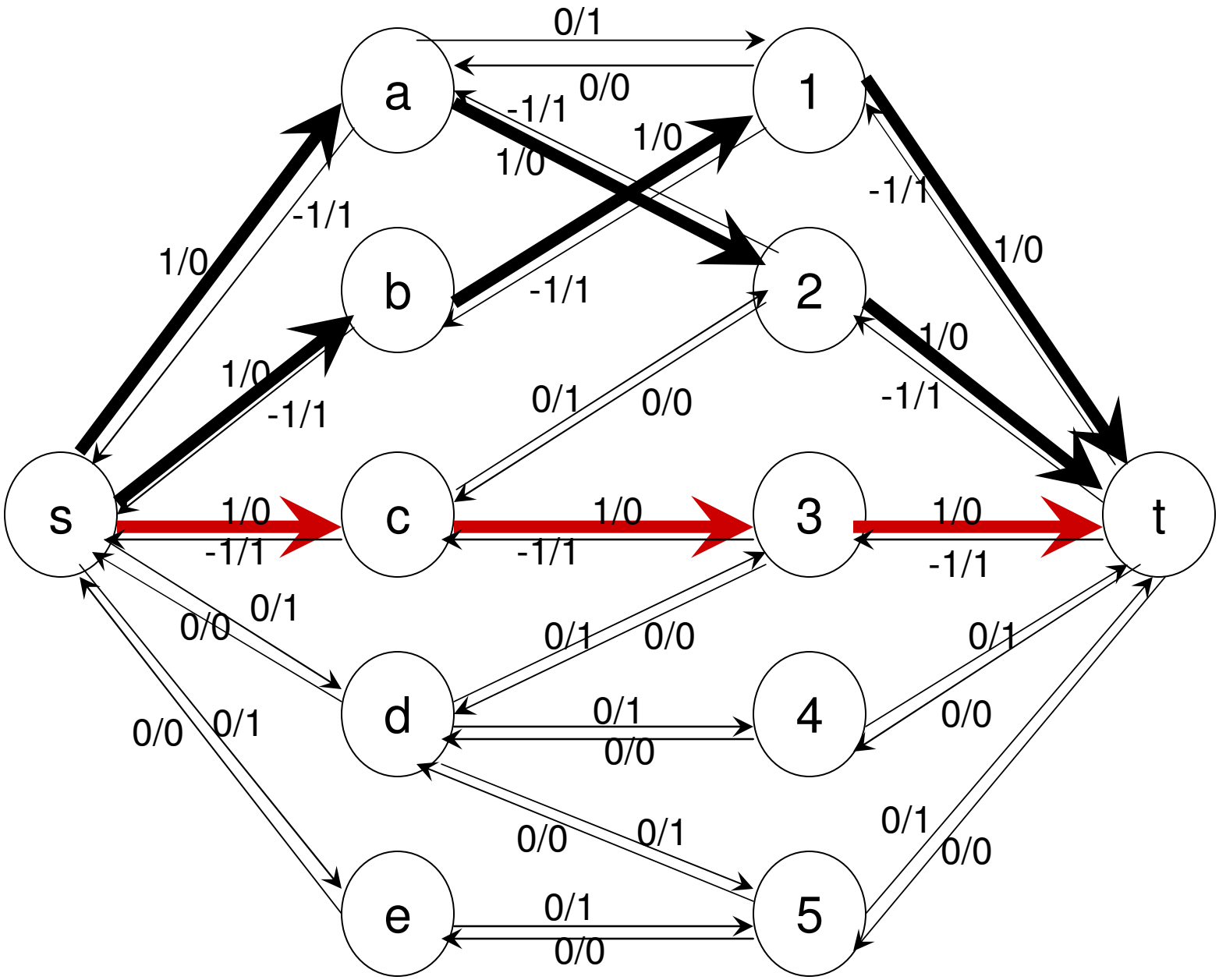
b
2
s
c
3
t
d
4
e
5
0/1
0/0
-1/1
1/0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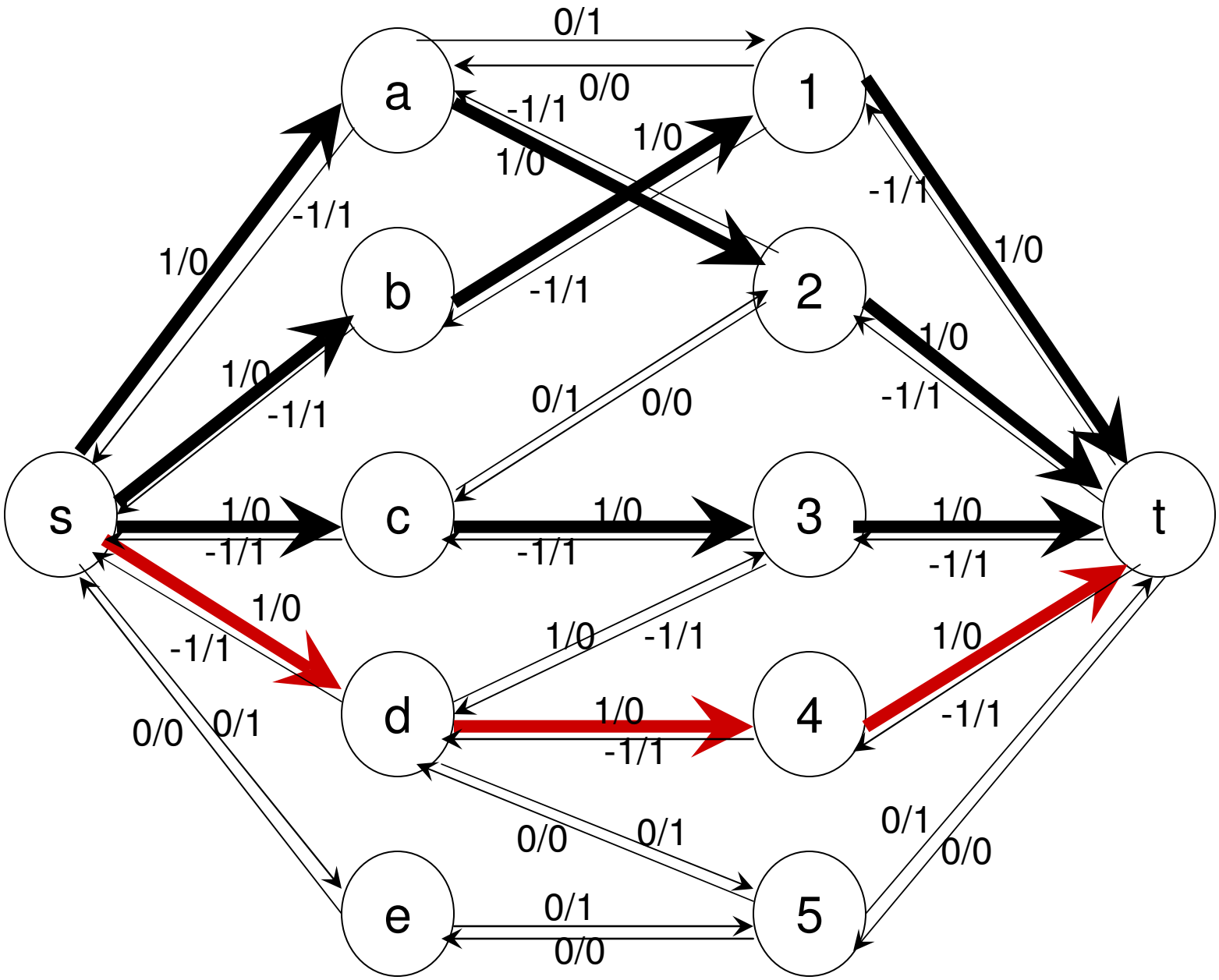
s
c
3
t
d
4
e
5
0/1
0/0
-1/1
1/0


