# Module 2:
# Divide and Conquer

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## Introduction to Divide and Conquer

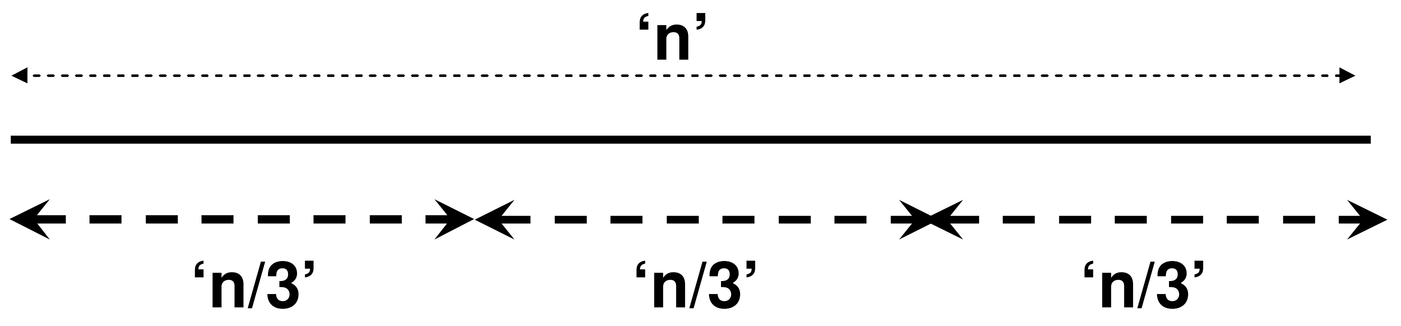
- Divide and Conquer is an algorithm design strategy of dividing a problem into sub problems, solving the sub problems and merging the solutions of the sub problems to get a solution for the larger problem.

- Let a problem space of size ‘n’ (for example: an n-element array used for sorting) be divided into sub problems of size ‘n/b’ each, which could be either overlapping or non-overlapping.
- Let us say we solve ‘a’ of these sub problems of size n/b.
- Let f(n) represent the time complexity of merging the solutions of the sub problems to get a solution for the larger problem.
- The general format of the recurrence relation can be then written as follows: where T(n/b) is the time complexity to solve a sub problem of size n/b and T(n) is the overall time complexity to solve a problem of size n.

$$T(n) = a * T(n/b) + f(n)$$

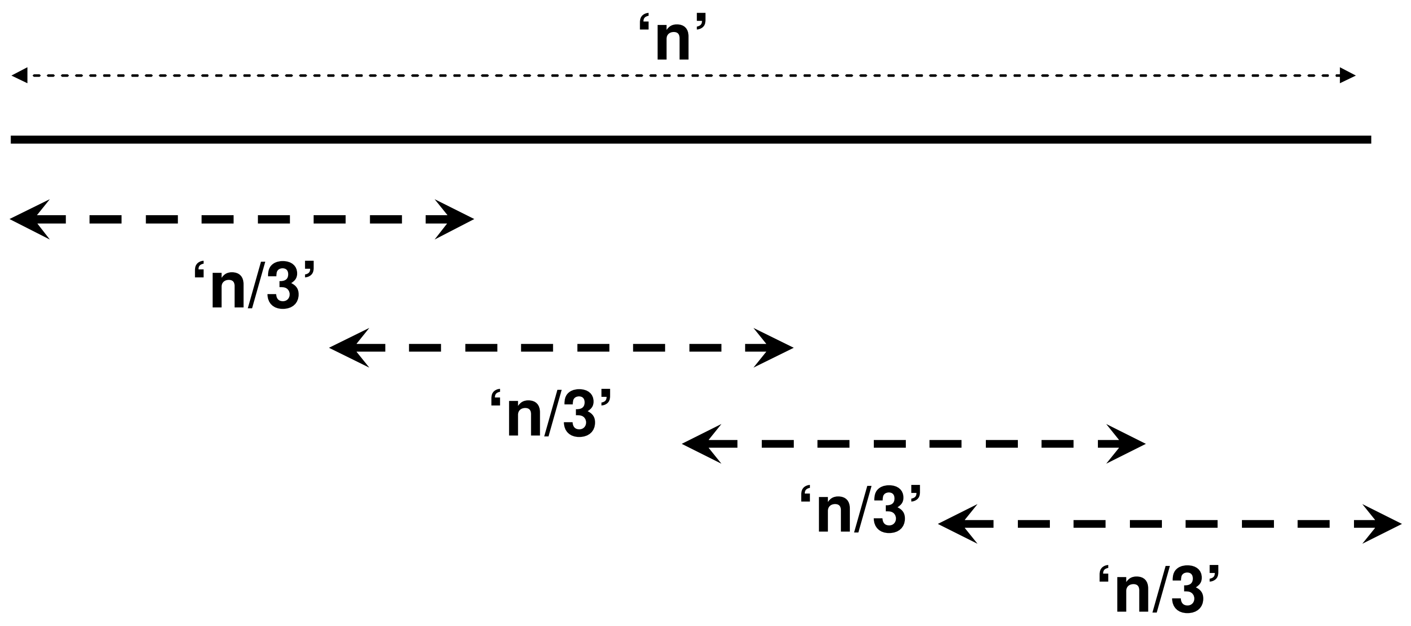
# Recurrence Relations for Divide and Conquer

## Non-Overlapping Sub Problems



T(n)
= 3 * T (n/3)
+ f(n)

## Overlapping Sub Problems (a ≠ b)



T(n)
= 4 * T (n/3)
+ f(n)

# Master Theorem to Solve Recurrence Relations: T(n) = a * T(n/b) + f(n)

**Master Theorem (Θ - version)**

If $f(n) \in \Theta(n^d)$ where $d \geq 0$, then

$$T(n) \in \begin{cases} \Theta(n^d) & \text{if } a < b^d, \\ \Theta(n^d \log n) & \text{if } a = b^d, \\ \Theta(n^{\log_b a}) & \text{if } a > b^d. \end{cases}$$

**Note: To satisfy the definition of a polynomial, 'd' should be a non-negative integer.**

Note: To apply Master Theorem, the function **f(n)** should be a **polynomial and should be monotonically increasing**

If $\begin{matrix} f(n) \notin \Theta(n^d); but \\ f(n) \in O(n^d) \end{matrix}$, then

$$T(n) \in \begin{cases} O(n^d) & \text{if } a < b^d, \\ O(n^d \log n) & \text{if } a = b^d, \\ O(n^{\log_b a}) & \text{if } a > b^d. \end{cases}$$

**Master Theorem (O - version)**

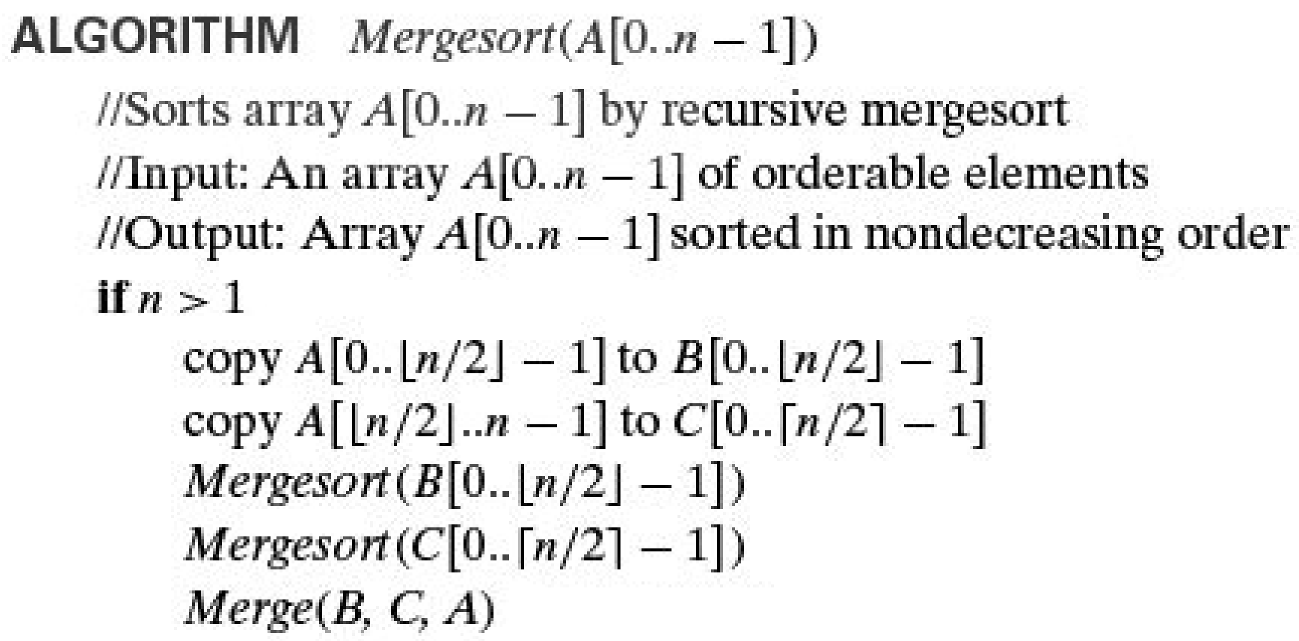
where d ≥ 0 and an integer

Note: We will try to apply the **Θ – version** wherever possible. If the **Θ – version** cannot be applied, we will try to apply the **O-version**.

