1.0 Introduction

- Epidemiology: The branch of science that deals with the study of the control and spread of diseases, viruses, ideas, etc., upon a population or system.
  - Epi: Upon or on
  - Demos: people
  - Logy – study

- Computational Epidemiology: is a field that focuses on the study and development of computational techniques and tools for modeling, simulating, predicting and mitigating the spread of diseases, viruses, etc.

Compartmental Models

- The population is assigned to non-overlapping compartments, identified with labels: such as, S – Susceptible, E – Exposed, I – Infected, R – Recovered, etc.
- The order of the labels in a model name typically follows the flow patterns between the compartments:
  - For example: SIR model means an individual moves from susceptible state to infected state and then to recovered state. This implies, a susceptible individual cannot directly move to recovered state in unit time. Also, a recovered individual is no longer susceptible.
  - SIS model means an individual moves from susceptible state to infected state and then moves back to susceptible state.
- The models try to predict the spread of a disease, the total number of infected at any time, the duration of an epidemic, etc.

SI
SIS
SIR
SIRS

All the models studied here do not take into consideration "vital dynamics" (also called "demographic data"): i.e., birth rate and death rate

1.1 SI Model

S(t) - # susceptible individuals
I(t) - # infected individuals

N = 1000
S(t = 0) = 990
I(t = 0) = 10

S(t = 1) = 950
I(t = 1) = 50

S(t = 2) = 900
I(t = 2) = 100

At any time t, S(t) + I(t) = N, the total # individuals

The probability that an infected individual will come into contact with a susceptible individual at time t is S(t)/N

The infected individual spreads the disease with probability β (measured as the probability for an infection per time unit).

Let <k> be the "average" number of contacts per infected individual.

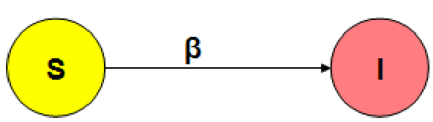
The probability that these contacts can be among the susceptible individuals is S(t)/N

$\frac{S(t)}{N} * < k >$ is the number of contacts who are also susceptible for the infected individual

Hence, the number of susceptible individuals (among the contacts) infected by an infected individual at time t is $\frac{S(t)}{N} * < k > * \beta$

<k>β is the number of contacts that can be infected by the infected individual and is also called the transmissibility or transmission rate.

Considering that there at I(t) infected individuals at time t, the total number of susceptible individuals infected by the I(t) infected individuals is I(t)* <k>β* S(t)/N



The change in the number of infected individuals, denoted as dI(t), occurring over a time period dt, is given by:

From time t ... t+dt
Change in the number of infected individuals = dI(t) = I(t+dt) - I(t) = I(t)* <k>β* S(t)/N * dt

Rate of change in the number of infected individuals I(t) with respect to time
// differential equation

$$\frac{dI(t)}{dt} = I(t) * <k> \beta * S(t)/N \qquad \text{.............................. (1)}$$

Let s(t) = S(t)/N and i(t) = I(t)/N be the fractions of susceptible and infected individuals at time t.
For simplicity, we can denote s(t) as s and i(t) as i.

Divide both sides of (1) by N, we get:

$$\frac{dI(t)}{dt} = I(t) * <k> \beta * S(t)/N$$

$$\frac{dI(t)}{Ndt} = \frac{I(t) * <k> \beta * S(t)/N}{N}$$

$$\frac{dI(t)}{Ndt} = I(t)/N * <k> \beta * S(t)/N$$

I(t) / N = i(t) = i

i.e., $\frac{di}{dt} = i * <k> \beta * s$

Hence,

$$\frac{di}{dt} = i * <k> \beta * (1-i) \qquad \text{.. (2)}$$

S(t) + I(t) = N // invariant - something that is maintained at all the time instants

$$\frac{S(t) + I(t)}{N} = \frac{N}{N} = 1$$

In the SI model, s(t) + i(t) = 1 at any time t.

$$\frac{ds}{dt} + \frac{di}{dt} = 0$$

$$\frac{ds}{dt} = -\frac{di}{dt}$$

$$\frac{ds}{dt} = -i * <k> \beta * s$$

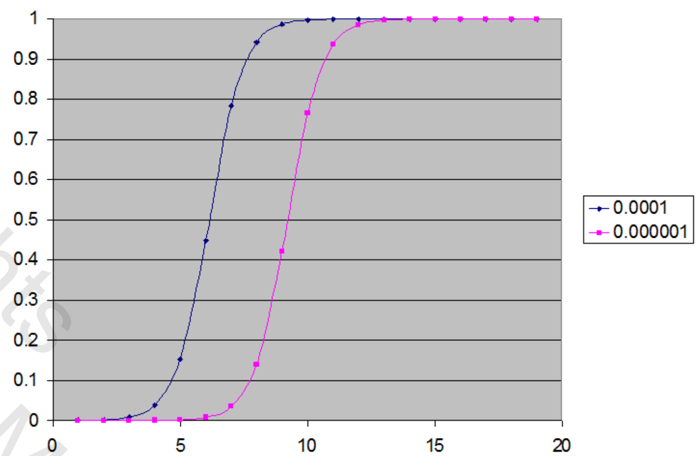
Let the initial condition be that $i = i_0$ at $t = 0$.

Final equation for SI Model

$$i = \frac{i_0 * e^{<k>\beta t}}{(1-i_0) + i_0 * e^{<k>\beta t}}$$

.... (6)

We plot aside, i as a function of t for two different values of i_0: 0.0001 and 0.000001. For both cases, <k>β (referred to as the transmission rate): the number of contacts that can be infected by the infected individual is assumed to 1.5.



For all practical purposes, i_0 is very small compared to 1. Hence 1- i_0 ~ 1.

Hence, $i = \frac{i_0 * e^{<k>\beta t}}{1 + i_0 * e^{<k>\beta t}}$

Observations:
(1) At the beginning, the fraction of infected individuals grows exponentially with time, as S(t)/N is close to 1 in the beginning and everyone an infected individual encounters is most likely to be a susceptible individual.
(2) Characteristic time is the time it takes for the fraction of infected individuals to be (1/e ~ 36%) of the entire population.

When *t* is small:

$$i = \frac{i_0 * e^{<k>\beta t}}{(1-i_0) + i_0 * e^{<k>\beta t}}$$

When t->0, the denominator tends to (1-i_0) + i0*e^0 ~ 1-i0+i0*1 = 1

==> $i = i_0 * e^{<k>\beta t}$

Thus, at the beginning, the fraction of infected individuals grows exponentially with time, as S(t)/N is close to 1 in the beginning and everyone an infected individual encounters is most likely to be a susceptible individual.

To estimate the characteristic time, we need to evaluate t for i = 0.36 (= 1/e).
e = 2.7183
1/2.7183 = 0.36

$$i = \frac{i_0 * e^{<k>\beta t}}{1 + i_0 * e^{<k>\beta t}} = 0.36$$

$$i_0 * e^{<k>\beta t} = 0.36 + 0.36 * i_0 * e^{<k>\beta t}$$
$$i_0 * e^{<k>\beta t}[1 - 0.36] = 0.36$$

$$0.64 * i_0 * e^{<k>\beta t} = 0.36$$

<u>If i_0 = 0.0001 and <k>β = 1.5,</u>
0.64*0.0001*e^(1.5*t) = 0.36
e^(1.5*t) = 5625

5625 = e^(1.5*t)

1.5t = ln(5625) = 8.635
t = 5.75 units

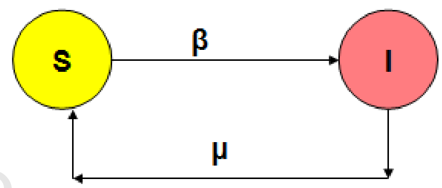
<u>If i_0 = 0.000001 and <k>β = 1.5,</u>
0.64*0.000001*e^(1.5*t) = 0.36
e^(1.5*t) = 562500
1.5t = ln(562500) = 13.24
t = 8.83 units

The SI model forms the basis for other epidemic models, but is however not realistic enough as most infected individuals recover with the body's immune system or through medical treatment.

Practice Problems:
(1) Analyze the impact of the term <u><k>β</u> (1.5 and 2.5) on the rate of increase in the fraction of infected individuals with time for a given i_0 (0.0001)

1.2 SIS Model

The SIS model comprises of the same two states (Susceptible-S and Infected-I) of the SI model, but considers the possibility of the infected individuals to recover (and thereby they again become susceptible).