a
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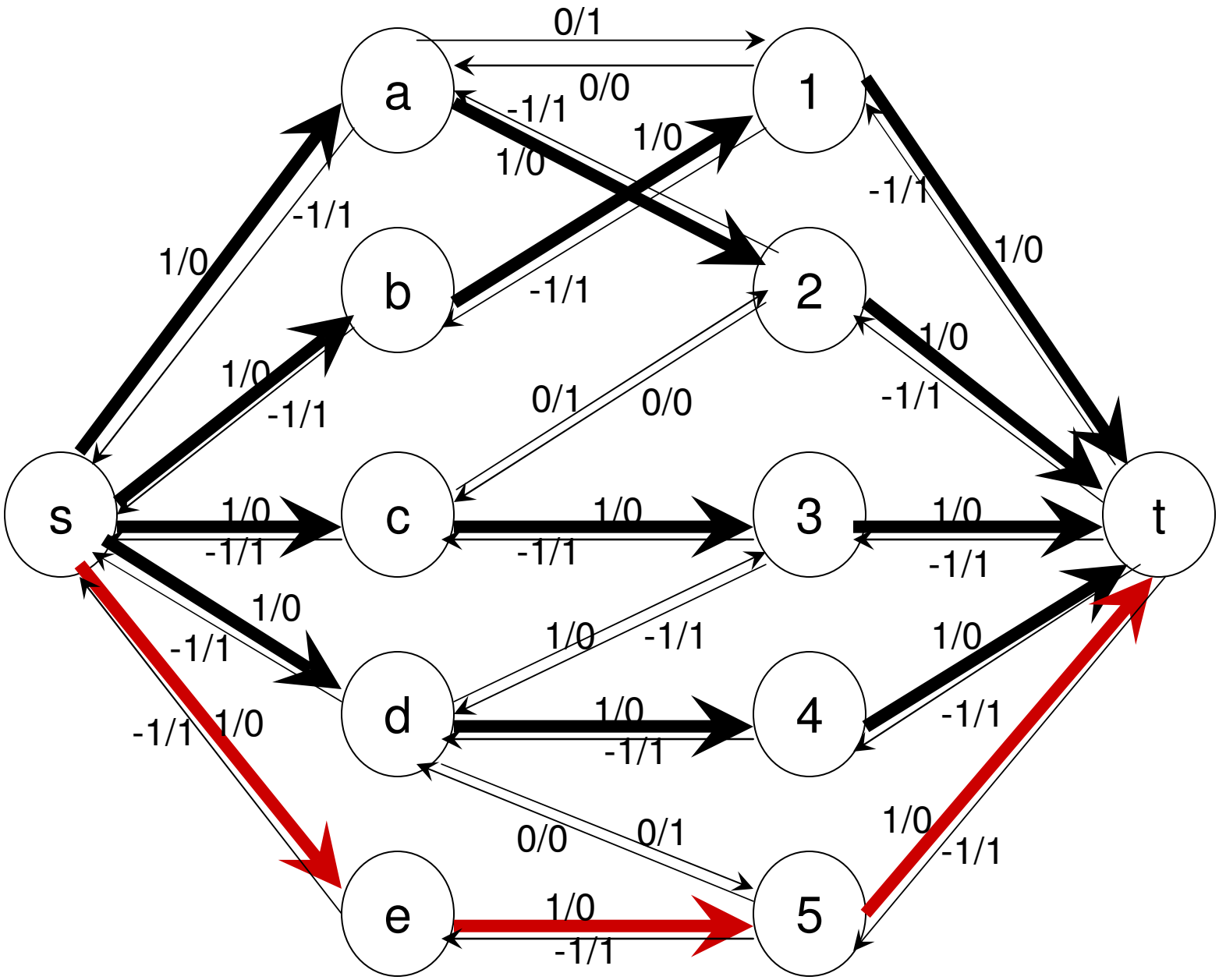