## Merge Sort

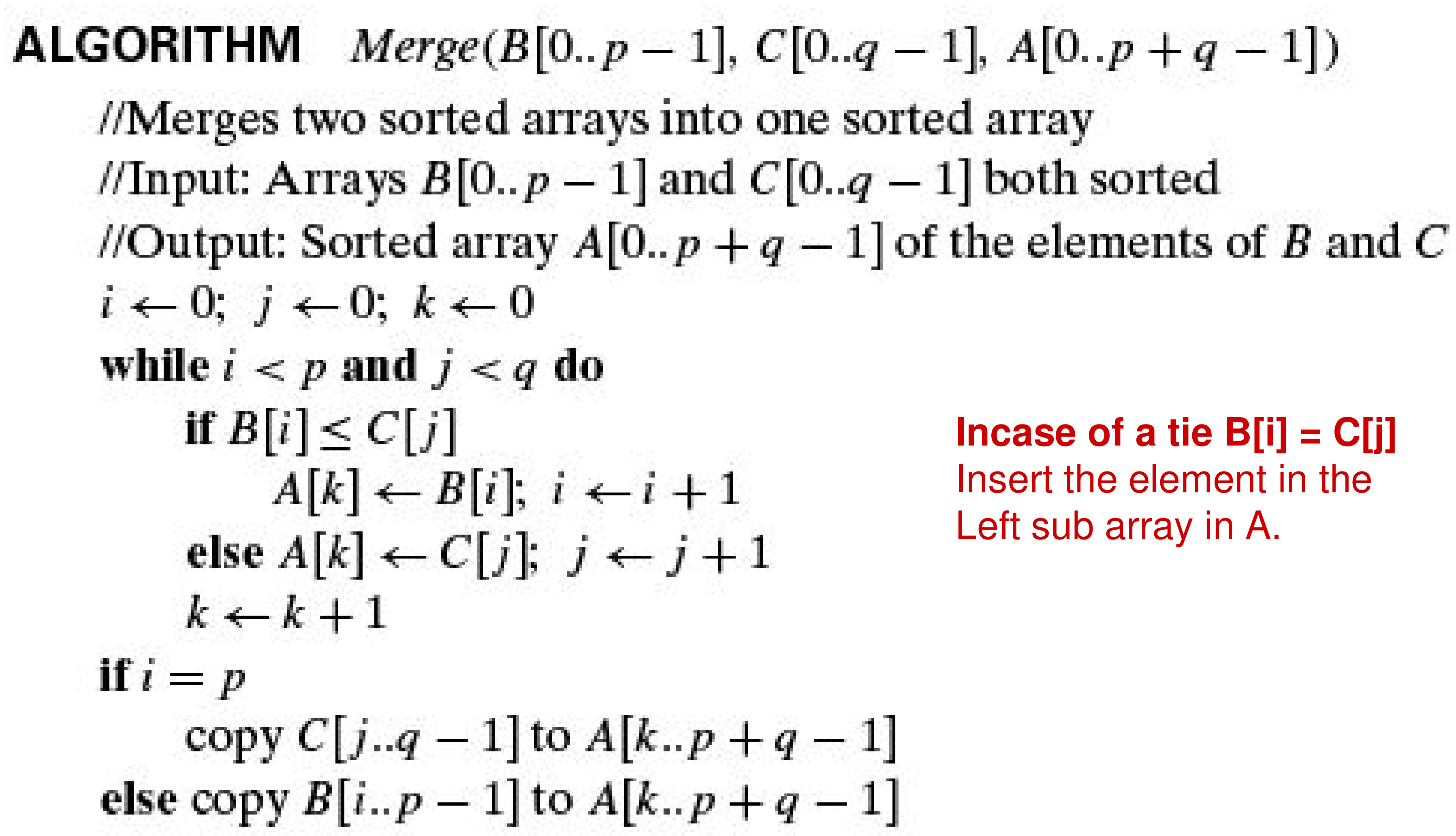
- Split array A[0..*n*-1] in two about equal halves and make copies of each half in arrays B and C
- Sort arrays B and C recursively
- Merge sorted arrays B and C into array A as follows:
  - Repeat the following until no elements remain in one of the arrays:
    - compare the first elements in the remaining unprocessed portions of the arrays
    - copy the smaller of the two into A, while incrementing the index indicating the unprocessed portion of that array
  - Once all elements in one of the arrays are processed, copy the remaining unprocessed elements from the other array into A.

# Merge Sort



```
ALGORITHM  Mergesort($A[0..n-1]$)
    //Sorts array $A[0..n-1]$ by recursive mergesort
    //Input: An array $A[0..n-1]$ of orderable elements
    //Output: Array $A[0..n-1]$ sorted in nondecreasing order
    if $n > 1$
        copy $A[0..\lfloor n/2 \rfloor - 1]$ to $B[0..\lfloor n/2 \rfloor - 1]$
        copy $A[\lfloor n/2 \rfloor..n-1]$ to $C[0..\lceil n/2 \rceil - 1]$
        Mergesort($B[0..\lfloor n/2 \rfloor - 1]$)
        Mergesort($C[0..\lceil n/2 \rceil - 1]$)
        Merge($B, C, A$)
```

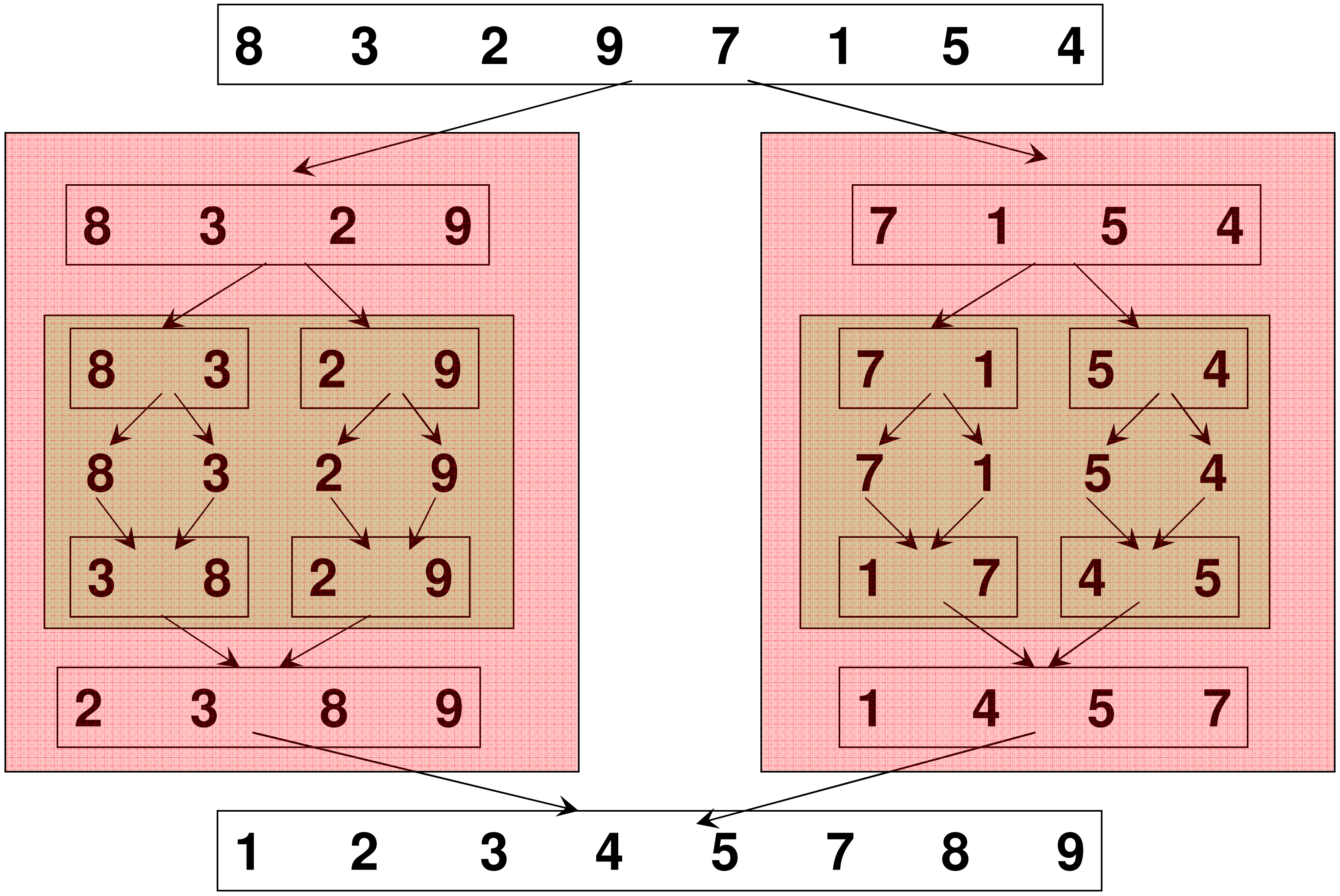
# Merge Algorithm



```
**ALGORITHM** $Merge(B[0..p-1], C[0..q-1], A[0..p+q-1])$
    //Merges two sorted arrays into one sorted array
    //Input: Arrays $B[0..p-1]$ and $C[0..q-1]$ both sorted
    //Output: Sorted array $A[0..p+q-1]$ of the elements of $B$ and $C$
    $i \leftarrow 0$;  $j \leftarrow 0$;  $k \leftarrow 0$
    **while** $i < p$ **and** $j < q$ **do**
        **if** $B[i] \leq C[j]$
            $A[k] \leftarrow B[i]$;  $i \leftarrow i + 1$
        **else** $A[k] \leftarrow C[j]$;  $j \leftarrow j + 1$
        $k \leftarrow k + 1$
    **if** $i = p$
        copy $C[j..q-1]$ to $A[k..p+q-1]$
    **else** copy $B[i..p-1]$ to $A[k..p+q-1]$
```