Diseases caused by bacteria are usually of SIS type.

The infected individual is assumed to spend an average of $1/\mu$ time units in the infected state, where μ is the probability that an infected individual will recover at any time unit (and will be considered susceptible).

$\mu = 0.2$ is the probability that an infected individual will recover on a particular day
$1/\mu = 5$ is the number of days (on average) an infected individual will recover

$1/\mu$ = average time period for an infected individual to recover = 5 days

For example, if $1/\mu = 5 \Longrightarrow \mu = 0.2$. This implies, if we generate 5 random numbers (one trial) in the range of 0 to 1, we can expect to see one among the 5 random numbers to be less than or equal to 0.2. To be more statistically thorough, we can repeat the trials a large number of times (say 100 trials) and obtain a total of 5*100 = 500 random numbers and sort them, we can expect to see around 100 random numbers out of the 500 random numbers (~100/500 = 0.2) to be less than or equal to 0.2.

As per the SI model, we have (eq. 2)

$$\frac{di}{dt} = i^* < k > \beta * (1 - i) \qquad \text{..} \ (2)$$

We enhance this model to mimic the SIS model as follows:

If μ is the probability with which an infected individual can recover at a particular time unit, the probability that an infected individual will recover in dt time units is μ*dt. There are I(t) infected individuals at time t and the number of infected individuals who would have recovered by time t +dt is: μ*dt * I(t).

Going from fundamentals, we can enhance and write the differential equation for I(t) as follows:

$$dI(t) = I(t)^* < k > \beta * S(t) / N * dt - \mu * I(t) * dt$$

$$\frac{dI(t)}{dt} = I(t)^* < k > \beta * S(t) / N - \mu * I(t)$$

Dividing by 'N' on both sides, we get:

$$\frac{dI(t)}{Ndt} = \frac{I(t)* < k > \beta * S(t)/N}{N} - \frac{\mu * I(t)}{N}$$

i(t) = I(t)/N
s(t) = S(t)/N

$$\frac{di}{dt} = i* < k > * \beta * s - \mu * i$$

s+i = 1
S(t) + I(t) = N

$$\frac{di}{dt} = i* < k > \beta * (1-i) - \mu * i \qquad \text{.................................(7)}$$

The term μ*i captures the rate at which the infected individuals recover from the disease.

The solution for the differential equation of (7) is:

$$i = \left(1 - \frac{\mu}{\beta < k >}\right) \frac{Ce^{(\beta<k>-\mu)t}}{1 + Ce^{(\beta<k>-\mu)t}} \qquad \text{........................... (8)}$$

where C is an integration constant whose value (obtained by setting $i = i_0$ @ t = 0) is as follows:

$$C = \frac{i_0}{(1 - i_0 - \mu / \beta < k >)} \qquad \text{............................. (9)}$$

For all practical purposes, C can be simplified as (considering that i_0 is much smaller than 1):

$$C = \frac{i_0}{(1 - \mu / \beta < k >)} \qquad \text{.. (10)}$$

<u>When t is closer to 0:</u>

From equation (8), we have: $i = \left(1 - \frac{\mu}{\beta < k >}\right) \frac{Ce^{(\beta<k>-\mu)t}}{1 + Ce^{(\beta<k>-\mu)t}}$

$1 + Ce^{(\beta<k>-\mu)t}$ can be approximated to 1: as $Ce^{(\beta<k>-\mu)t}$ ~ C and C can be considered to be reasonably smaller than 1 (i.e., the term $Ce^{(\beta<k>-\mu)t}$ becomes greater than 1, only if $e^{(\beta<k>-\mu)t}$ becomes greater than 1 and not due to C).

Hence, Equation (8) simplifies to,

$$i = \left(1 - \frac{\mu}{\beta < k >}\right) C e^{(\beta < k > - \mu)t}$$

Substituting for C from equation (10) in the above equation, we get:

$$i = \left(1 - \frac{\mu}{\beta < k >}\right) \frac{i_0}{(1 - \mu / \beta < k >)} e^{(\beta < k > - \mu)t}$$

i.e., $i = i_0 * e^{(\beta < k > - \mu)t}$ .... (11) contributing to an exponential outbreak at the early stages.

<u>When t is closer to ∞:</u>

$$i = \left(1 - \frac{\mu}{\beta < k >}\right) \frac{C e^{(\beta < k > - \mu)t}}{1 + C e^{(\beta < k > - \mu)t}}$$

$C e^{(\beta < k > - \mu)t}$ will be much greater than 1 and hence,

$$\frac{C e^{(\beta < k > - \mu)t}}{1 + C e^{(\beta < k > - \mu)t}} \sim 1$$

$$\frac{C e^{(\beta < k > - \mu)t}}{1 + C e^{(\beta < k > - \mu)t}} = \frac{C e^{(\beta < k > - \mu)t}}{C e^{(\beta < k > - \mu)t}} \sim 1$$

Therefore, i of equation (8) reduces to:

$$i = \left(1 - \frac{\mu}{\beta < k >}\right) \frac{C e^{(\beta < k > - \mu)t}}{1 + C e^{(\beta < k > - \mu)t}}$$

$i = \left(1 - \frac{\mu}{\beta < k >}\right)$ .... (12), the spread is considered to have reached an endemic state (i.e., saturated: the fraction of infected individuals is a constant).