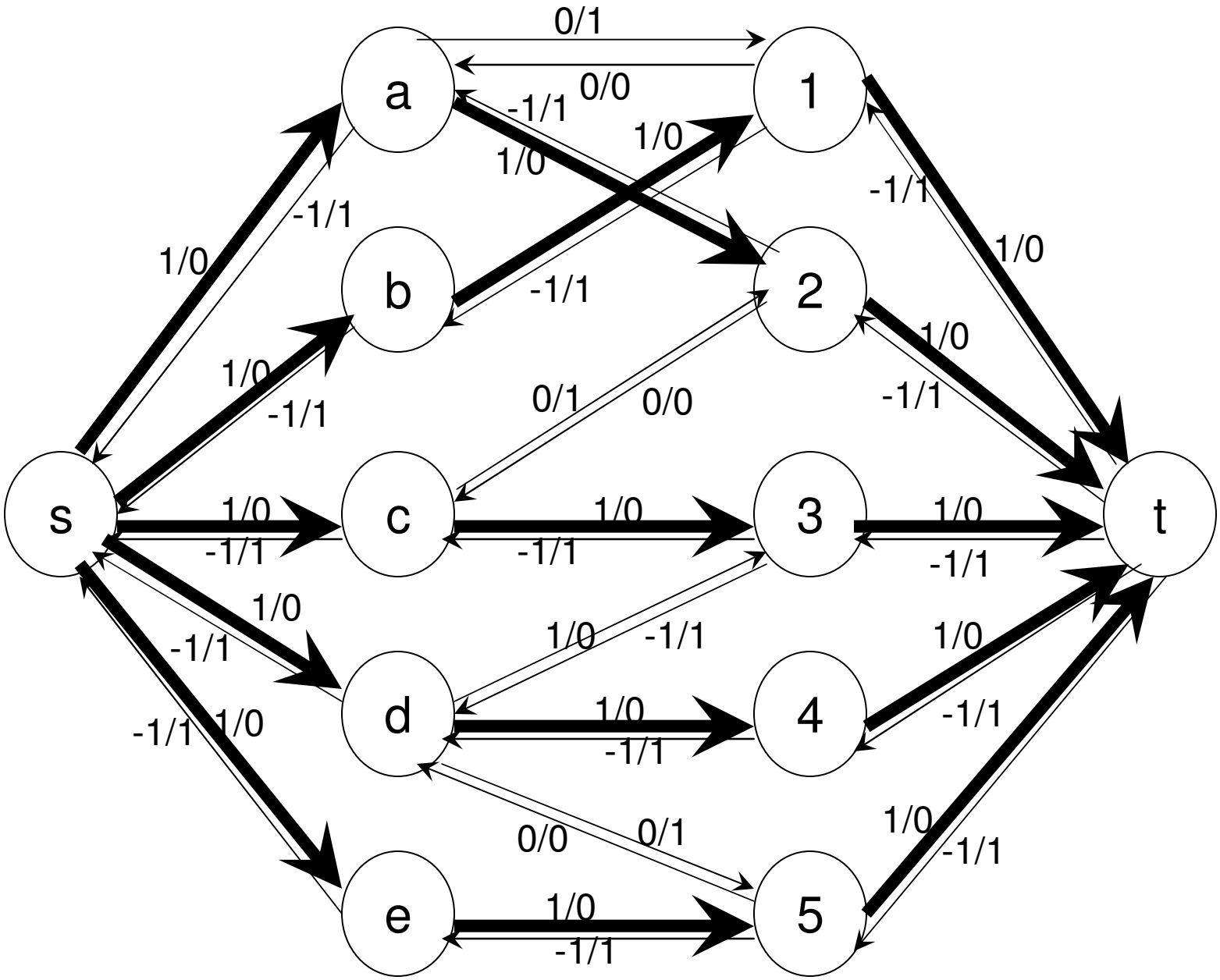
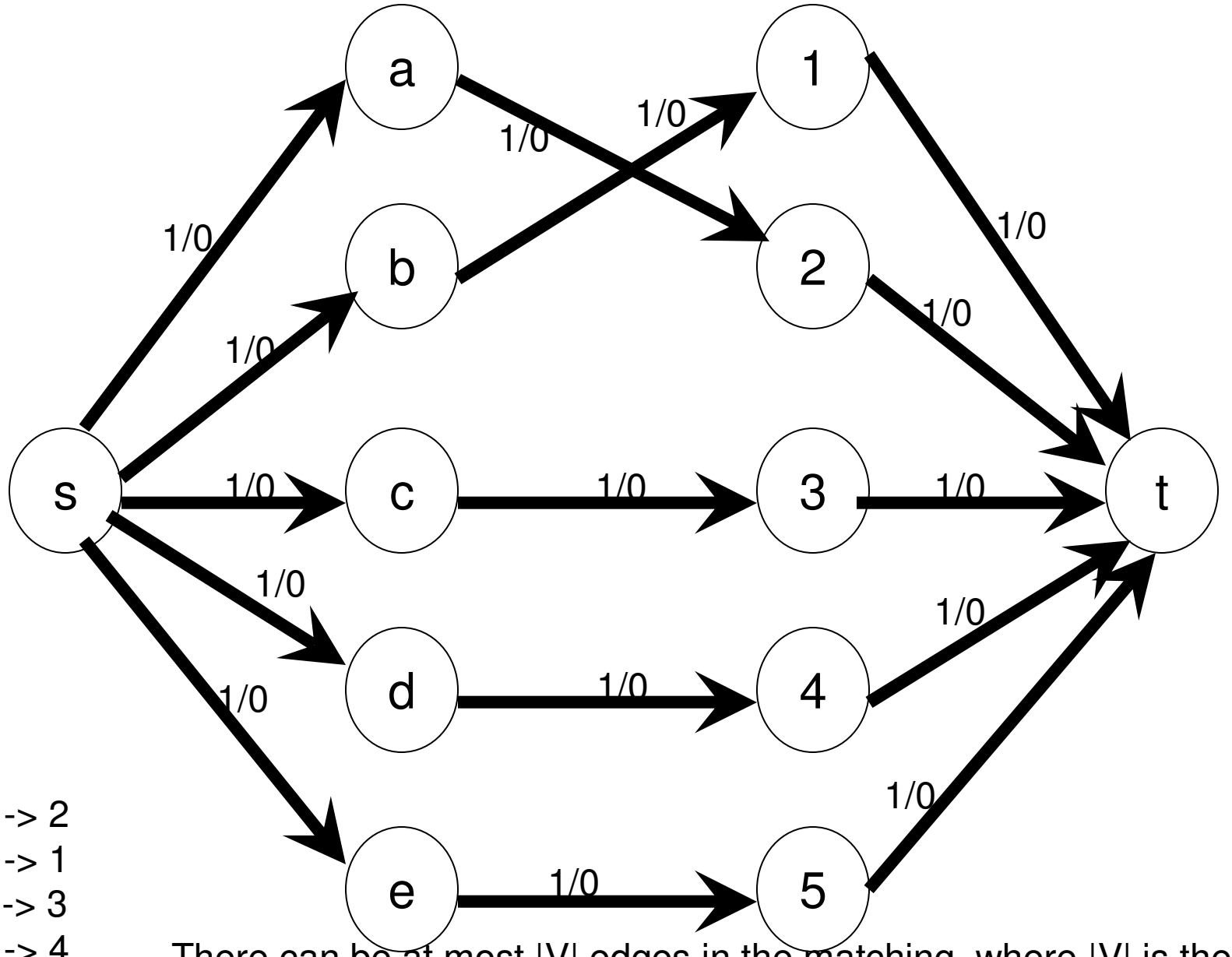
b
2
s
c
3
t
d
4
e
5
0/1
0/0
-1/1
1/0