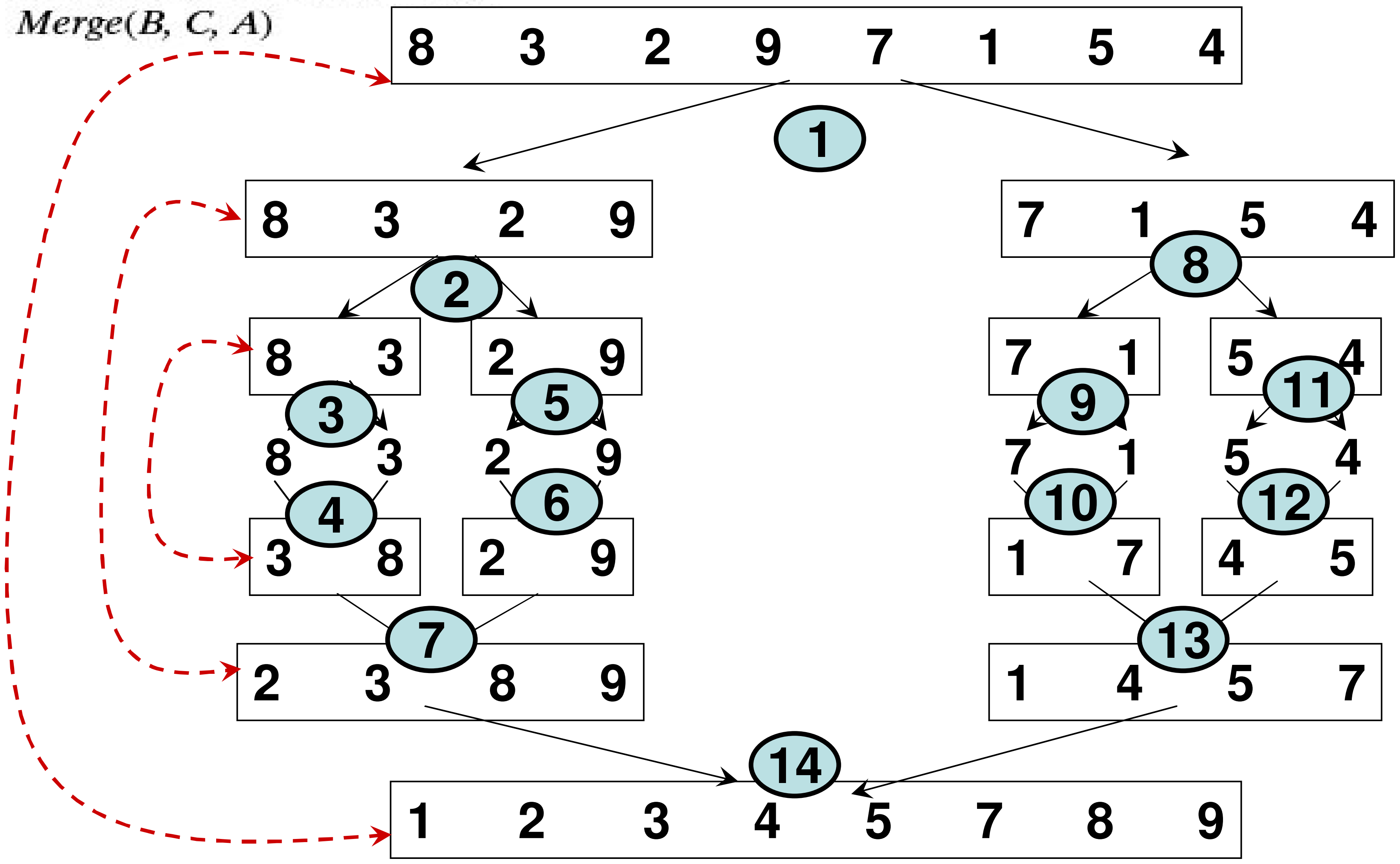
**Incase of a tie B[i] = C[j]**
Insert the element in the
Left sub array in A.

## Example for Merge Sort



# The order recursion runs

```
if $n > 1$
    copy $A[0..\lfloor n/2 \rfloor - 1]$ to $B[0..\lfloor n/2 \rfloor - 1]$
    copy $A[\lfloor n/2 \rfloor..n-1]$ to $C[0..\lceil n/2 \rceil - 1]$
    $Mergesort(B[0..\lfloor n/2 \rfloor - 1])$
    $Mergesort(C[0..\lceil n/2 \rceil - 1])$
    $Merge(B, C, A)$
```



# Analysis of Merge Sort

The recurrence relation for the number of key comparisons C(n) is

$$C(n) = 2C(n/2) + C_{merge}(n) \quad \text{for } n > 1,\ C(1) = 0$$

At each step, exactly one comparison is made, after which the total number of elements in the two arrays still needed to be processed is reduced by one.

**Best case:** We will encounter n/2 comparisons (when every element in the left sorted sub array is less than or equal to the first element in right sorted sub array)

**Worst case:** We will encounter (n-1) comparisons (when smaller elements come from the alternating sub arrays; neither of the two sub arrays will become empty before the other sub array contains just one element.

As best case is different from worst case,
the time complexity to merge: $C_{merge}(n) = O(n-1) = O(n)$

$C(n) = 2*C(n/2) + O(n)$ for $n > 1$ and $C(1) = 0$

$a = 2;\ b = 2;\ d = 1$

$a = b^d$

$\boxed{\text{Hence, } C(n) = O(n\log n)}$

## Merge Sort: Space-time Tradeoff

- Unlike the sorting algorithms (insertion sort, bubble sort, selection sort) we saw in Module 1, Merge sort incurs a time-complexity of O(nlogn), whereas the other sorting algorithms we have seen incur a time complexity of $O(n^2)$ or $\Theta(n^2)$ .

- The tradeoff is Merge sort requires additional space proportional to the size of the array being sorted. That is, the space-complexity of merge sort is $\Theta(n)$, whereas the other sorting algorithms we have seen incur a space-complexity of $\Theta(1)$.
  - Algorithms that incur a $\Theta(1)$ space complexity are said to be "**in place**"

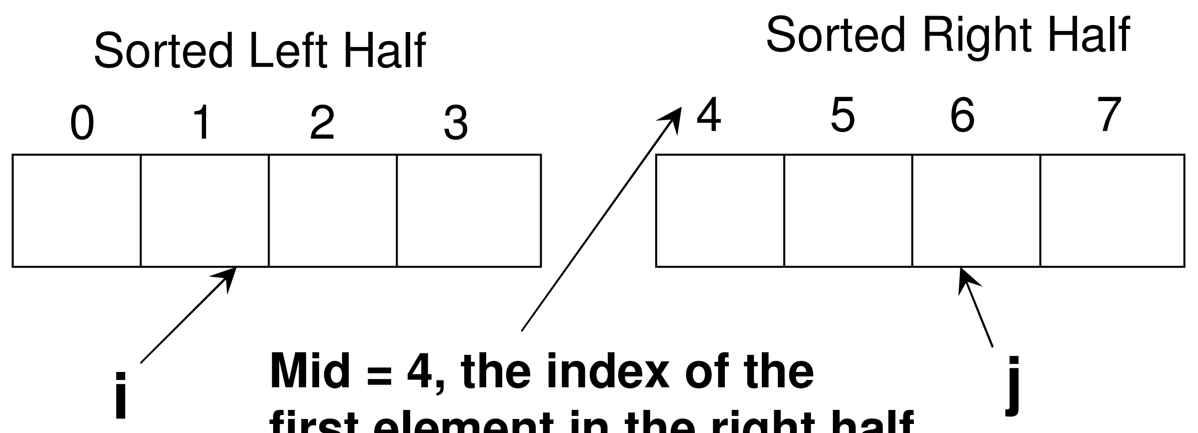
# Number of Inversions in an Array

- Given an array A, an inversion is said to have occurred if i < j and A[i] > A[j].
- **Example**

| 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| **2** | **8** | **1** | **9** | **3** | **7** |

Inverted Pairs
(2, 1)
(8, 1)
(8, 3)
(8, 7)
(9, 3)
(9, 7)

**The number of inversions in an array can be computed as the Sum of the number of inversions encountered in each of the Merging steps of the Merge Sort algorithm.**



If A[i] > A[j], then everything to the right of Index in the sorted left half are also going to be greater than A[j]. Hence, the number of inversions due to A[i] > A[j] is:
**Mid – i.**

# # Inversions in the Merge Step (Ex.2)

**Mid = 5**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **14** | **17** | **19** | **22** | **25** | **13** | **16** | **18** | **20** | **27** |

| | | | | | | | Sorted Array | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Index i | Index j | A[i] | A[j] | Inv Y/N | # Inv | | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 0 | 5 | 14 | 13 | Y | 5 - 0 = 5 | | 13 | | | | | | | | | |
| 0 | 6 | 14 | 16 | N | - | | 13 | 14 | | | | | | | | |
| 1 | 6 | 17 | 16 | Y | 5 - 1 = 4 | | 13 | 14 | 16 | | | | | | | |
| 1 | 7 | 17 | 18 | N | - | | 13 | 14 | 16 | 17 | | | | | | |
| 2 | 7 | 19 | 18 | Y | 5 - 2 = 3 | | 13 | 14 | 16 | 17 | 18 | | | | | |
| 2 | 8 | 19 | 20 | N | - | | 13 | 14 | 16 | 17 | 18 | 19 | | | | |
| 3 | 8 | 22 | 20 | Y | 5 - 3 = 2 | | 13 | 14 | 16 | 17 | 18 | 19 | 20 | | | |
| 3 | 9 | 22 | 27 | N | - | | 13 | 14 | 16 | 17 | 18 | 19 | 20 | 22 | | |
| 4 | 9 | 25 | 27 | N | - | | 13 | 14 | 16 | 17 | 18 | 19 | 20 | 22 | 25 | |
| - | 9 | - | 27 | N | - | | 13 | 14 | 16 | 17 | 18 | 19 | 20 | 22 | 25 | 27 |