$$i = \left(1 - \frac{1}{R_0}\right)$$

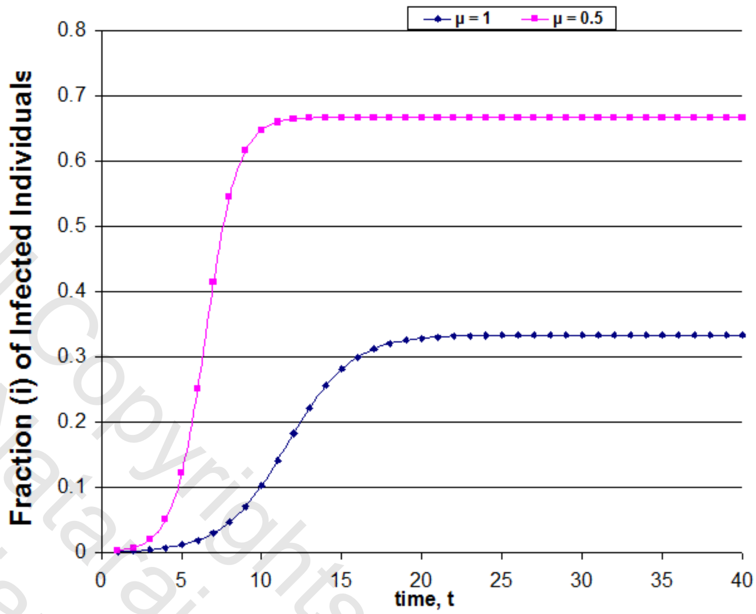


Figure: Simulation of the SIS Model, β<k> = 1.5

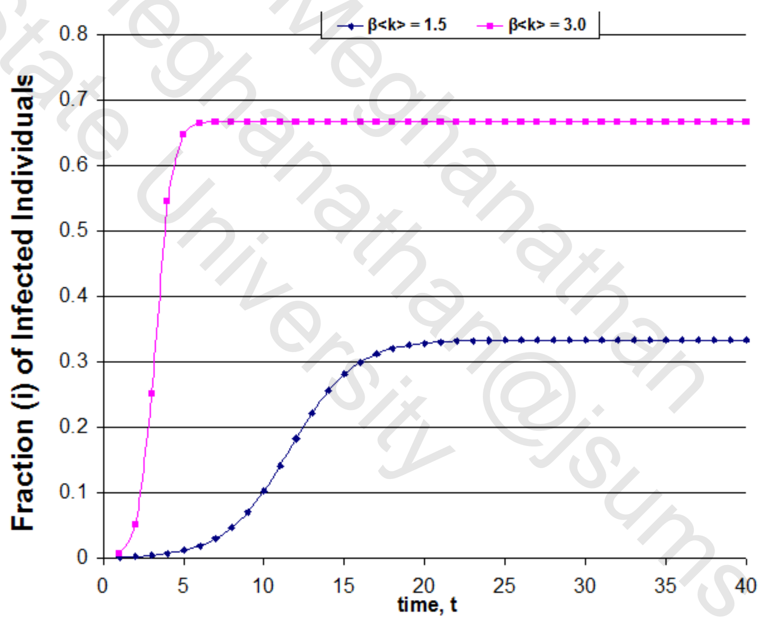


Figure: Simulation of the SIS Model, μ = 1.0

Basic Reproductive Number

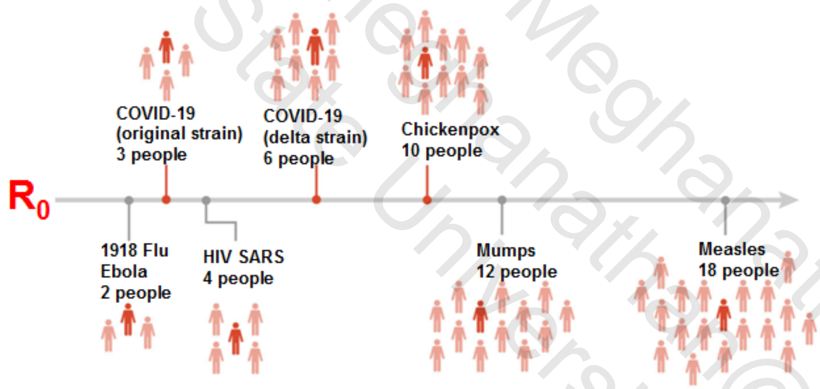
For the SIS model, we define the *basic reproductive number* as $R_0 = \dfrac{\beta < k >}{\mu}$ and define the characteristic time as $\tau = \dfrac{1}{\mu(R_0 - 1)}$ for diseases with R0 > 1.

infectious period (period for which an infection exists) = $\frac{1}{\mu}$

The basic reproductive number represents the average number of susceptible individuals infected by an infected individual during its infectious period in a fully susceptible population. One important assumption is that R0 represents the number of individuals an infected individual can infect if all its contacts are susceptible. In other words, R_0 is the number of new infections each infected individual causes under ideal circumstances.

If $R_0 > 1$, it implies that each infected individual on average can infect more than one healthy/susceptible person and/or the average infectious period per person is larger; the pathogen is predicted to exist and eventually reach an endemic state where the fraction of infected individuals is $1-1/R_0$. The larger the R_0, the faster the spread (i.e., the pathogen/epidemic quickly reaches the endemic state) and the larger the fraction of infected individuals.

If $R_0 < 1$, an infected individual on average can infect less than one susceptible person and/or the average infectious period person is smaller, and the pathogen eventually disappears from the population. The lower is the $R_0 < 1$, the more sooner the pathogen/epidemic will disappear.



(adapted from [https://www.npr.org/sections/goatsandsoda/2021/08/11/1026190062/covid-delta-variant-transmission-cdc-chickenpox])

Disease	Transmission	R_0
Measles	Airborne	12-18
Pertussis	Airborne droplet	12-17
Diptheria	Saliva	6-7
Smallpox	Social contact	5-7
Polio	Fecal oral route	5-7
Rubella	Airborne droplet	5-7
Mumps	Airborne droplet	4-7
HIV/AIDS	Sexual contact	2-5
SARS	Airborne droplet	2-5
Influenza	Airborne droplet	2-3
COVID-19 (original strain)	Airborne droplet, Social contact, Saliva	2-3
COVID-19 (delta variant)	Airborne droplet, Social contact, Saliva	5-6

Practice Problems:

(1) Determine the value of the transmission rate (term <u><k>β</u>) for a given R0, t, i0 and i. Also, determine the value of μ.

i0 = 0.0001, R0 = 3, i = 0.5, t = 5

$$i = \left(1 - \frac{\mu}{\beta < k >}\right) \frac{Ce^{(\beta < k > - \mu)t}}{1 + Ce^{(\beta < k > - \mu)t}} \qquad R_0 = \frac{< k > * \beta}{\mu} = 3$$

$$i = \left(1 - \frac{1}{\frac{\beta < k >}{\mu}}\right) \frac{Ce^{\mu\left(\frac{\beta < k >}{\mu} - 1\right)t}}{1 + Ce^{\mu\left(\frac{\beta < k >}{\mu} - 1\right)t}}$$

$$0.5 = \left(1 - \frac{1}{3}\right) \frac{Ce^{\mu(3-1)5}}{1 + Ce^{\mu(3-1)5}}$$

$$C = \frac{i_0}{(1 - \mu / \beta < k >)} = 0.0001 / [1 - 1/3] = 0.00015$$

$$0.5 = \left(1 - \frac{1}{3}\right) \frac{0.00015e^{10\mu}}{1 + 0.00015e^{10\mu}}$$

$$0.5 = \frac{0.0001e^{10\mu}}{1 + 0.00015e^{10\mu}}$$

0.5 + 0.000075e^(10μ) = 0.0001e^(10μ)

0.5 = 0.000025e^(10μ)

e^(10μ) = 0.5/0.000025 = 20000

$$\log_e(e^{10\mu}) = \ln(20000)$$

10μ = ln(20000) = 9.9035

μ = 9.9035/10 = 0.99

$$R_0 = \frac{< k > * \beta}{\mu} = 3 = \frac{< k > * \beta}{0.99}$$

<k>β = 2.97

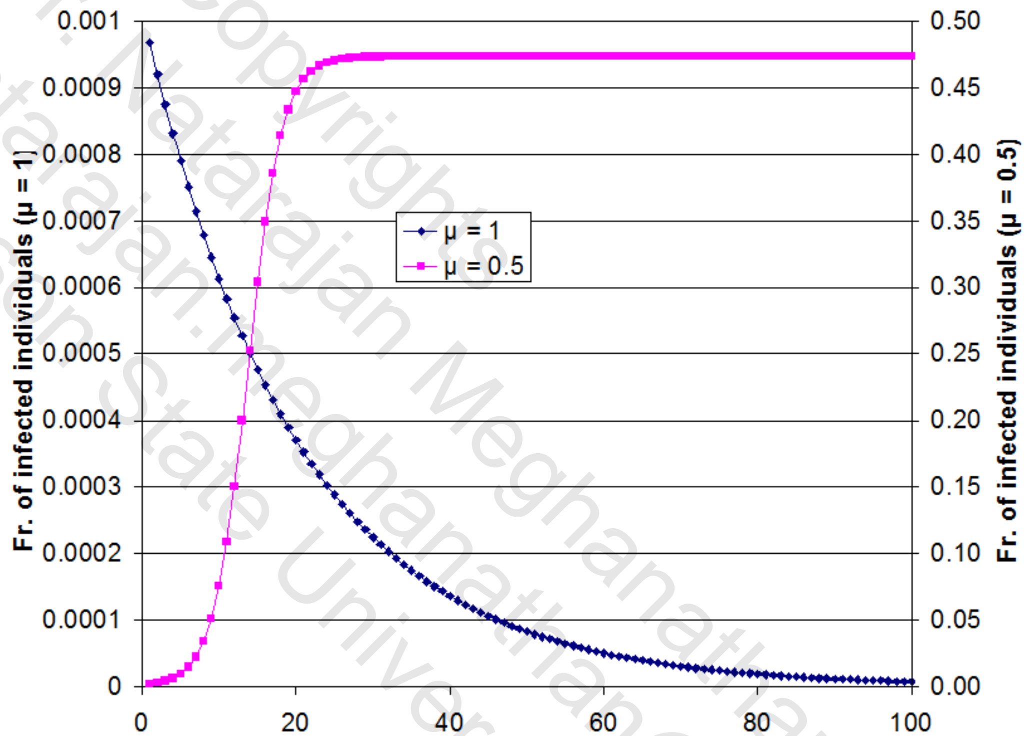
<u>What happens when μ > β<k></u>

i.e., the recovery rate greater than the number of individuals who can get infected when in contact with one infected individual

$$i = \left(1 - \frac{\mu}{\beta < k >}\right) \frac{Ce^{(\beta<k>-\mu)t}}{1 + Ce^{(\beta<k>-\mu)t}} \quad \text{.........................} (8)$$

$$C = \frac{i_0}{(1 - \mu / \beta < k >)}$$

the exponent in equation (8) becomes negative; the number of infected individuals decreases exponentially and the disease soon dies out.