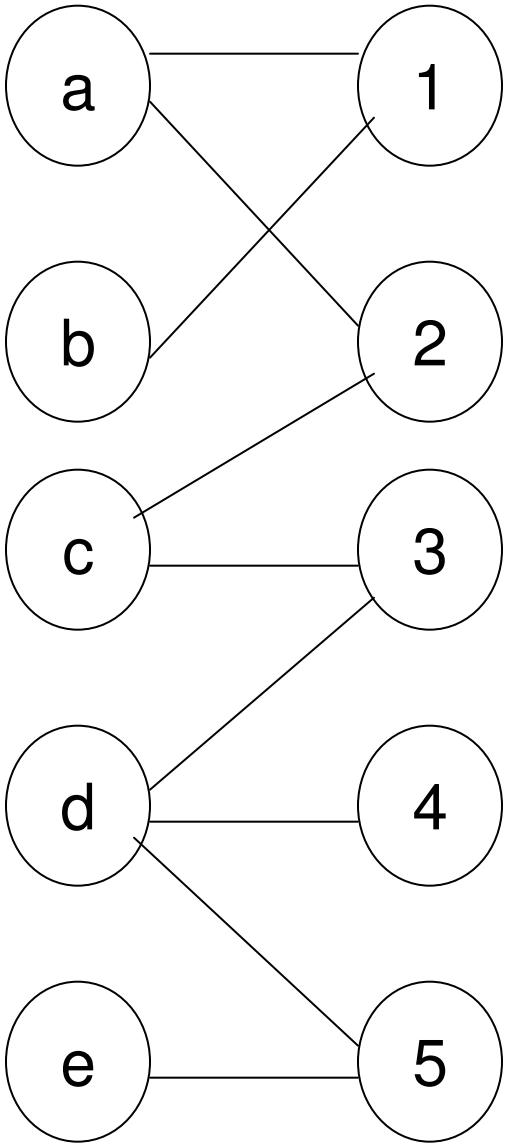

a
1
b
2
c
3
d
4
e
5
s
t
0/1
0/0
-1/1
1/0


a
1
b
2
c
3
d
4
e
5
s
t
0/1
0/0
-1/1
1/0

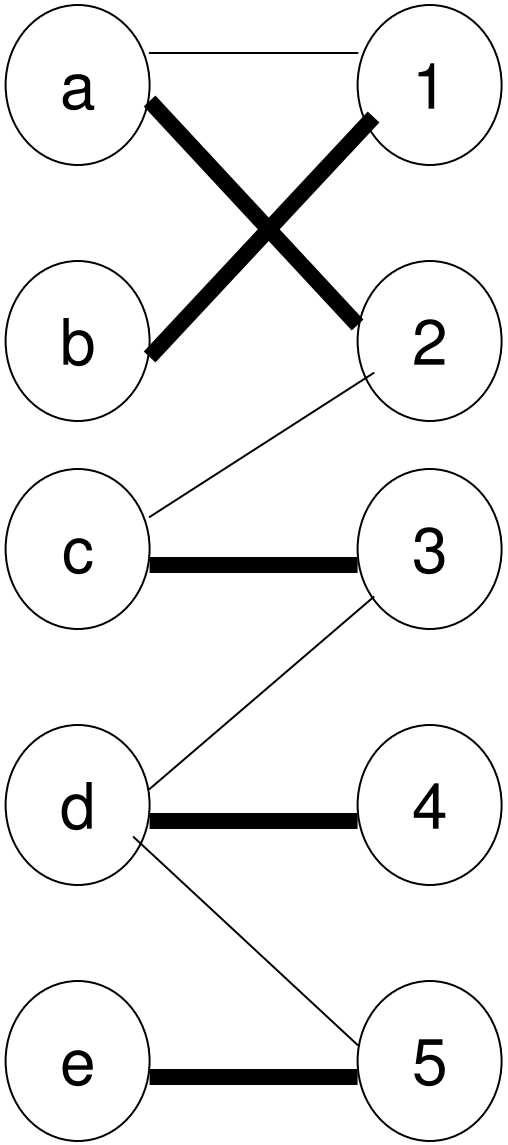


a -> 2
b -> 1
c -> 3
d -> 4
e -> 5

There can be at most |V| edges in the matching, where |V| is the maximum cardinality of the two disjoint sets that form the bi-partite graph. The Θ(V+E) BFS algorithm is run at most V times, leading to an overall time complexity of $\Theta(V*(V+E)) = \Theta(V^2 + EV)$



Bipartite Graph

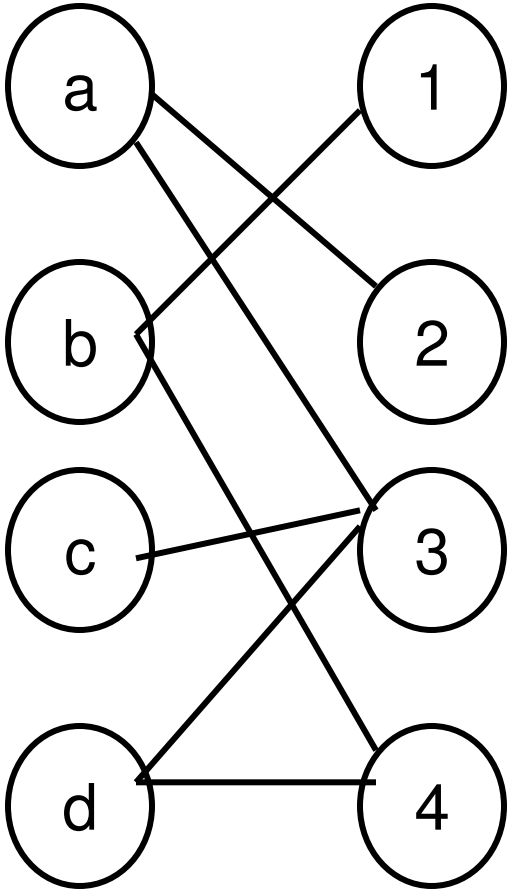


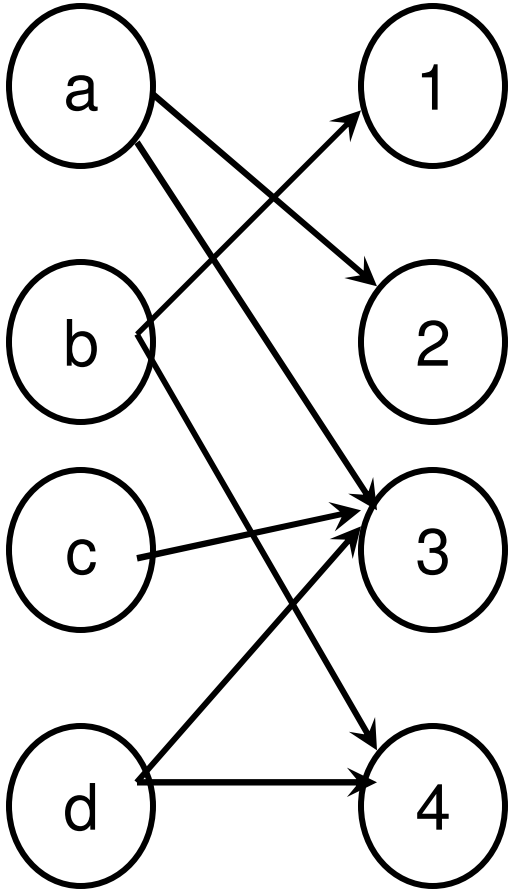
Maximum Matching on the Bipartite Graph (5 edges)