**# Inversions in the Merging Step** **= 5 + 4 + 3 + 2**
**= 14**

# # Inversions in the Merge Step (Ex.2)

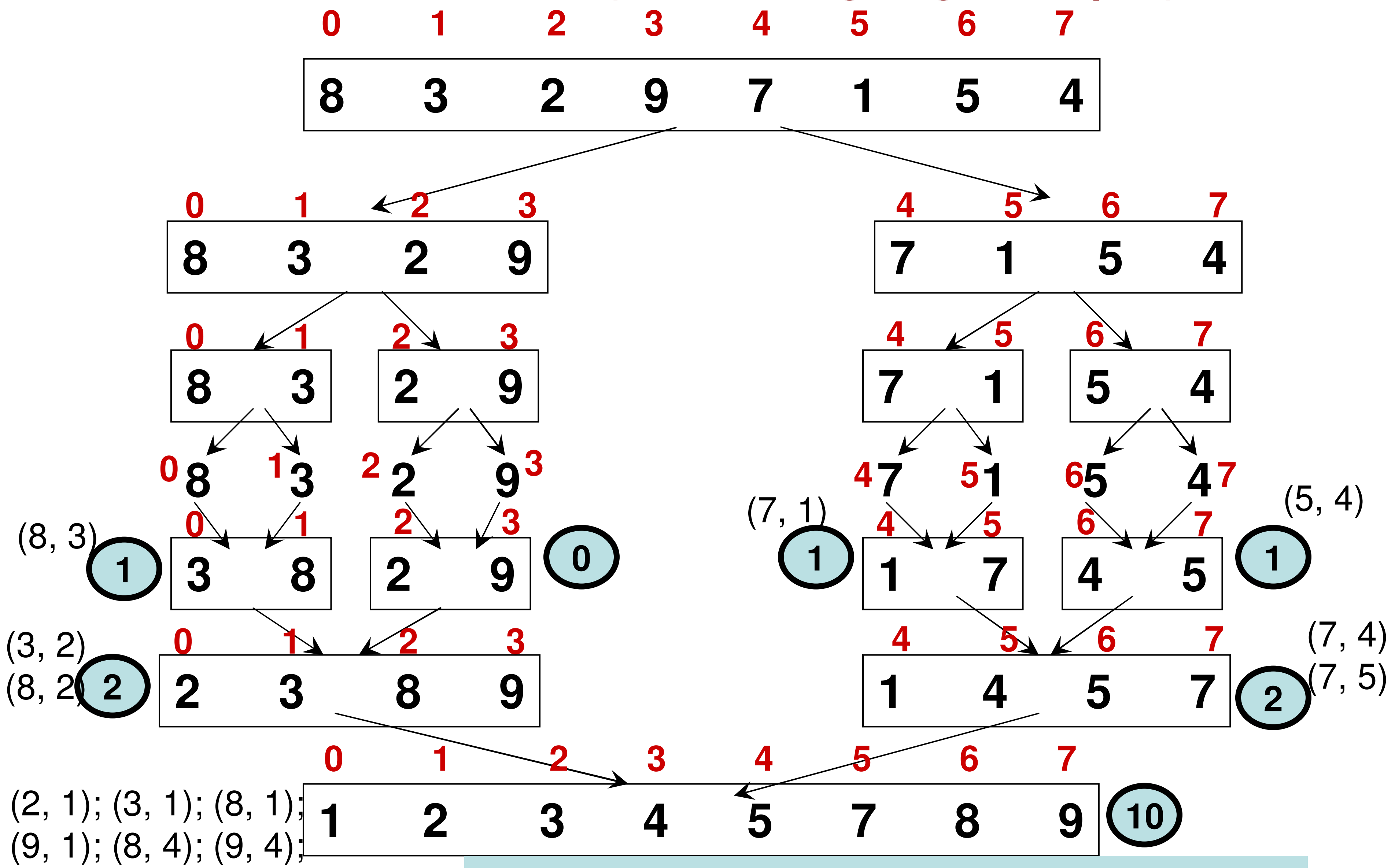
**Mid = 5**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **2** | **3** | **8** | **9** | **10** | **1** | **4** | **5** | **7** | **8** |

| | | | | | | | Sorted Array | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Index i | Index j | A[i] | A[j] | Inv Y/N | # Inv | | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 0 | 5 | 2 | 1 | Y | 5 - 0 = 5 | | 1 | | | | | | | | | |
| 0 | 6 | 2 | 4 | N | - | | 1 | 2 | | | | | | | | |
| 1 | 6 | 3 | 4 | N | - | | 1 | 2 | 3 | | | | | | | |
| 2 | 6 | 8 | 4 | Y | 5 - 2 = 3 | | 1 | 2 | 3 | 4 | | | | | | |
| 2 | 7 | 8 | 5 | Y | 5 - 2 = 3 | | 1 | 2 | 3 | 4 | 5 | | | | | |
| 2 | 8 | 8 | 7 | Y | 5 - 2 = 3 | | 1 | 2 | 3 | 4 | 5 | 7 | | | | |
| 2 | 9 | 8 | 8 | N | - | | 1 | 2 | 3 | 4 | 5 | 7 | 8 | | | |
| 3 | 9 | 9 | 8 | Y | 5 - 3 = 2 | | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 8 | | |
| 3 | - | 9 | - | - | - | | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 8 | 9 | 10 |

**# Inversions in the Merging Step** **= 5 + 3 + 3 + 3 + 2**
**= 16**

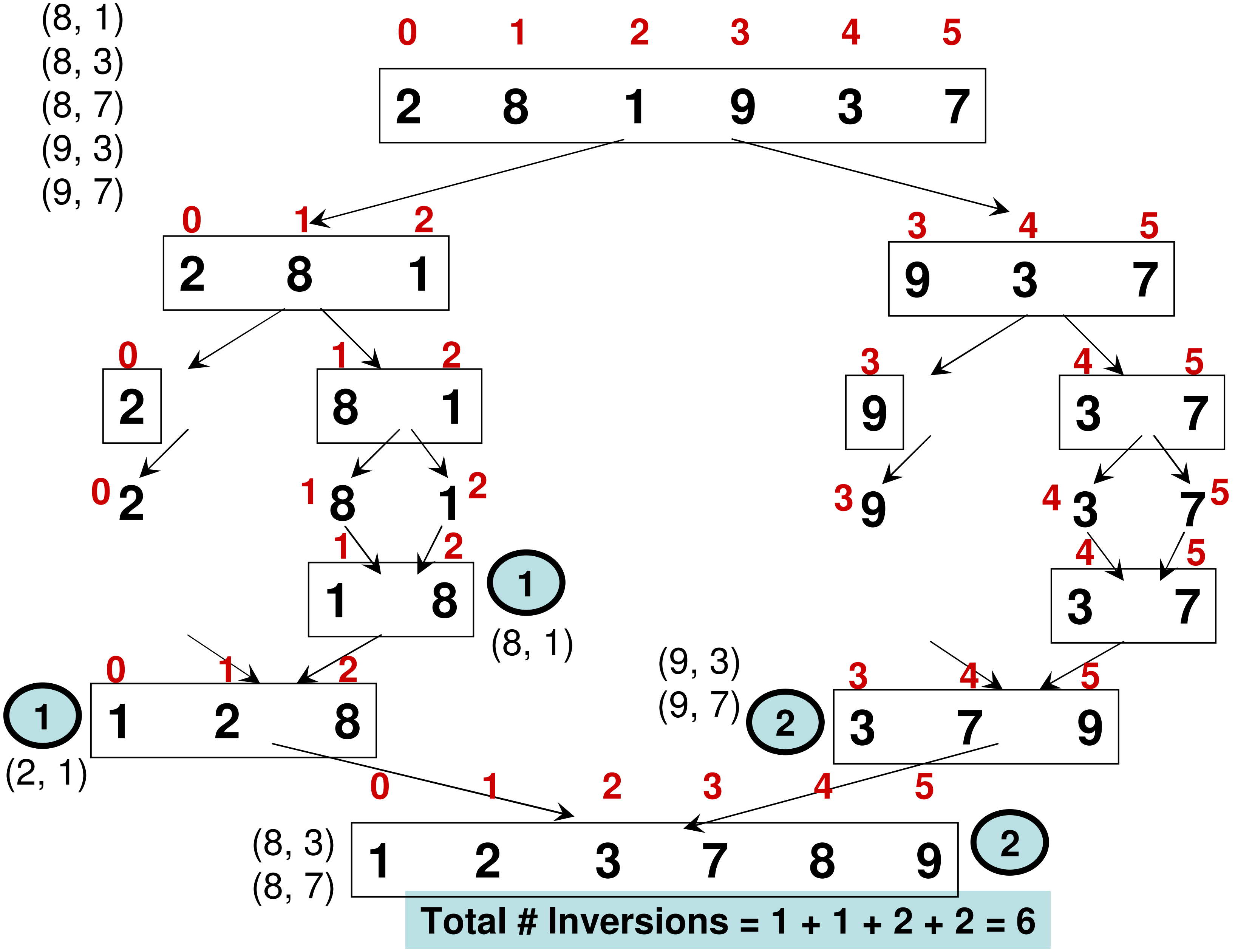
# Total # Inversions (all Merging Steps): Ex. 3



**Total # Inversions = 1 + 0 + 2 + 1 + 1 + 2 + 10 = 17**

# Total # Inversions: Ex. 4

Inverted Pairs
(2, 1)
(8, 1)
(8, 3)
(8, 7)
(9, 3)
(9, 7)



Total # Inversions = 1 + 1 + 2 + 2 = 6

# Finding the Maximum Integer in an Array: Recursive Divide and Conquer

Algorithm FindMaxIndex(Array A, int leftIndex, int rightIndex)

// returns the index of the maximum left in the array A for //index positions ranging from leftIndex to rightIndex

```
if (leftIndex = rightIndex)
        return leftIndex
middleIndex = (leftIndex + rightIndex)/2
leftMaxIndex = FindMaxIndex(A, leftIndex, middleIndex)
rightMaxIndex = FindMaxIndex(A, middleIndex + 1, rightIndex)
if A[leftMaxIndex] ≥ A[rightMaxIndex]
        return leftMaxIndex
else
        return rightMaxIndex
```

**Terminating Condition**

**Getting ready to divide**

**Divide part**

**Conquer part**

**Since we keep track of the index positions of the maximum element in the sub arrays, We do not need to create additional space. So, this algorithm is in-place.**