SIS model with μ = 1 and μ = 0.5 vs. β<k> = 0.95, i0 = 0.001

Practice Problem 2:
Consider the spread of an epidemic under the SIS model. The R0 and characteristic time for the epidemic are 3 and 10 days respectively.
(a) What is the average duration of infection for a person?
(b) If the fraction of the people infected during the 30th day of the epidemic is 0.5, draw a plot that presents the fraction of the people infected during each of the first 100 days of the epidemic.

Solution:
Characteristic Time: Is the time it takes for the disease to reach an Equilibrium state (steady-state)

Characteristic time for SIS model, $\tau = \frac{1}{\mu(R_0 - 1)}$ = 10 days

$$\tau = \frac{1}{\mu(3-1)} = 10$$

$$\frac{1}{\mu} = 10 * 2 = 20$$

$\frac{1}{\mu} = 10*(3-1) = 20$ days is the average duration of infection for a person.

$\mu = 1/20 = 0.05$ is the probability with which an infected person will recover on a particular day

$R0 = 3 = <k>\beta / \mu$; $<k>$ is the average # of susceptible contacts per infected individual
β is the probability with which an infected individual can spread the disease to a susceptible individual per unit time when comes into contact with the latter

$<k>\beta = R0 * \mu = 3*0.05 = 0.15.$

$R0 = <k>\beta / \mu$

$1/R0 = \mu / <k>\beta$

Per the SIS model, $i(t) = \left(1 - \frac{\mu}{\beta < k >}\right) \frac{Ce^{(\beta<k>-\mu)t}}{1 + Ce^{(\beta<k>-\mu)t}}$ .... (8)

Given that i(t = 30) = 0.5, we have:

$$i(t=30) = 0.5 = \left(1 - \frac{1}{3}\right) \frac{Ce^{(0.15-0.05)30}}{1 + Ce^{(0.15-0.05)30}}$$

The value for e is 2.7183

$$0.5 = \left(\frac{2}{3}\right) \frac{C*20.086}{1 + C*20.086} = \frac{C*13.391}{1 + C*20.086}$$

$$\frac{C*13.391}{1 + C*20.086} = 0.5$$

C*13.391 = 0.5*[1+C*20.086]

C*13.391 = 0.5 + C*10.043
C*[13.391-10.043] = 0.5

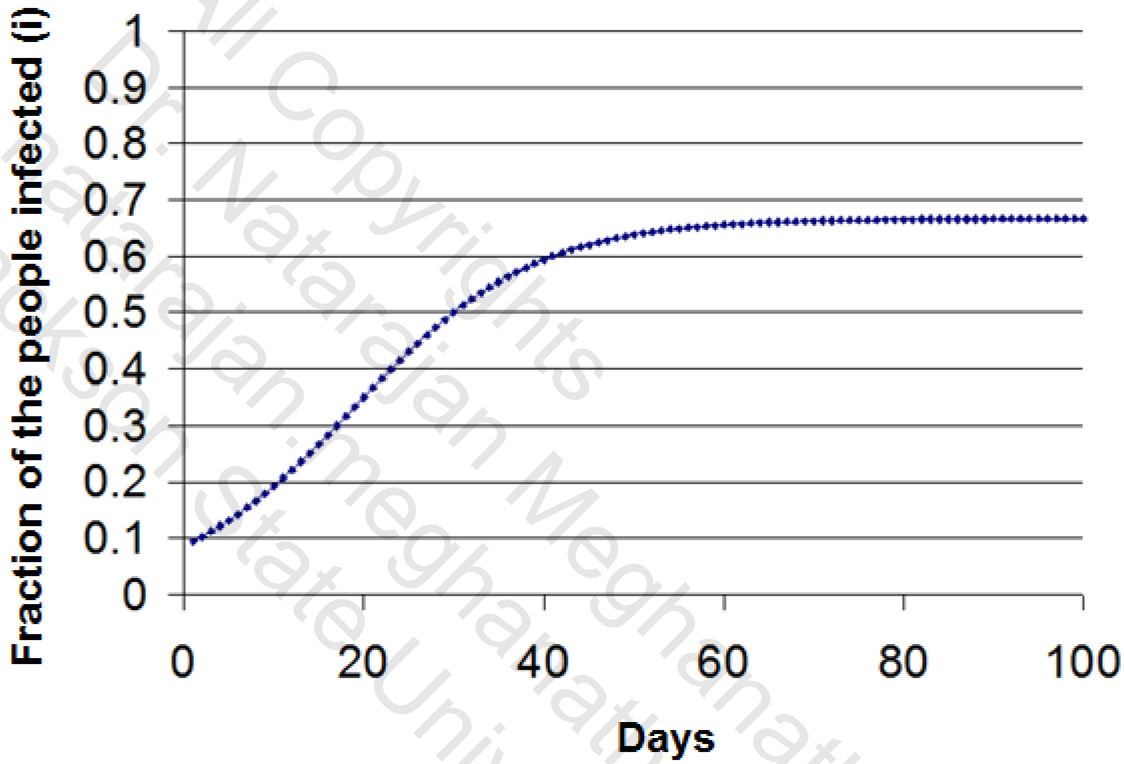
C = 0.5 / (13.391 - 10.043) = 0.1493

The epidemic becomes an endemic when the fraction of infected people i = 1 - 1/R0. = 1 - 1/3 = 0.6667.

Substituting back for C = 0.1493 in equation (8) for the SIS model

$$i(t) = \left(1 - \frac{\mu}{\beta < k >}\right) \frac{Ce^{(\beta<k>-\mu)t}}{1 + Ce^{(\beta<k>-\mu)t}}$$

$$i(t) = \left(1 - \frac{1}{3}\right) \frac{0.1493 * e^{(0.15-0.05)t}}{1 + 0.1493 * e^{(0.15-0.05)t}}$$



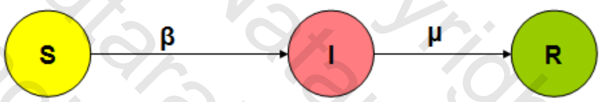
1.3 SIR Model

The infected people recover at rate μ and enter to the Recovered (R) state. The dead people are also considered to have moved to the R state.

Diseases caused by viruses are usually of SIR type.

Let R(t) represents the number of recovered people at time t out of a population of N.

If i(t) (simply represented as *i*) represents the fraction of infected people: I(t)/N at time t,
the rate at which the fraction of recovered people r(t) (simply represented as *r*) = R(t)/N changes in a time period dt given by:



$r(t+dt) = r(t) + i*\mu*dt$
$r(t+dt) - r(t) = dr(t) = i*\mu*dt$

$$\frac{dr(t)}{dt} = i*\mu$$

In time dt, the probability with which an infected person can recover is μ*dt

i(t) is the fraction of infected people at time t
Fraction of infected people who will recover in time 'dt' is i(t) * μ*dt
= i*μ*dt

$$\frac{dr}{dt} = \mu * i \qquad \text{..............................} \ (9)$$

Let S(t) represents the number of susceptible people at time t out of a population of N and s(t) = S(t)/N represents the fraction of susceptible people at time t (simply represented as s).

At any time t, s + i + r = 1

s = 1-i-r

An infected individual can infect its contacts (<k> in number) with a probability β. The average fraction of susceptible contacts that can get infected per infected individual is s*<k>*β. With *i* as the fraction of infected people, the fraction of susceptible population getting infected at time dt is i*s*<k>*β, which is: i*(1-i-r) *<k>*β.

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s(t)

s(t+dt)

β is the probability with which a susceptible person can get infected by an infected individual per time unit (per day)

fraction of susceptible contacts (people) who can get infected by an infected individual per time unit = s*<k>*β

There are 'i' fraction of infected people

Fraction of susceptible people who can get infected by 'i' fraction of infected people per time unit is i*s*<k>*β

Fraction of susceptible people who can get infected by 'i' fraction of infected people for 'dt' time units is i*s*<k>*β*dt

s(t) > s(t+dt)
s(t) - s(t+dt) = i*s*<k>*β*dt

ds(t) = s(t+dt) - s(t) = - i*s*<k>*β*dt

ds(t): change in the fraction of susceptible people

ds(t)/dt = - i*s*<k>*β

$$\frac{ds}{dt} = -<k>\beta * i * (1 - i - r) \quad \text{..................................} (10)$$

Note that i*(1-i-r) *<k>*β is the fraction of newly infected people in time dt and μ*i is the fraction of the infected people who have recovered in time dt.