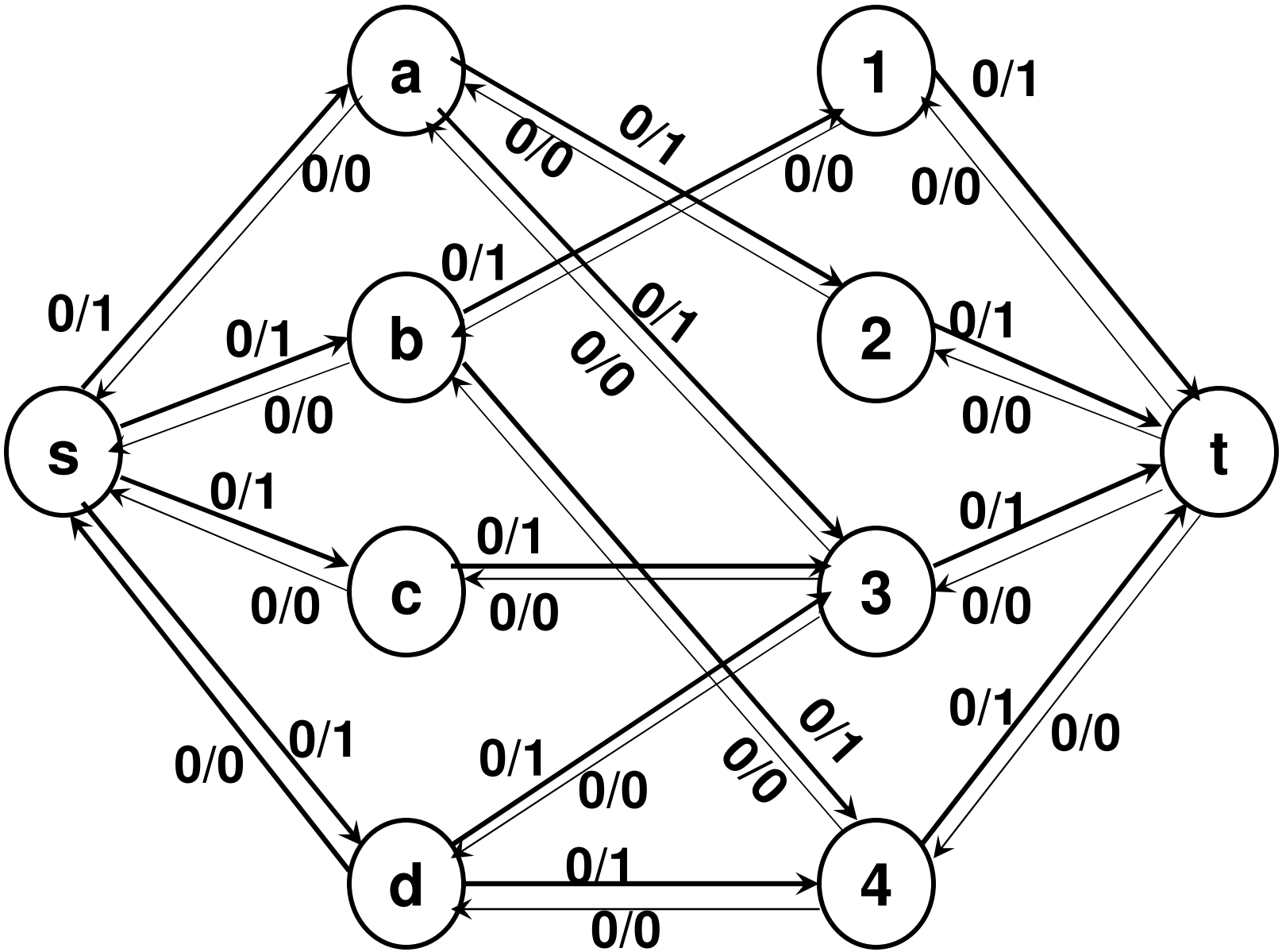
EXAMPLE 2

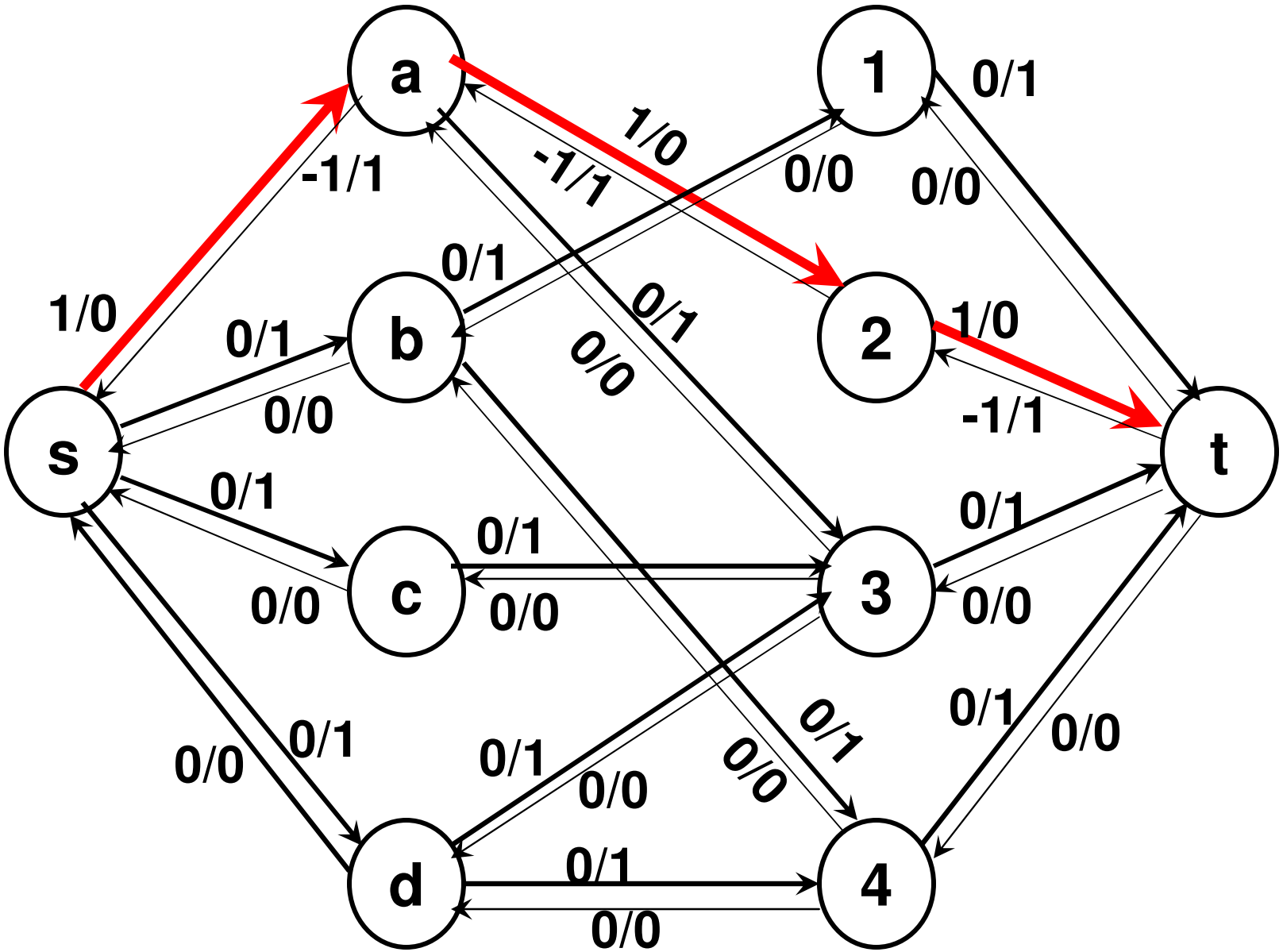
Given the following edges, identify a maximum bipartite matching