## Max Integer Index Problem: Time Complexity

$T(n) = 2{*}T(n/2) + 1$

i.e., $T(n) = 2{*}T(n/2) + \Theta(n^0)$

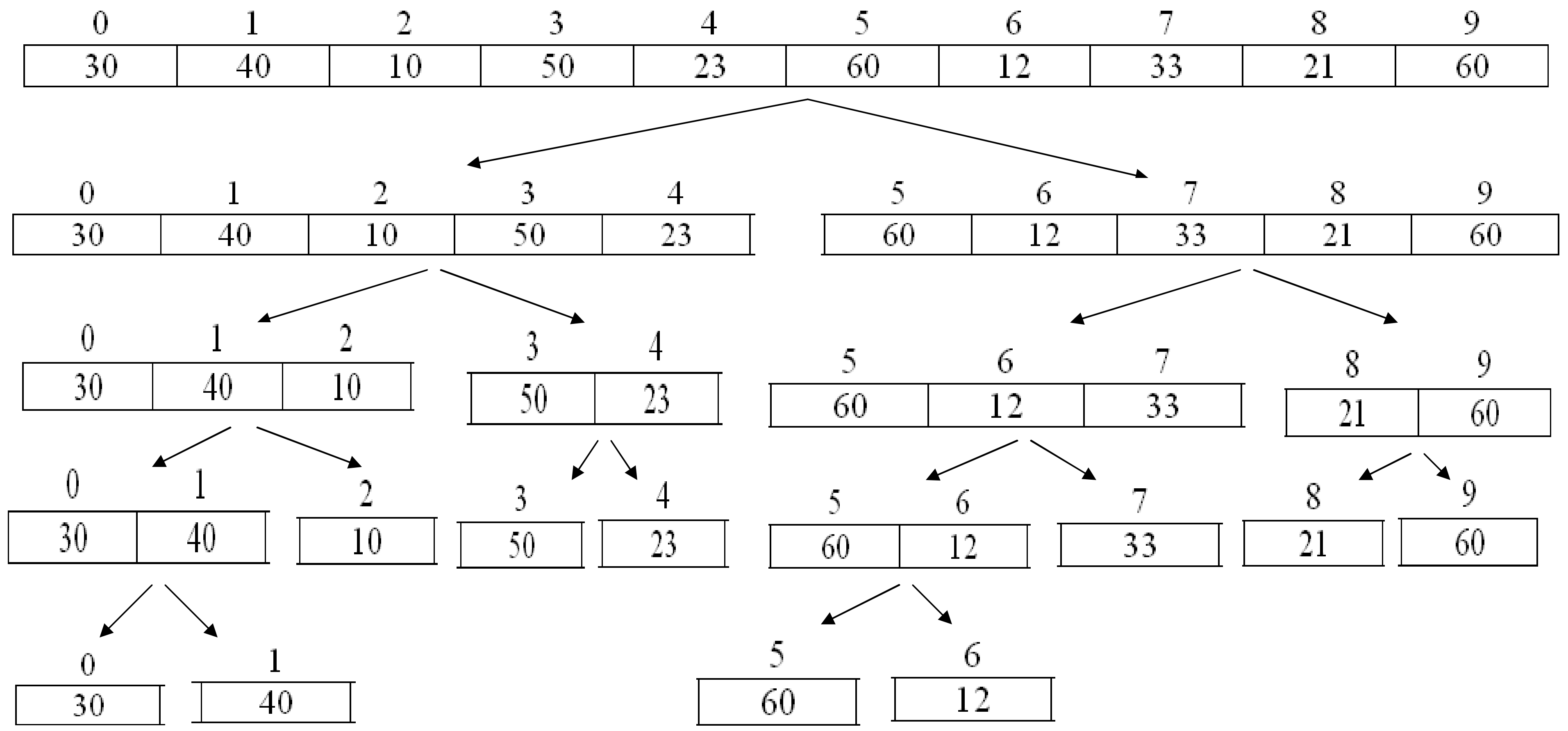
$a = 2, b = 2, d = 0$

$b^d = 2^0 = 1$. Hence, $a > b^d$

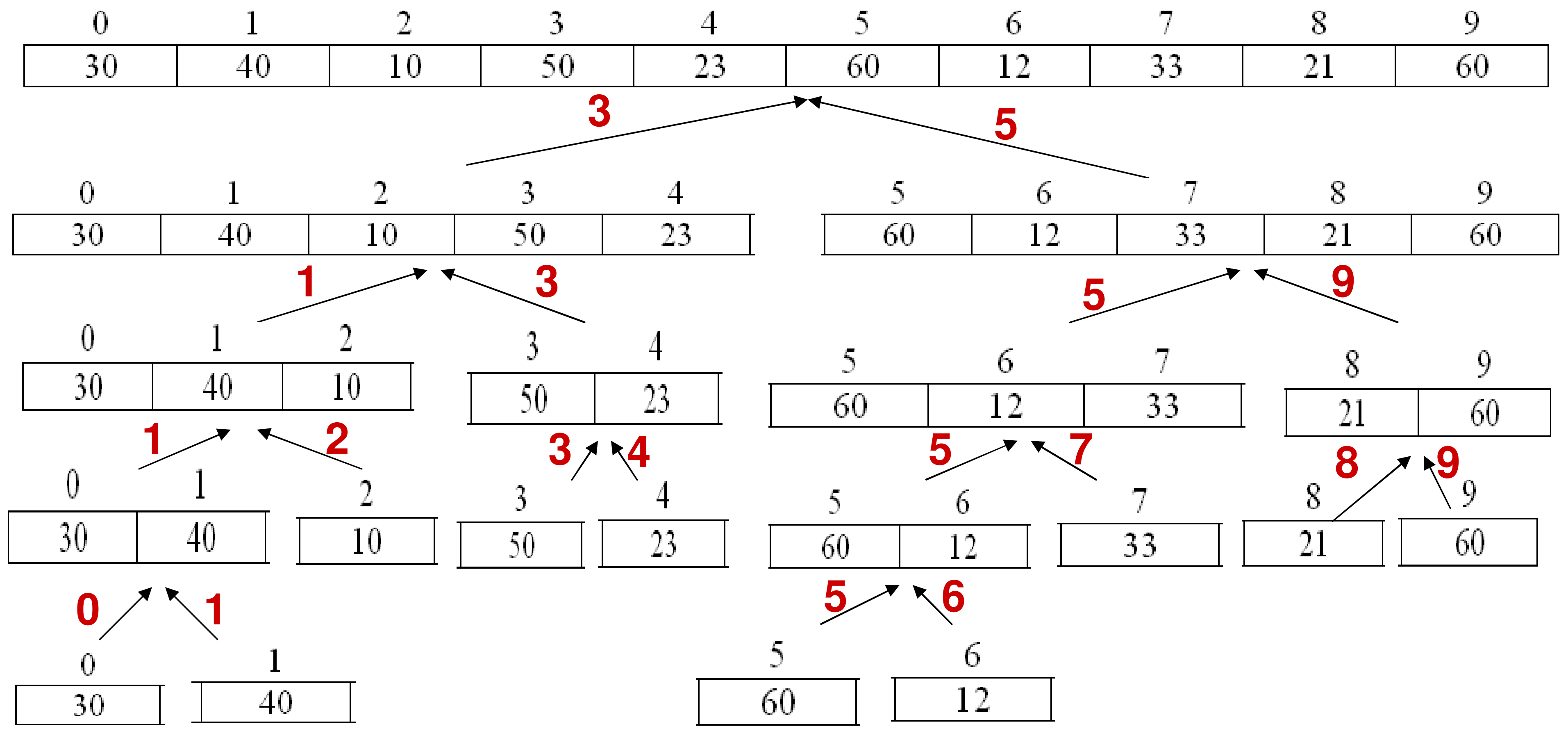
$$\mathbf{T(n) = \Theta(n^{\log_b(a)}) = = \Theta(n^{\log 2(2)}) = \Theta(n)}$$

Note that even an iterative approach would take Θ(n) time to compute the time-complexity. The overhead comes with recursion.

# FindMaxIndex: Example



# FindMaxIndex: Example (contd..)



# Binary Search

$$\begin{array}{ccc} & K & \\ & \updownarrow & \\ \underbrace{A[0]\ldots A[m-1]}_{\substack{\text{search here if}\\ K<A[m]}} & A[m] & \underbrace{A[m+1]\ldots A[n-1]}_{\substack{\text{search here if}\\ K>A[m]}} \end{array}$$

- Binary search is a Θ(log n) algorithm
  - **the array needs to be sorted**.
- Working Principle
  - **Define a range of indices, left index to right index, within which the search key could be there.**
    - For an array, left index = 0, right index = n-1
  - Run Iterations: In each iteration
    - find the middle index = (left index + right index) / 2
- It works by comparing a search key K with the array's middle element A[m]. If they match, the algorithm stops; otherwise, the same operation is repeated recursively for the first half of the array if K < A[m], and for the second half if K > A[m].
- Though binary search in based on a recursive idea, it can be easily implemented as a non-recursive algorithm.

# Binary Search

**Example**

**Search Key**
**K = 70**

| | | |
|---|---|---|
| l=0 | r=12 | m=6 |
| l=7 | r=12 | m=9 |
| l=7 | r=8 | m=7 |

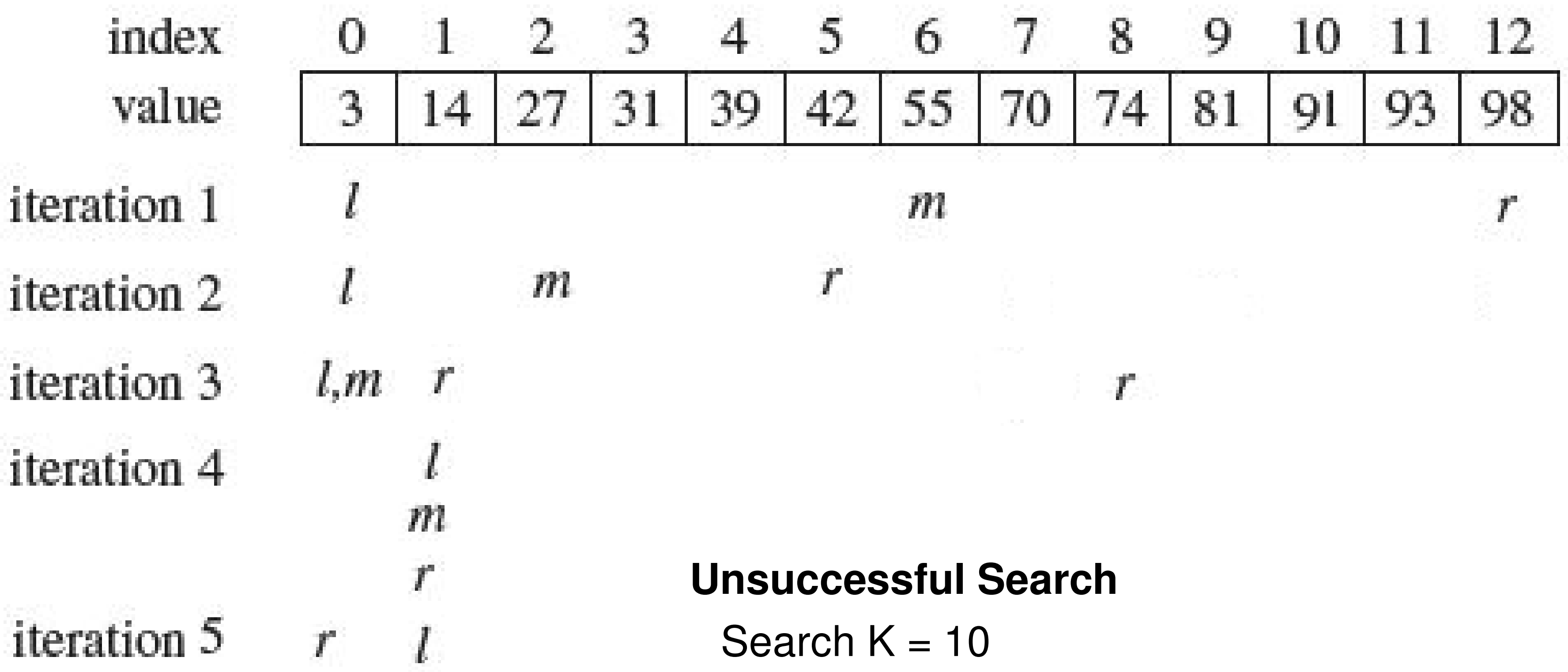
| index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| value | 3 | 14 | 27 | 31 | 39 | 42 | 55 | 70 | 74 | 81 | 91 | 93 | 98 |
| iteration 1 | $l$ | | | | | | $m$ | | | | | | $r$ |
| iteration 2 | | | | | | | | $l$ | | $m$ | | | $r$ |
| iteration 3 | | | | | | | | $l,m$ | $r$ | | | | |

```
ALGORITHM  BinarySearch($A[0..n-1], K$)
    //Implements nonrecursive binary search
    //Input: An array $A[0..n-1]$ sorted in ascending order and
    //        a search key $K$
    //Output: An index of the array's element that is equal to $K$
    //        or $-1$ if there is no such element
    $l \leftarrow 0$;  $r \leftarrow n-1$
    while $l \le r$ do
        $m \leftarrow \lfloor (l+r)/2 \rfloor$
        if $K = A[m]$ return $m$
        else if $K < A[m]$ $r \leftarrow m-1$
        else $l \leftarrow m+1$
    return $-1$
```