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i(t) is the fraction of infected people at time t

susceptible --> infected: i(t)*s(t)*<k>*β*dt
infected --> recovered: i(t)*μ*dt

i(t+dt) = i(t) + i(t)*s(t)*<k>*β*dt - i(t)*μ*dt
di(t) = i(t+dt) - i(t) = i(t)*s(t)*<k>*β*dt - i(t)*μ*dt

di(t)/dt = i*s*<k>*β - i*μ

Hence, $\frac{di}{dt} = <k> \beta * i * (1-i-r) - \mu * i$ .... (11)

$$\frac{di}{dt} = i * [<k> \beta * (1-i-r) - \mu]$$

Let s_0 and i_0 represent the fraction of susceptible people and infected people respectively at time 0. At time 0, there are no recovered people. Hence, $s_0 + i_0 = 1$.

At any time t > 0, the fraction of susceptible people can only decrease as they get infected. Hence,
s(t) <= s0
s <= s0

$(1-i-r)$ (which is s) is always less than or equal to s_0.

Hence, $i * [<k> \beta * (1-i-r) - \mu] \leq i * [<k> \beta * s_0 - \mu]$

Therefore, $\frac{di}{dt} \leq i * [<k> \beta * s_0 - \mu]$ .... (12)

<u>Basic Reproductive Number</u>

For the rate of infection to decrease over time, di/dt < 0

$$\frac{di}{dt} = i * [<k> \beta * (1-i-r) - \mu]$$

$$\frac{di}{dt} = i * [<k> \beta * (1-i-r) - \mu] < 0$$

Since i is a fraction $0 \leq i \leq 1$,

for di/dt < 0: $[<k> \beta * (1-i-r) - \mu] < 0$

$$[<k> \beta * (1-i-r) - \mu] < 0$$

i.e., $<k> \beta * (1-i-r) < \mu$

$$(1-i-r) < \mu / <k> \beta$$

$$s < \mu / <k> \beta$$

Note that $R_0 = <k> \beta / \mu$ is the basic reproductive number for the SIS model (also applicable for the SIR model).

Hence, for the SIR model if $s < 1/R_0$, the rate of infection spread with time will decrease and will eventually die down.
For any time t, if di/dt has to be less than 0, we need s(t) < 1/R0

i.e., if at any time, the fraction of susceptible people is < the inverse of the basic reproductive number, the pathogen will NOT spread and die down eventually from the population.

Herd Immunity:
If the fraction of susceptible people, s(t) < 1/R0, then the epidemic will eventually die down on its own

Fraction of susceptible people + Fraction of non-susceptible people = 1
Fraction of susceptible people + Fraction of non-susceptible (vaccinated) people = 1
Fraction of vaccinated people = 1 - Fraction of susceptible people

0+s < 0+1/R0

s < 1/R0

-s >= -1/R0
1-s >= 1-1/R0
v = fraction of vaccinated people

If v >= 1 - 1/R0, then the epidemic will die down on its own
Achieving Herd Immunity

<u>Determining a Closed Form Expression for the fraction of infected people *i* at any time</u>

From equations (10) and (11), we have:

$$\frac{ds}{dt} = -<k>\beta * i * (1 - i - r) \qquad \text{.................................} (10)$$

$$\frac{di}{dt} = i * [<k>\beta * (1 - i - r) - \mu] \qquad \text{.................................} (11)$$

We can write *s* as $(1 - i - r)$.

Hence, we can write equations (10) and (11) as:

$$\frac{ds}{dt} = -<k>\beta * i * s \qquad \text{................................} (12)$$

$$\frac{di}{dt} = i * [<k>\beta * s - \mu] \qquad \text{................................} (13)$$

Divide (13) by (12), we get:

$$\frac{di}{ds} = -\left[\frac{<k>\beta * s - \mu}{<k>\beta * s}\right]$$

i.e., $\frac{di}{ds} = -1 + \frac{\mu}{<k>\beta * s}$ .... (14)

R0 = <k>β/μ
1/R0 = μ/<k>β

$$di = -ds + \frac{1}{R_0} * \frac{ds}{s}$$ .... (14)

Integrating both sides of (14): the LHS with respect to *i* and the RHS with respect to *s*, we get:

$$\int_{i_0}^{i} di = -\int_{s_0}^{s} ds + \frac{1}{R_0} * \int_{s_0}^{s} \frac{ds}{s}$$

$$i_{i_0}^{i} = -s_{s_0}^{s} + \frac{1}{R_0} * \ln(s)_{s_0}^{s}$$

$$(i - i_0) = -(s - s_0) + \frac{1}{R_0} * (\ln s - \ln s_0)$$

$$(i - i_0) = -(s - s_0) + \frac{1}{R_0} * \ln\left(\frac{s}{s_0}\right)$$

Hence, $i = i_0 + s_0 - s + \frac{1}{R_0} * \ln\left(\frac{s}{s_0}\right)$

But, $i_0 + s_0 = 1$

Therefore, $i = 1 - s + \frac{1}{R_0} * \ln\left(\frac{s}{s_0}\right)$ .... (15)

Equation (15) models the fraction of infected people (i) at any time on the basis of only the fraction of suspected people at any time (s), even though there is another variable (r) in the SIR model.

Determining the Maximum Number of Infected People at any time per the SIR model

From equation (14),

$$\frac{di}{ds} = -1 + \frac{\mu}{<k>\beta * s}$$

The maximum occurs when the slope of the curve (modeled by the equation (15)) given by equation 14 reaches 0.

i.e., $\frac{di}{ds} = -1 + \frac{\mu}{<k> \beta * s} = 0$

Solving for s, $\frac{\mu}{<k> \beta * s} = 1$

we get: $s = \frac{\mu}{<k> \beta} = \frac{1}{R_0}$

Hence, for i to be the maximum, the fraction of susceptible people s has to be $1/R_0$.

Substituting for $s = 1/R_0$ in equation (15), we get:

$$i_{\max} = 1 - \frac{1}{R_0} + \frac{1}{R_0} * \ln\left(\frac{1}{R_0 * s_0}\right)$$

$$i_{\max} = 1 - \frac{1}{R_0} + \frac{1}{R_0} * [\ln(1) - \ln(R_0 * s_0)]$$

$$i_{\max} = 1 - \frac{1}{R_0} + \frac{1}{R_0} * [0 - \ln(R_0 * s_0)]$$

$$i_{\max} = 1 - \frac{1}{R_0} - \frac{1}{R_0} * [\ln(R_0 * s_0)]$$

$$i_{\max} = 1 - \frac{1}{R_0} * [1 + \ln(R_0 * s_0)] \qquad \text{.................................... (16)}$$

s+i+r = 1

Note that from equation (16), we can infer that the minimum number of people who are not infected at any time $(s+r)_{\min}$ is $1 - i_{\max} = \frac{1}{R_0} * [1 + \ln(R_0 * s_0)]$

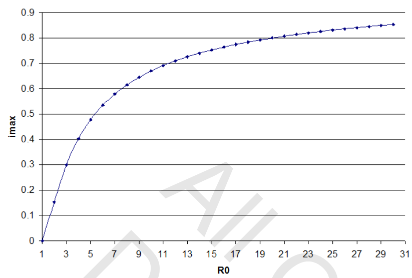
i.e., $(s+r)_{\min} = \frac{1}{R_0} * [1 + \ln(R_0 * s_0)]$ .... (17)

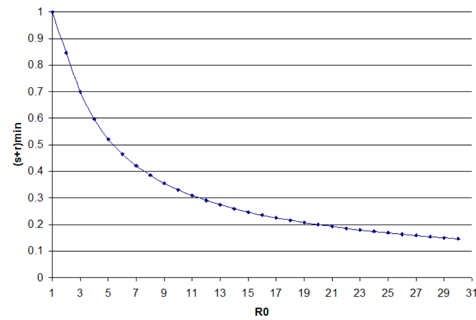
Consider $s_0 = 1$,

For $R_0 = 1$, $i_{\max} = 1 - \frac{1}{1} * [1 + \ln(1)] = 0$ $\quad (s+r)_{\min} = 1$

For $R_0 = 5$, $i_{\max} = 1 - \frac{1}{5} * [1 + \ln(5)] = 0.4781$ $\quad (s+r)_{\min} = 0.5219$

For $R_0 = 10$, $i_{\max} = 1 - \frac{1}{10} * [1 + \ln(10)] = 0.6697$ $\quad (s+r)_{\min} = 0.3303$





s0 = 1

1.4 Discrete Event Simulation of the SIR Model

We will start with $s_0 > 0$, $i_0 > 0$ and $r_0 = 0$. i.e., s0+i0 = 1

we will proceed with discrete times, t = 0, 1, 2, 3, ...