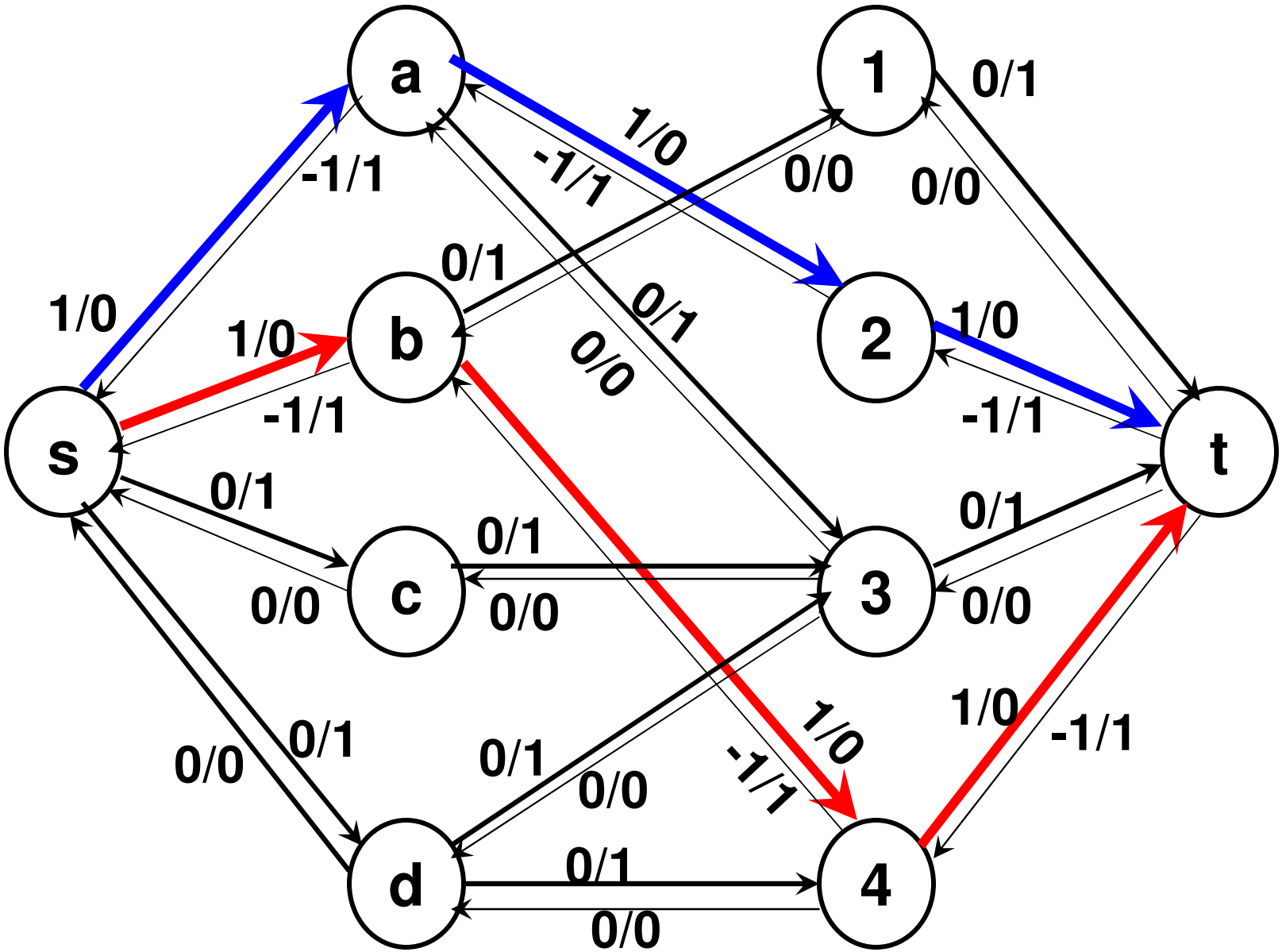
a: 2, 3
b: 1, 4
c: 3
d: 3, 4






a
1
b
2
s
t
c
3
d
4
0/1
0/0


a
1
s
b
2
t
c
3
d
4
1/0
-1/1
0/1
0/0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1
s
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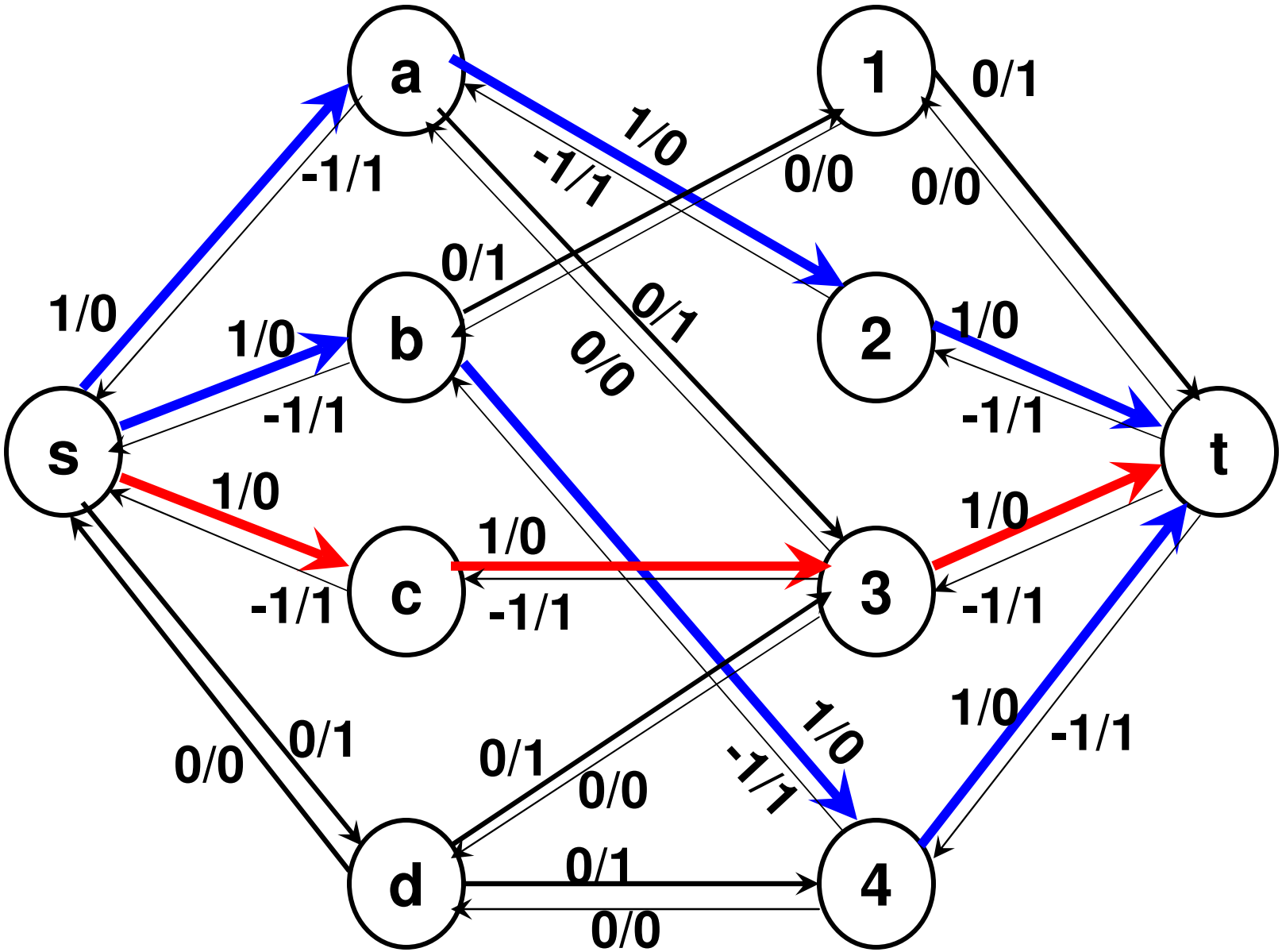
2
t
c
3
d
4
1/0
0/1
-1/1
0/0