**$C(n) = C(n/2) + 1$ for $n > 1$**
**$C(1) = 1$**

**$C(n) = C(n/2) + \Theta(1)$ for $n > 1$**
**$a = 1, b = 2, d = 0$**
**$a = b^d$**
**$C(n) = \Theta(n^0 \log n) = \Theta(\log n)$**

# Binary Search



| index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| value | 3 | 14 | 27 | 31 | 39 | 42 | 55 | 70 | 74 | 81 | 91 | 93 | 98 |
| iteration 1 | *l* | | | | | | *m* | | | | | | *r* |
| iteration 2 | *l* | | *m* | | | *r* | | | | | | | |
| iteration 3 | *l*,*m* | *r* | | | | | | | *r* | | | | |
| iteration 4 | | *l* *m* *r* | | | | | | | | | | | |
| iteration 5 | *r* | *l* | | | | | | | | | | | |

**Unsuccessful Search**

Search K = 10

l=0 r=12 m=6

l=0 r=5 m=2

l=0 r=1 m=0

l=1 r=1 m=1

l=1 r=0 STOP!!

# Applications of Binary Search (1)
## Searching for a Threshold Value for a Function

- <u>Sample Scenario</u>
- Consider a monotonically decreasing function $f(n) = 2/n^2$, where n is a positive integer ($n > 0$).
- We need to develop a Θ(logn) algorithm that would determine the smallest value of n (called the threshold value) for which f(n) would be <u>less than</u> a target value 't' (say, t = 0.01).

| ***n*** | **f(*n*) = 2/*n*²** | ***n*** | **f(*n*) = 2/*n*²** |
|---|---|---|---|
| 1 | 2 | 9 | 0.0247 |
| 2 | 0.5 | 10 | 0.02 |
| 3 | 0.222 | 11 | 0.0165 |
| 4 | 0.125 | 12 | 0.0139 |
| 5 | 0.08 | 13 | 0.0118 |
| 6 | 0.0556 | 14 | 0.0102 |
| 7 | 0.0408 | 15 | 0.0089 |
| 8 | 0.0313 | 16 | 0.0078 |

- <u>Solution Approach (for optimization problems)</u>
- **Invariants** (something that will remain true throughout the algorithm):
  - We will keep the **Left Index as an 'n' value for which f(n) is always going to be greater than or equal to the threshold value**
  - We will keep the **Right Index as an 'n' value for which f(n) is always going to be less than the threshold value.**
- We will go through a sequence of iterations of Binary Search until the difference between the Right Index and Left Index is greater than ONE (note: we are dealing with integers here).
  - In each iteration, the middle index is the average of the Left Index and Right Index.
    - If f(Middle Index) < target, we set: Right Index = Middle Index
    - If f(Middle Index) >= target, we set: Left Index = Middle Index
    - In each iteration, either the Left Index increases or the Right Index decreases.
  - The moment the difference between the Left Index and Right Index is equal to 1, we will exit from the loop and say that the value of the Right Index is the threshold (smallest integer) value of 'n' for which the function value is less than the target.

$$\text{\# Iterations} = \log_2 \frac{\text{Initial Right Index - Initial Left Index}}{\text{Min. Allowable Diff. between the Right Index and Left Index}}$$

Function $f(n) = 2/n^2$; for $n > 0$
Target = 0.01

Left Index = 1; f(Left Index) = 2 > target
Right Index = 100; f(Right Index) = $2/100^2$ = 0.0002 < target

| It # | Left Index | Right Index | Middle Index | f(Middle Index) |
|---|---|---|---|---|
| 1 | 1 | 100 | (1 + 100)/2 = 50 | 0.0008 < target |
| 2 | 1 | 50 | (1 + 50)/2 = 25 | 0.0032 < target |
| 3 | 1 | 25 | (1 + 25)/2 = 13 | 0.0118 > target |
| 4 | 13 | 25 | (13 + 25)/2 = 19 | 0.0055 < target |
| 5 | 13 | 19 | (13 + 19)/2 = 16 | 0.0078 < target |
| 6 | 13 | 16 | (13 + 16)/2 = 14 | 0.0102 > target |
| 7 | 14 | 16 | (14 + 16)/2 = 15 | 0.0089 < target |
| 8 | 14 | 15 | STOP! | |

Threshold = Value of Right Index when we stop the iterations
= 15

# Iterations = log2 ( (100-1) / 1) = log2(99) ~ 7

**Solution:**
15 is the smallest integer for which the function $f(n) = 2/n^2$ is less than 0.01

## Practice Problem

- Given a monotonically increasing function $f(n) = n^2/10000$ (where ‘n’ is an integer), use a binary search algorithm to find the largest value of ‘n’ for which f(n) is less than a target (say, 0.01).

## (Maximum) Unimodal Array

- A (maximum) unimodal array is an array that has a sequence of monotonically increasing integers followed by a sequence of monotonically decreasing integers.
- All elements in the array are unique
- Examples
  - {4, 5, 8, 9, 10, 11, 7, 3, 2, 1}: Max. Element: 11
    - There is an increasing seq. followed by a decreasing seq.
  - {11, 9, 8, 7, 5, 4, 3, 2, 1}: Max. Element: 11
    - There is no increasing seq. It is simply a decreasing seq.
  - {1, 2, 3, 4, 5, 7, 8, 9, 11}: Max. Element: 11
    - There is an increasing seq., but there is no decreasing seq.

## Applications of Binary Search (2)

### Finding the Maximum Element in a Unimodal Array

```
Search Range: L = 0; R = n-1
while (L < R) do
    m = (L+R)/2
    if A[m] < A[m+1]
        L = m+1  // max. element is from m+1 to R
    else if A[m] > A[m+1]
        R = m // max. element is from L to m
end while
return A[L]
```

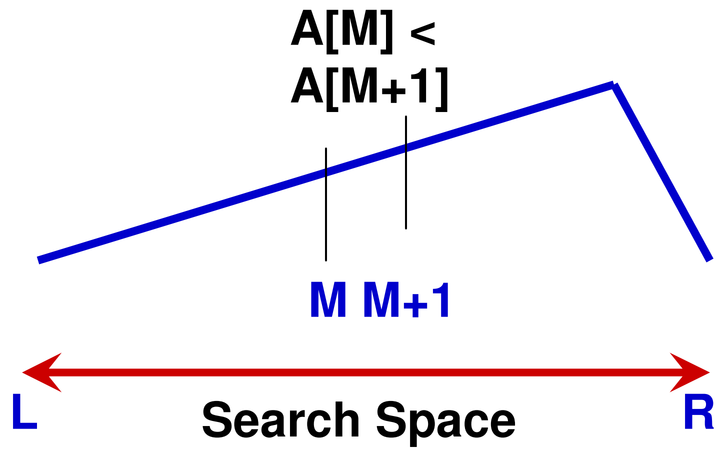
**$C(n) = C(n/2) + 2$**
**Using Master Theorem,**
**$C(n) = \Theta(\log n)$**

**Space complexity: $\Theta(1)$**

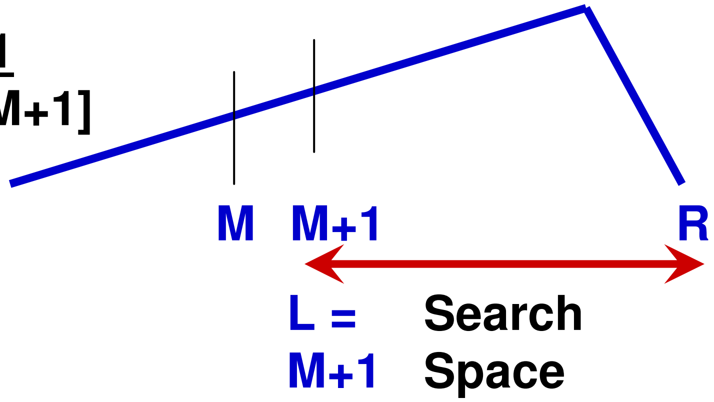
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 5 | 8 | 9 | 10 | 14 | 11 | 4 | 2 | 1 |

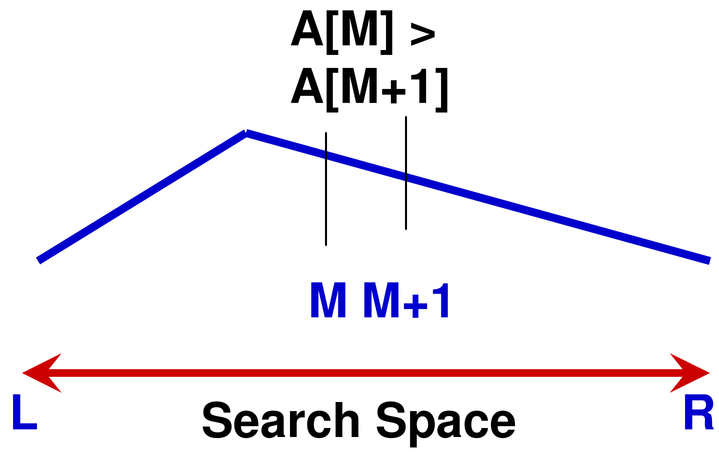
L = 0; R = 9; m = 4: A[m] < A[m+1]
L = 5; R = 9; m = 7: A[m] > A[m+1]
L = 5; R = 7; m = 6: A[m] > A[m+1]
L = 5; R = 6; m = 5: A[m] > A[m+1]
L = 5; R = 5; return A[5] = 14