Let s(t), i(t) and r(t) be respectively the fractions of susceptible, infected and recovered people/nodes at time t, and be simply represented as s, i an r. Note that s + i + r = 1 for any time instant t.

The infected individual spreads the disease with probability β (per time unit).

Let <k> be the average number of contacts per infected individual.

<k>β is the average number of contacts that can be infected by the infected individual and is also called the transmissibility or transmission rate.

<k>β is the average number of *susceptible* contacts that can be infected by the infected individual.

Fraction of nodes that became newly infected at t+1 = $< k > \beta * i * s$
// Note: To get infected, a contact node has to be currently susceptible
Fraction of nodes that became newly infected at t+1 can also be written as = $< k > \beta * i * (1 - i - r)$

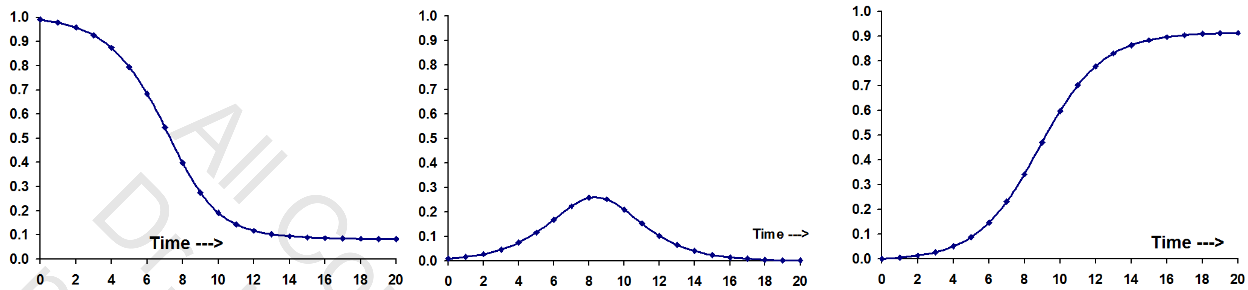
Fraction of nodes that specifically recover at t+1 = $i * \mu$
// Note: To recover, a node has to be currently infected

Fraction of total nodes that are susceptible at the end of t+1 is: $s - < k > \beta * i * s$
Fraction of total nodes that are infected at the end of t+1 is: $i + < k > \beta * i * s - i * \mu$
Fraction of total nodes that have recovered at the end of t+1 is: $r + i * \mu$

Example 1: Let <k> = 4, β = 0.3, μ = 0.5, $s_0 = 0.99$, $i_0 = 0.01$

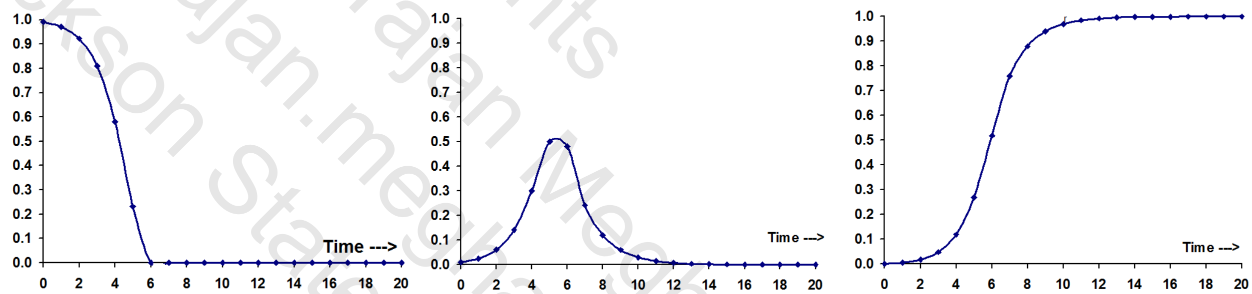
Time, t	Newly infected	Newly recovered	s	i	r	Total fraction
0	-	-	0.99	0.01	0	1
1	0.01188	0.005	0.97812	0.01688	0.005	1
2	0.019813	0.00844	0.958307	0.028253	0.01344	1
3	0.03249	0.014126	0.925817	0.046616	0.027566	1
4	0.05179	0.023308	0.874028	0.075098	0.050875	1
5	0.078765	0.037549	0.795263	0.116314	0.088423	1
6	0.111	0.058157	0.684262	0.169157	0.14658	1
7	0.138898	0.084579	0.545365	0.223476	0.231159	1
8	0.146251	0.111738	0.399114	0.257989	0.342897	1
9	0.12356	0.128995	0.275553	0.252555	0.471892	1
10	0.083511	0.126278	0.192042	0.209788	0.598169	1
11	0.048346	0.104894	0.143696	0.15324	0.703064	1
12	0.026424	0.07662	0.117272	0.103044	0.779684	1
13	0.014501	0.051522	0.102771	0.066023	0.831206	1
14	0.008142	0.033012	0.094629	0.041154	0.864217	1
15	0.004673	0.020577	0.089956	0.02525	0.884794	1
16	0.002726	0.012625	0.08723	0.015351	0.897419	1
17	0.001607	0.007675	0.085623	0.009282	0.905095	1
18	0.000954	0.004641	0.084669	0.005595	0.909736	1
19	0.000568	0.002797	0.084101	0.003366	0.912533	1
20	0.00034	0.001683	0.083761	0.002023	0.914216	1

Sum	**0.906239**	**0.914216**	**8.591581**	**1.830455**	**10.57796**	
Std Dev.	**0.050263**	**0.044342**	**0.369433**	**0.088612**	**0.380764**	



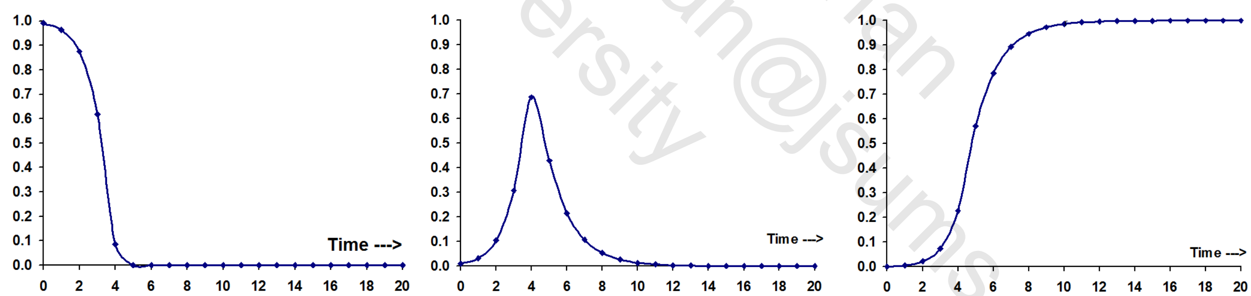
Frac. of total nodes that are susceptible — Frac. of total nodes that are infected — Frac. of total nodes that have recovered

SIR Model (Ex. 1): Fraction of the total nodes in the susceptible (s), infected (i) and recovered (r) status
<k> = 4, β = 0.3, μ = 0.5, s_0 = 0.99, i_0 = 0.01



Frac. of total nodes that are susceptible — Frac. of total nodes that are infected — Frac. of total nodes that have recovered

SIR Model (Ex. 2): Fraction of the total nodes in the susceptible (s), infected (i) and recovered (r) status
<k> = 4, β = 0.5, μ = 0.5, s_0 = 0.99, i_0 = 0.01



Frac. of total nodes that are susceptible — Frac. of total nodes that are infected — Frac. of total nodes that have recovered

SIR Model (Ex. 3): Fraction of the total nodes in the susceptible (s), infected (i) and recovered (r) status
<k> = 4, β = 0.7, μ = 0.5, s_0 = 0.99, i_0 = 0.01

Example 2: Let <k> = 4, β = 0.5, μ = 0.5, s_0 = 0.99, i_0 = 0.01

Time, t	Newly infected	Newly recovered	s	i	r	Total fraction
0			0.99	0.01	0	1
1	0.0198	0.005	0.9702	0.0248	0.005	1
2	0.048122	0.0124	0.922078	0.060522	0.0174	1