a
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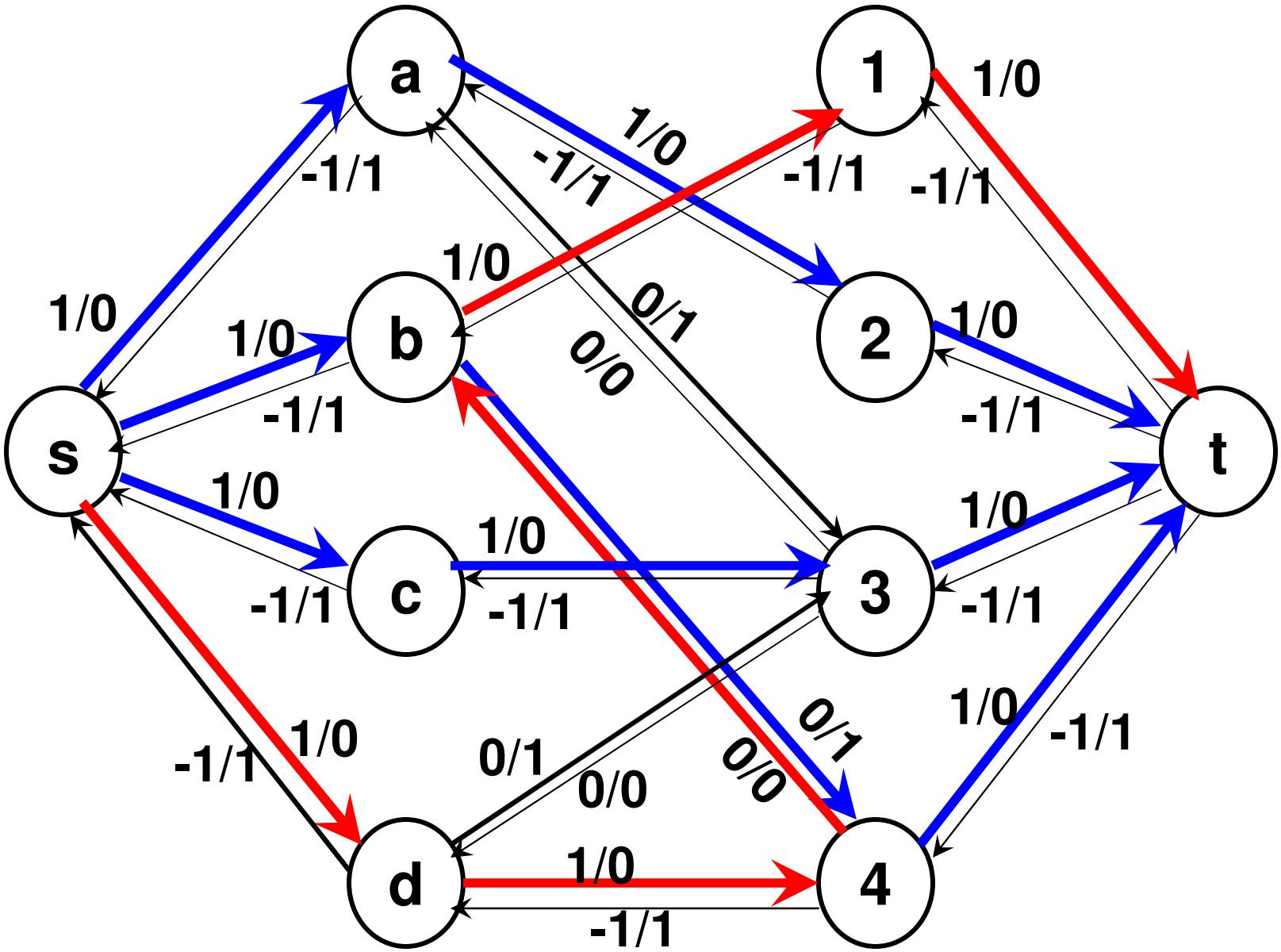
s
b
2
t
c
3
d
4
1/0
-1/1
0/1
0/0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s
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t
c
3
d
4
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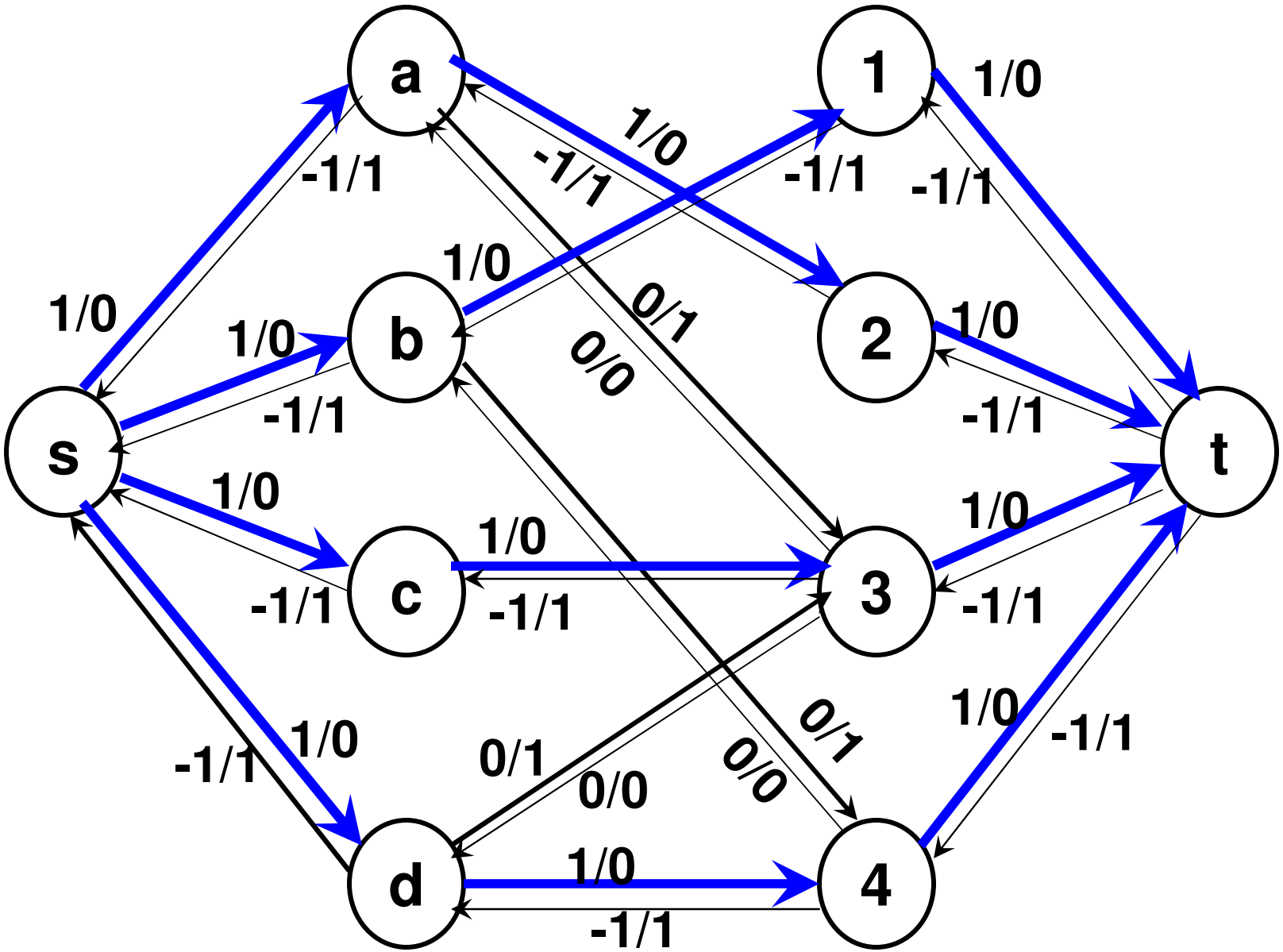
0/1
0/0