# Two Scenarios

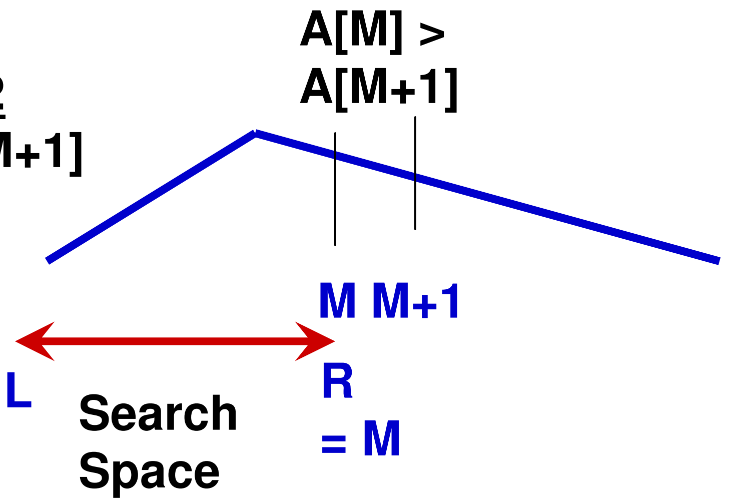


**<u>Scenario 1</u>**
**A[M] < A[M+1]**





**<u>Scenario 2</u>**
**A[M] > A[M+1]**



## Applications of Binary Search (2)

### Finding the Maximum Element in a Unimodal Array

- Proof of Correctness
  - We always maintain the invariant that the maximum element lies in the range of indexes: L…R.
  - If A[m] < A[m+1], then, the maximum element has to be either at index m+1 or to the right of index m+1. Hence, we set L = m+1 and retain R as it is, maintaining the invariant that the maximum element is in the range L…R.
  - If A[m] > A[m+1], then, the maximum element is either at index m or before index m. Hence, we set R = m and retain L as it is, maintaining the invariant that the maximum element is in the range L...R.
  - The loop runs as long as L < R. Once L = R, the loop ends and we return the maximum  element.

# Applications of Binary Search (2)

## Finding the Maximum Element in a Unimodal Array

```
L = 0; R = n-1
while (L < R) do
    m = (L+R)/2
   if A[m] < A[m+1]
       L = m+1  // max. element is from m+1 to R
   else if A[m] > A[m+1]
       R = m // max. element is from L to m
end while
return A[L]
```

| 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| **3** | **5** | **8** | **9** | **10** | **14** |

L = 0; R = 5; m = 2: A[m] < A[m+1]
L = 3; R = 5; m = 4: A[m] < A[m+1]
L = 5; R = 5; return A[5] = 14

# Applications of Binary Search (3)

## Local Minimum in an Array

- Problem: Given an array A[0,…, n-1], an element at index i (0 < i < n-1) is a local minimum if A[i] < A[i-1] as well as A[i] < A[i+1]. That is, the element is lower than the element to the immediate left as well as to the element to the immediate right.
- Constraints:
  - The arrays has at least three elements
  - The first two numbers are decreasing and the last two numbers are increasing.
  - The numbers are unique
- Example:
  - Let A = {8, **5**, 7, **2**, 3, 4, **1**, 9}; the array has several local minimum. These are: 5, 2 and 1.
- Algorithm: Do a binary search and see if every element we index into is a local minimum or not.
  - If the element we index into is not a local minimum, then we search on the half corresponding to the smaller of its two neighbors.

# Applications of Binary Search (3)

## Local Minimum in an Array

**Examples**

**1)**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 8 | 5 | 7 | 2 | 3 | 4 | 1 | 9 |

**Iteration 1:** L = 0; R = 7; M = (L+R)/2 = 3   Element at A[3] is a local minimum.

**2)**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 8 | 5 | 2 | 7 | 3 | 4 | 1 | 9 |

**Iteration 1:** L = 0; R = 7; M = (L+R)/2 = 3   Element at A[3] is NOT a local minimum.
Search in the space [0…2] corresponding to the smaller neighbor ‘2’

**Iteration 2:** L = 0; R = 2; M = (L+R)/2 = 1   Element at A[1] is NOT a local minimum.
Search in the space [2…2] corresponding to the smaller neighbor ‘2’

**Iteration 3:** L = 2; R = 2; M = (L+R)/2 = 2. Element at A[2] is a local minimum.

# Applications of Binary Search (3)

## Local Minimum in an Array

**Examples**

**3)**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| **-2** | **-5** | **5** | **2** | **4** | **7** | **1** | **8** | **0** | **-8** | **10** |

Iteration 1: L = 0; R = 10; M = (L+R)/2 = 5  Element at A[5] is NOT a local minimum.
Search in the space [6…10] corresponding to the smaller neighbor ‘1’
Iteration 2: L = 6; R = 10; M = (L+R)/2 = 8  Element at A[8] is NOT a local minimum.
Search in the space [9…10] corresponding to the smaller neighbor ‘-8’
Iteration 3: L = 9; R = 10; M = (L+R)/2 = 9. Element at A[9] is a local minimum. STOP

### Time-Complexity Analysis

Recurrence Relation: T(n) = T(n/2) + 3 for n > 3
Basic Condition: T(3) = 2
Using Master Theorem, we have
a = 1, b = 2, d = 0 ➔ a = $b^d$.
Hence, T(n) = Θ($n^d$ logn) = Θ($n^0$ logn) = Θ(logn)

$\rightarrow$ One comparison for A[M] with A[M+1]
$\rightarrow$ One comparison for A[M] with A[M-1]
$\rightarrow$ One comparison for A[M-1] with A[M+1]

**Space Complexity:** As all evaluations are done on the input array itself, no extra space proportional to the input is needed. Hence, space complexity is Θ(1).

# Applications of Binary Search (3)

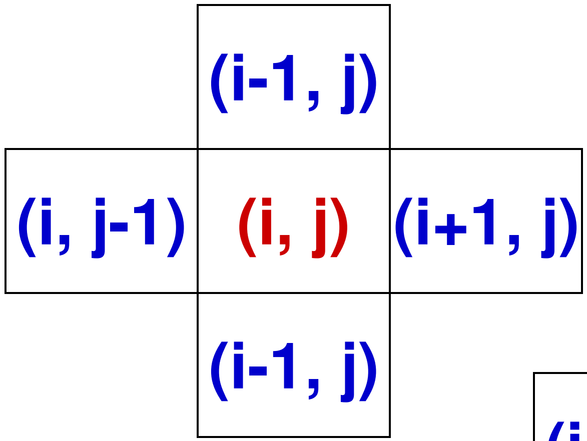
## Local Minimum in an Array

- Constraints:
  - The arrays has at least three elements
  - The first two numbers are decreasing and the last two numbers are increasing.
  - The numbers are unique
- Theorem: If the above three constraints are met for an array, then the array has to have at least one local minimum.
- Proof: Let us prove by contradiction.
  - If the second number is not to be a local minimum, then the third number in the array has to be less than the second number.
  - Continuing like this, if the third number is not to be a local minimum, then the fourth number has to be less than the third number and so on.
  - Again, continuing like this, if the penultimate number is not to be a local minimum, then the last number in the array has to be smaller than the penultimate number. This would mean the second constraint is violated (and also the array is basically a monotonically decreasing sequence). A contradiction.

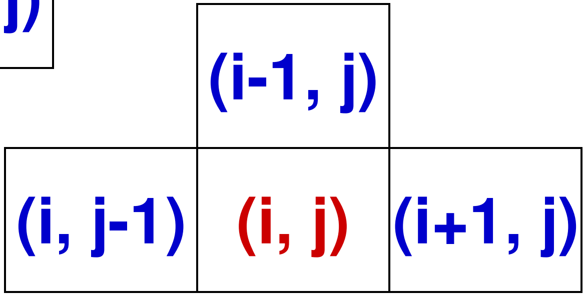
# Applications of Binary Search (4)

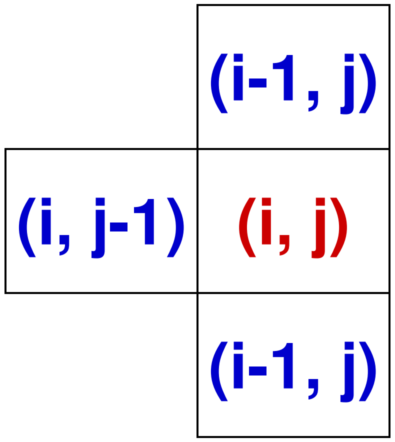
## Local Minimum in a Two-Dimensional Array

- An element is a local minimum in a two-dim array if the element is the minimum compared to the elements to its immediate left and right as well as to the elements to its immediate top and bottom.
  - If an element is in the edge row or column, it is compared only to the elements that are its valid neighbors.