3	0.111612	0.030261	0.810466	0.141873	0.047661	1
4	0.229966	0.070936	0.5805	0.300903	0.118597	1
5	0.349348	0.150451	0.231152	0.499799	0.269049	1
6	0.231059	0.2499	9.28E-05	0.480959	0.518948	1
7	8.92E-05	0.240479	3.53E-06	0.240569	0.759428	1
8	1.7E-06	0.120284	1.83E-06	0.120286	0.879712	1
9	4.41E-07	0.060143	1.39E-06	0.060143	0.939855	1
10	1.67E-07	0.030072	1.22E-06	0.030072	0.969927	1
11	7.37E-08	0.015036	1.15E-06	0.015036	0.984963	1
12	3.46E-08	0.007518	1.12E-06	0.007518	0.992481	1
13	1.68E-08	0.003759	1.1E-06	0.003759	0.99624	1
14	8.27E-09	0.00188	1.09E-06	0.00188	0.998119	1
15	4.1E-09	0.00094	1.09E-06	0.00094	0.999059	1
16	2.04E-09	0.00047	1.09E-06	0.00047	0.999529	1
17	1.02E-09	0.000235	1.08E-06	0.000235	0.999764	1
18	5.09E-10	0.000117	1.08E-06	0.000117	0.999881	1
19	2.55E-10	5.87E-05	1.08E-06	5.87E-05	0.99994	1
20	1.27E-10	2.94E-05	1.08E-06	2.94E-05	0.99997	1
Sum	**0.989999**	**0.99997**	**4.504508**	**1.999968**	**14.49552**	
Std Dev.	**0.101171**	**0.078919**	**0.377668**	**0.155381**	**0.416386**	

Example 3: Let <k> = 4, β = 0.7, μ = 0.5, s_0 = 0.99, i_0 = 0.01

Time, t	**Newly infected**	**Newly recovered**	**s**	**i**	**r**	**Total fraction**
0			0.99	0.01	0	1
1	0.02772	0.005	0.96228	0.03272	0.005	1
2	0.08816	0.01636	0.87412	0.10452	0.02136	1
3	0.255817	0.05226	0.618303	0.308077	0.07362	1
4	0.533358	0.154039	0.084945	0.687396	0.227659	1
5	**0.084945** (0.1634)	0.343698	0	0.428643	0.571357	1
6	0	0.214322	0	0.214322	0.785678	1
7	0	0.107161	0	0.107161	0.892839	1
8	0	0.05358	0	0.05358	0.94642	1
9	0	0.02679	0	0.02679	0.97321	1
10	0	0.013395	0	0.013395	0.986605	1
11	0	0.006698	0	0.006698	0.993302	1
12	0	0.003349	0	0.003349	0.996651	1
13	0	0.001674	0	0.001674	0.998326	1
14	0	0.000837	0	0.000837	0.999163	1
15	0	0.000419	0	0.000419	0.999581	1
16	0	0.000209	0	0.000209	0.999791	1
17	0	0.000105	0	0.000105	0.999895	1
18	0	5.23E-05	0	5.23E-05	0.999948	1
19	0	2.62E-05	0	2.62E-05	0.999974	1
20	0	1.31E-05	0	1.31E-05	0.999987	1
Sum	**0.99**	**0.999987**	**3.529647**	**1.999987**	**15.47037**	
Std Dev.	**0.129099**	**0.090649**	**0.351171**	**0.178049**	**0.399462**	

Note that if the fraction of total nodes that became newly infected at time instant t+1: $<k> \beta * i * (1 - i - r)$ exceeds the fraction of total nodes that are susceptible at the end of time instant t, then the fraction of total nodes that became newly infected at time instant t+1 is set equal to the fraction of total nodes that are susceptible at the end of time instant t, and the calculations are continued. If that is the case, then the fraction of the total nodes that are susceptible at the end of time instant t+1 becomes 0. In the above table (Example 3), such a situation arises at time instant t+1 = 5 where in the fraction of total nodes that could become newly infected (0.1634) at time instant 5 exceeds the fraction of total nodes that are susceptible (0.084945) at the end of time instant 4.

Example 4: Consider the Rubella disease that lasts for 11 days. A city of 30,000 people is exposed to the disease wherein a person is on average in contact with 6.8 other people. Let S(0) = 29,000 and I(0) = 1,000. Assume R0 for Rubella is 5.0. Assume Rubella follows a SIR model.
(1) Determine the number of people who are infected in the next two days.
(2) Determine the day when the maximum number of people will be infected. What is the maximum number of infected people?
(3) Determine the day when less than 1% of the population remains infected with Rubella.

Given: $\langle k \rangle = 6.8$ R0 = 5.0 $1/\mu = 11 \implies \mu = 1/11 = 0.091$

$R0 = \langle k \rangle\beta/\mu \implies \beta = R0*\mu/\langle k \rangle = 5*0.091/6.8 = 0.067$

Day	new infec fr	new recov fr	new infec	new rec	S	I	R	Total	s	i	r
0					29000	1000	0	30000	0.966667	0.033333	0
1	0.01468044	0.003033333	440.4133	91	28559.59	1349.413	91	30000	0.951986	0.04498	0.003033
2	0.01950914	0.00409322	585.2742	122.7966	27974.31	1811.891	213.7966	30000	0.932477	0.060396	0.007127
3	0.02565858	0.005496069	769.7575	164.8821	27204.55	2416.766	378.6787	30000	0.906818	0.080559	0.012623
4	0.03328262	0.007330858	998.4786	219.9257	26206.08	3195.319	598.6044	30000	0.873536	0.106511	0.019953
5	0.04238942	0.009692468	1271.683	290.774	24934.39	4176.228	889.3785	30000	0.831146	0.139208	0.029646
6	0.05271378	0.012667891	1581.414	380.0367	23352.98	5377.604	1269.415	30000	0.778433	0.179253	0.042314
7	0.06357295	0.016312067	1907.189	489.362	21445.79	6795.431	1758.777	30000	0.71486	0.226514	0.058626
8	0.07377348	0.020612807	2213.205	618.3842	19232.59	8390.251	2377.161	30000	0.641086	0.279675	0.079239
9	0.08168718	0.025450429	2450.615	763.5129	16781.97	10077.35	3140.674	30000	0.559399	0.335912	0.104689
10	0.08561122	0.030567973	2568.337	917.0392	14213.64	11728.65	4057.713	30000	0.473788	0.390955	0.135257
11	0.08439067	0.035576909	2531.72	1067.307	11681.92	13193.06	5125.021	30000	0.389397	0.439769	0.170834
12	0.0780191	0.040018961	2340.573	1200.569	9341.342	14333.07	6325.59	30000	0.311378	0.477769	0.210853
13	0.06777814	0.043476974	2033.344	1304.309	7307.998	15062.1	7629.899	30000	0.2436	0.50207	0.25433
→ 14	0.05572181	0.04568838	1671.654	1370.651	5636.344	15363.11	9000.55	30000	0.187878	0.512104	0.300018
15	0.04383467	0.046601422	1315.04	1398.043	4321.304	15280.1	10398.59	30000	0.144043	0.509337	0.34662
16	0.03342584	0.046349648	1002.775	1390.489	3318.528	14892.39	11789.08	30000	0.110618	0.496413	0.392969
17	0.02501792	0.045173581	750.5375	1355.207	2567.991	14287.72	13144.29	30000	0.0856	0.476257	0.438143
18	0.01857367	0.043339415	557.21	1300.182	2010.781	13544.75	14444.47	30000	0.067026	0.451492	0.481482
19	0.01378723	0.041085732	413.6168	1232.572	1597.164	12725.79	15677.04	30000	0.053239	0.424193	0.522568
20	0.01028906	0.038601568	308.6717	1158.047	1288.493	11876.42	16835.09	30000	0.04295	0.395881	0.56117
21	0.00774655	0.036025129	232.3966	1080.754	1056.096	11028.06	17915.85	30000	0.035203	0.367602	0.597195
22	0.00589581	0.033451779	176.8744	1003.553	879.2216	10201.38	18919.4	30000	0.029307	0.340046	0.630647
23	0.00454045	0.030944186	136.2134	928.3256	743.0082	9409.268	19847.72	30000	0.024767	0.313642	0.661591
24	0.00353908	0.028541446	106.1725	856.2434	636.8357	8659.197	20703.97	30000	0.021228	0.28864	0.690132
25	0.00279156	0.026266231	83.74666	787.9869	553.0891	7954.957	21491.95	30000	0.018436	0.265165	0.716398
26	0.00222728	0.024130035	66.81829	723.9011	486.2708	7297.874	22215.86	30000	0.016209	0.243262	0.740529
27	0.00179645	0.022136884	53.89357	664.1065	432.3772	6687.661	22879.96	30000	0.014413	0.222922	0.762665
28	0.00146379	0.020285905	43.91365	608.5771	388.4636	6122.997	23488.54	30000	0.012949	0.2041	0.782951
29	0.00120408	0.018573092	36.12242	557.1928	352.3411	5601.927	24045.73	30000	0.011745	0.186731	0.801524
30	0.00099918	0.016992512	29.97528	509.7754	322.3659	5122.127	24555.51	30000	0.010746	0.170738	0.818517
31	0.00083587	0.015537119	25.07621	466.1136	297.2897	4681.09	25021.62	30000	0.00991	0.156036	0.834054
32	0.00070448	0.014199305	21.13437	425.9792	276.1553	4276.245	25447.6	30000	0.009205	0.142541	0.848253
33	0.0005978	0.012971276	17.93405	389.1383	258.2212	3905.041	25836.74	30000	0.008607	0.130168	0.861225
34	0.00051046	0.01184529	15.31369	355.3587	242.9075	3564.996	26192.1	30000	0.008097	0.118833	0.87307