a
1
s
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c
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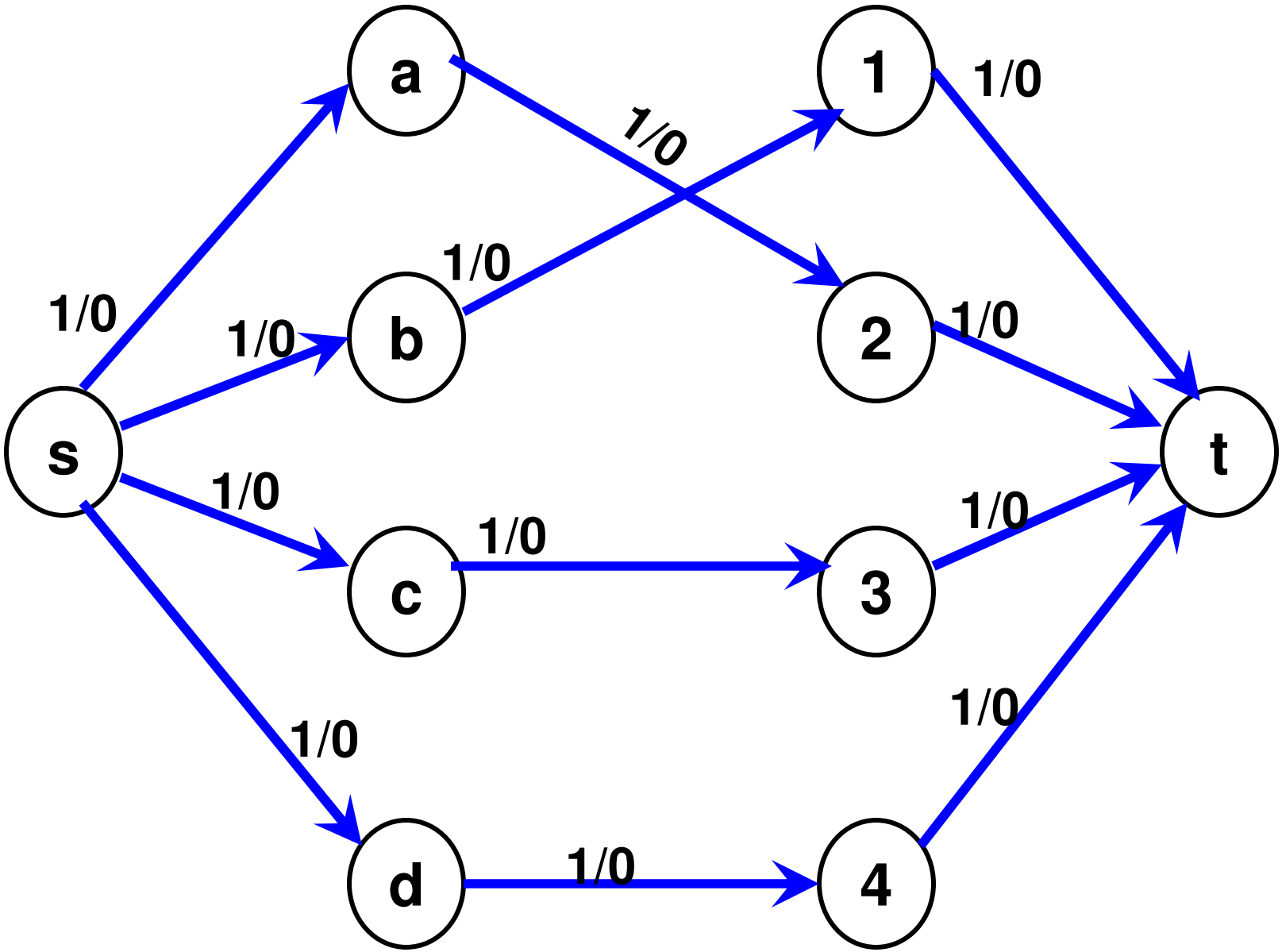
d
4
1/0
-1/1
0/1
0/0


a
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1/0
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1/0
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s
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1/0
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c
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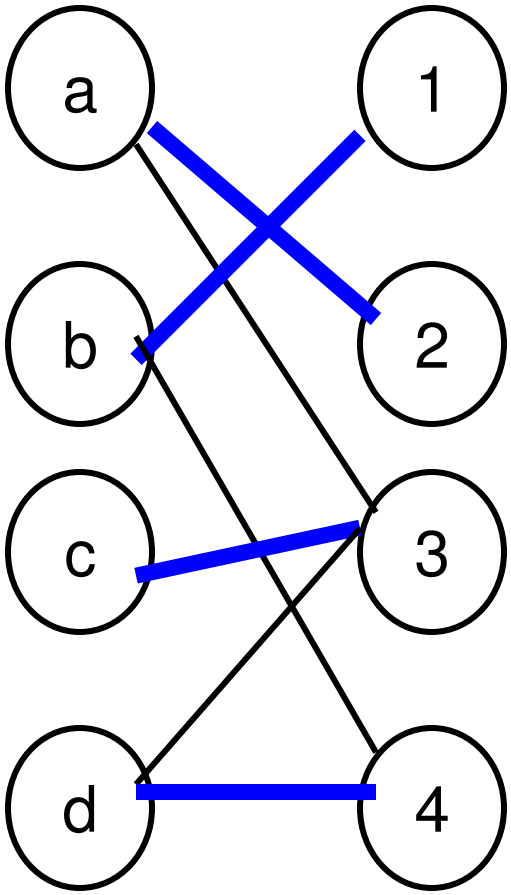
1/0
d
4



Maximum Bipartite Matching