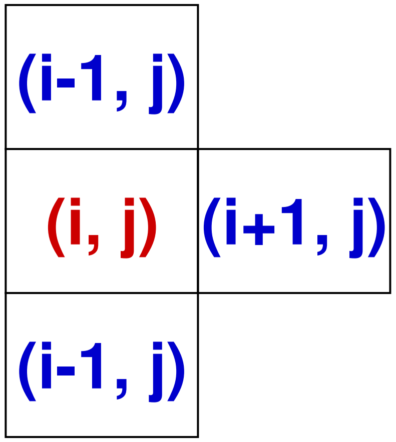


Bottommost Row

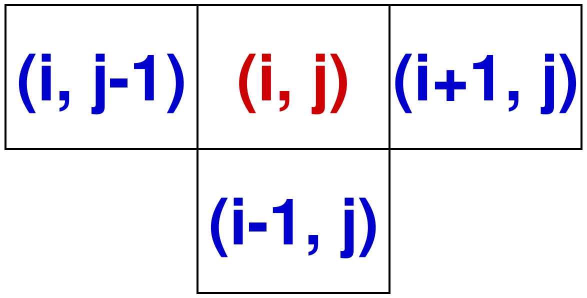




Rightmost column



Leftmost Column



Topmost Row

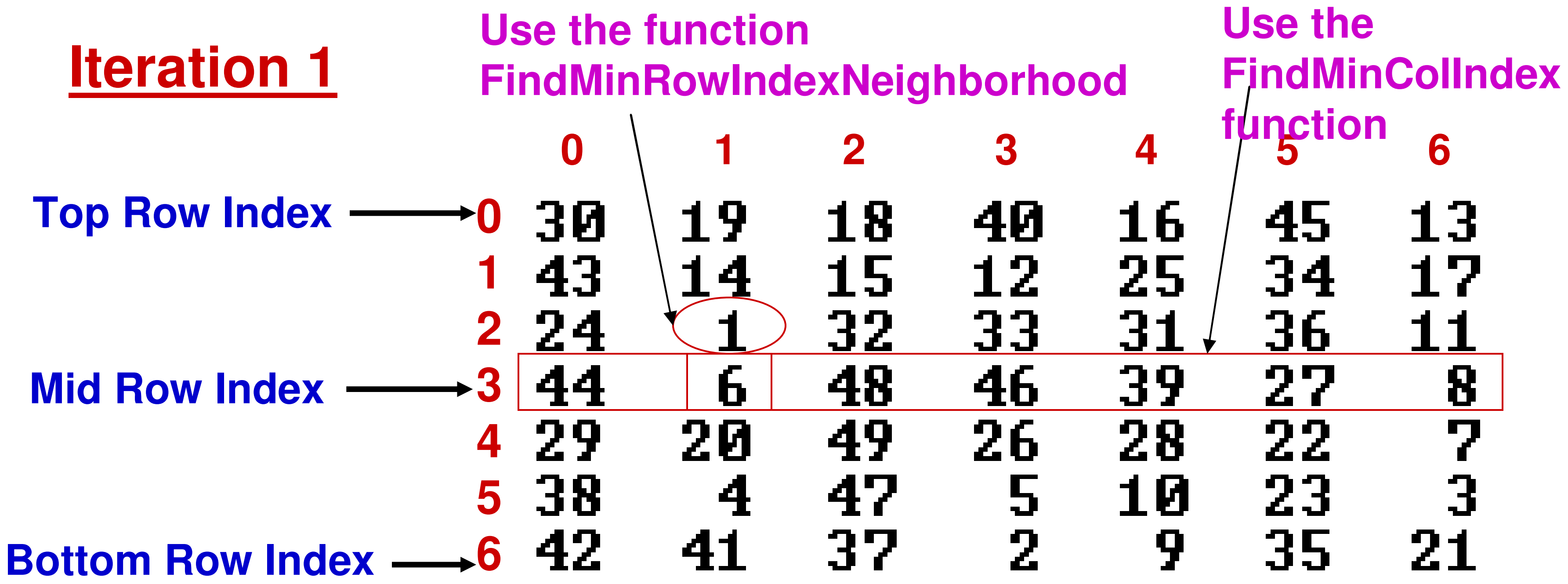
# Applications of Binary Search (4)
## Local Minimum in a Two-Dimensional Array

```
Given an array A[0…numRows-1][0…numCols-1]
TopRowIndex = 0
BottomRowIndex = numRows – 1
while (TopRowIndex ≤ BottomRowIndex) do
    MidRowIndex = (TopRowIndex + BottomRowIndex) / 2
    MinColIndex = FindMinColIndex( A[MidRowIndex][ ] )
            /* Finds the col index with the minimum element in the row
            corresponding to MidRowIndex */
    MinRowIndex = FindMinRowIndexNeighborhood (A, MidRowIndex,
                                                                MinColIndex)
            /* Finds the min entry in the column represented by MinColIndex
            and the rows MidRowIndex, MidRowIndex – 1,
            MidRowIndex + 1, as appropriate */
    if (MinRowIndex == MidRowIndex)
            return A[MinRowIndex][MinColIndex]
    else if (MinRowIndex < MidRowIndex)
            BottomRowIndex = MidRowIndex – 1
    else if (MinRowIndex > MidRowIndex)
            TopRowIndex = MidRowIndex + 1
end While
```

## Local Minimum in a Two-Dim Array: Ex. 1

$$\begin{array}{c|ccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
0 & 30 & 19 & 18 & 40 & 16 & 45 & 13 \\
1 & 43 & 14 & 15 & 12 & 25 & 34 & 17 \\
2 & 24 & 1 & 32 & 33 & 31 & 36 & 11 \\
3 & 44 & 6 & 48 & 46 & 39 & 27 & 8 \\
4 & 29 & 20 & 49 & 26 & 28 & 22 & 7 \\
5 & 38 & 4 & 47 & 5 & 10 & 23 & 3 \\
6 & 42 & 41 & 37 & 2 & 9 & 35 & 21
\end{array}$$

**Iteration 1**


Use the function FindMinRowIndexNeighborhood
Use the FindMinColIndex function
Top Row Index
Mid Row Index
Bottom Row Index

$$\begin{array}{c|ccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
0 & 30 & 19 & 18 & 40 & 16 & 45 & 13 \\
1 & 43 & 14 & 15 & 12 & 25 & 34 & 17 \\
2 & 24 & 1 & 32 & 33 & 31 & 36 & 11 \\
3 & 44 & \boxed{6} & 48 & 46 & 39 & 27 & 8 \\
4 & 29 & 20 & 49 & 26 & 28 & 22 & 7 \\
5 & 38 & 4 & 47 & 5 & 10 & 23 & 3 \\
6 & 42 & 41 & 37 & 2 & 9 & 35 & 21
\end{array}$$

# Local Minimum in a Two-Dim Array: Ex. 1 (1)

**Iteration 2**

| | | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| **Top Row Index** → | 0 | 30 | 19 | 18 | 40 | 16 | 45 | 13 |
| | 1 | 43 | 14 | 15 | 12 | 25 | 34 | 17 |
| **Bottom Row Index** → | 2 | 24 | 1 | 32 | 33 | 31 | 36 | 11 |
| | 3 | 44 | 6 | 48 | 46 | 39 | 27 | 8 |
| | 4 | 29 | 20 | 49 | 26 | 28 | 22 | 7 |
| | 5 | 38 | 4 | 47 | 5 | 10 | 23 | 3 |
| | 6 | 42 | 41 | 37 | 2 | 9 | 35 | 21 |

| | | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| **Top Row Index** → | 0 | 30 | 19 | 18 | 40 | 16 | 45 | 13 |
| **Mid Row Index** → | 1 | 43 | 14 | 15 | $\boxed{12}$ | 25 | 34 | 17 |
| **Bottom Row Index** → | 2 | 24 | 1 | 32 | 33 | 31 | 36 | 11 |
| | 3 | 44 | 6 | 48 | 46 | 39 | 27 | 8 |
| | 4 | 29 | 20 | 49 | 26 | 28 | 22 | 7 |
| | 5 | 38 | 4 | 47 | 5 | 10 | 23 | 3 |
| | 6 | 42 | 41 | 37 | 2 | 9 | 35 | 21 |

The minimum element 12 in Mid Row is smaller than its immediate top (40) and bottom (33) neighbors

**12 at (1, 3) is a local minimum**

# Local Minimum in a Two-Dim Array: Ex. 2

$$\begin{array}{c|cccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & 17 & 16 & 32 & 15 & 23 & 36 \\
1 & 20 & 3 & 18 & 35 & 11 & 9 \\
2 & 26 & 5 & 8 & 30 & 13 & 22 \\
3 & 10 & 31 & 2 & 1 & 7 & 14 \\
4 & 28 & 12 & 6 & 24 & 25 & 34 \\
5 & 29 & 21 & 27 & 19 & 4 & 33
\end{array}$$

## Iteration 1

$$\begin{array}{rc|cccccc}
 & & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
\text{Top Row Index} \longrightarrow & 0 & 17 & 16 & 32 & 15 & 23 & 36 \\
 & 1 & 20 & \bigcirc\!\!\!\!3 & 18 & 35 & 11 & 9 \\
\text{Mid Row Index} \longrightarrow & 2 & \boxed{26} & \boxed{5} & \boxed{8} & \boxed{30} & \boxed{13} & \boxed{22} \\
 & 3 & 10 & 31 & 2 & 1 & 7 & 14 \\
 & 4 & 28 & 12 & 6 & 24 & 25 & 34 \\
\text{Bottom Row Index} \longrightarrow & 5 & 29 & 21 & 27 & 19 & 4 & 33
\end{array}$$

# Local Minimum in a Two-Dim Array: Ex. 2 (1)

**Iteration 2**

$$
\begin{array}{lcrrrrrr}
 & & 0 & 1 & 2 & 3 & 4 & 5 \\
\text{Top Row Index} \longrightarrow & 0 & 17 & 16 & 32 & 15 & 23 & 36 \\
\text{Bottom Row Index} \longrightarrow & 1 & 20 & 3 & 18 & 35 & 11 & 9 \\
\text{Mid Row Index} \longrightarrow & 2 & 26 & 5 & 8 & 30 & 13 & 22 \\
 & 3 & 10 & 31 & 2 & 1 & 7 & 14 \\
 & 4 & 28 & 12 & 6 & 24 & 25 & 34 \\
 & 5 & 29 & 21 & 27 & 19 & 4 & 33
\end{array}
$$

$$
\begin{array}{lcrrrrrr}
 & & 0 & 1 & 2 & 3 & 4 & 5 \\
\text{Mid Row Index} \searrow,\ \text{Top Row Index} \longrightarrow & 0 & 17 & 16 & 32 & \boxed{15} & 23 & 36 \\
\text{Bottom Row Index} \longrightarrow & 1 & 20 & 3 & 18 & 35 & 11 & 9 \\
\text{Bottom Row Index} \longrightarrow & 2 & 26 & 5 & 8 & 30 & 13 & 22 \\
 & 3 & 10 & 31 & 2 & 1 & 7 & 14 \\
 & 4 & 28 & 12 & 6 & 24 & 25 & 34 \\
 & 5 & 29 & 21 & 27 & 19 & 4 & 33
\end{array}
$$

The minimum element 15 in Mid Row is smaller than its immediate top bottom (35) neighbor

**15 at (0, 3) is a local minimum**

# Applications of Binary Search (4)

## Local Minimum in a Two-Dimensional Array

- Time Complexity Analysis

$T(n^2) = T(n^2/ 2) + \Theta(n)$

**Time complexity to search for the minimum element in a row**

**The search space reduces by half**

Let $N = n^2$.

$T(N) = T(N/2) + \Theta(N^{1/2})$

Use Master Theorem: $a = 1$, $b = 2$, $d = \frac{1}{2}$

We have $a < b^d$. Hence, $T(N) = \Theta(N^{1/2}) = \Theta(n)$

Space Complexity: $\Theta(1)$