Answers:
(1) Determine the number of people who are infected in the next two days = 1811.891 ~ 1812
(2) Determine the day when the maximum number of people will be infected. What is the maximum number of infected people? = 14, 15363.11 ~ 15363 = 15363/30000 = 51.21% of the population
(3) Determine the day when less than 1% of the population remains infected with Rubella. = 61

Day	new infec fr	new recov fr	new infec	new rec	S	I	R	Total	s	i	r
35	0.00043837	0.01081382	13.15111	324.4146	229.7564	3253.732	26516.51	30000	0.007659	0.108458	0.883884
36	0.00037843	0.009869654	11.35303	296.0896	218.4034	2968.996	26812.6	30000	0.00728	0.098967	0.893753
37	0.00032825	0.009005953	9.847623	270.1786	208.5558	2708.665	27082.78	30000	0.006952	0.090289	0.902759
38	0.00028597	0.008216283	8.579064	246.4885	199.9767	2470.755	27329.27	30000	0.006666	0.082359	0.910976
39	0.00025012	0.007494624	7.503633	224.8387	192.4731	2253.42	27554.11	30000	0.006416	0.075114	0.91847
40	0.00021956	0.006835374	6.586802	205.0612	185.8863	2054.946	27759.17	30000	0.006196	0.068498	0.925306
41	0.00019337	0.006233335	5.801097	187.0001	180.0852	1873.747	27946.17	30000	0.006003	0.062458	0.931539
42	0.00017082	0.005683698	5.124498	170.5109	174.9607	1708.36	28116.68	30000	0.005832	0.056945	0.937223
43	0.00015131	0.005182026	4.539232	155.4608	170.4214	1557.439	28272.14	30000	0.005681	0.051915	0.942405
44	0.00013436	0.004724231	4.03086	141.7269	166.3906	1419.743	28413.87	30000	0.005546	0.047325	0.947129
45	0.00011959	0.004306553	3.587574	129.1966	162.803	1294.134	28543.06	30000	0.005427	0.043138	0.951435
46	0.00010666	0.003925539	3.199661	117.7662	159.6034	1179.567	28660.83	30000	0.00532	0.039319	0.955361
47	9.5303E-05	0.00357802	2.859085	107.3406	156.7443	1075.086	28768.17	30000	0.005225	0.035836	0.958939
48	8.5305E-05	0.003261093	2.559158	97.83279	154.1851	979.812	28866	30000	0.00514	0.03266	0.9622
49	7.6476E-05	0.002972096	2.294286	89.16289	151.8908	892.9434	28955.17	30000	0.005063	0.029765	0.965172
50	6.8659E-05	0.002708595	2.059766	81.25785	149.8311	813.7453	29036.42	30000	0.004994	0.027125	0.967881
51	6.1721E-05	0.002468361	1.851624	74.05082	147.9794	741.5461	29110.47	30000	0.004933	0.024718	0.970349
52	5.555E-05	0.002249356	1.666487	67.48069	146.3129	675.7319	29177.96	30000	0.004877	0.022524	0.972599
53	5.0049E-05	0.00204972	1.50148	61.4916	144.8115	615.7418	29239.45	30000	0.004827	0.020525	0.974648
54	4.5138E-05	0.00186775	1.354141	56.0325	143.4573	561.0634	29295.48	30000	0.004782	0.018702	0.976516
55	4.0745E-05	0.001701892	1.222354	51.05677	142.235	511.229	29346.54	30000	0.004741	0.017041	0.978218
56	3.681E-05	0.001550728	1.104293	46.52184	141.1307	465.8114	29393.06	30000	0.004704	0.015527	0.979769
57	3.3279E-05	0.001412961	0.998376	42.38884	140.1323	424.421	29435.45	30000	0.004671	0.014147	0.981182
58	3.0108E-05	0.00128741	0.903228	38.62231	139.2291	386.7019	29474.07	30000	0.004641	0.01289	0.982469
59	2.7255E-05	0.001172996	0.817652	35.18987	138.4114	352.3297	29509.26	30000	0.004614	0.011744	0.983642
60	2.4687E-05	0.001068733	0.7406	32.062	137.6708	321.0083	29541.32	30000	0.004589	0.0107	0.984711
61	2.2372E-05	0.000973725	0.671152	29.21175	136.9997	292.4677	29570.53	30000	0.004567	0.009749	0.985684
62	2.0283E-05	0.000887152	0.608499	26.61456	136.3912	266.4616	29597.15	30000	0.004546	0.008882	0.986572
63	1.8398E-05	0.000808267	0.551929	24.24801	135.8392	242.7655	29621.4	30000	0.004528	0.008092	0.98738
64	1.6694E-05	0.000736389	0.500812	22.09166	135.3384	221.1747	29643.49	30000	0.004511	0.007372	0.988116
65	1.5153E-05	0.000670897	0.454589	20.1269	134.8838	201.5024	29663.61	30000	0.004496	0.006717	0.988787
66	1.3759E-05	0.000611224	0.412765	18.33672	134.4711	183.5784	29681.95	30000	0.004482	0.006119	0.989398
67	1.2497E-05	0.000556855	0.374898	16.70564	134.0962	167.2477	29698.66	30000	0.00447	0.005575	0.989955
68	1.1353E-05	0.000507318	0.340596	15.21954	133.7556	152.3687	29713.88	30000	0.004459	0.005079	0.990463
69	1.0317E-05	0.000462185	0.309507	13.86556	133.4461	138.8127	29727.74	30000	0.004448	0.004627	0.990925
70	9.3773E-06	0.000421065	0.281318	12.63196	133.1648	126.4621	29740.37	30000	0.004439	0.004215	0.991346
71	8.5249E-06	0.000383602	0.255748	11.50805	132.909	115.2098	29751.88	30000	0.00443	0.00384	0.991729

Day	new infec fr	new recov fr	new infec	new rec	S	I	R	Total	s	i	r
72	7.7515E-06	0.00034947	0.232545	10.48409	132.6765	104.9582	29762.37	30000	0.004423	0.003499	0.992079
73	7.0494E-06	0.000318373	0.211482	9.551198	132.465	95.6185	29771.92	30000	0.004415	0.003187	0.992397
74	6.4119E-06	0.000290043	0.192356	8.701283	132.2726	87.10957	29780.62	30000	0.004409	0.002904	0.992687
75	5.8328E-06	0.000264232	0.174984	7.926971	132.0976	79.35758	29788.54	30000	0.004403	0.002645	0.992951
76	5.3067E-06	0.000240718	0.159201	7.22154	131.9384	72.29525	29795.77	30000	0.004398	0.00241	0.993192
77	4.8286E-06	0.000219296	0.144858	6.578867	131.7936	65.86124	29802.35	30000	0.004393	0.002195	0.993412
78	4.3941E-06	0.000199779	0.131822	5.993372	131.6618	59.99969	29808.34	30000	0.004389	0.002	0.993611
79	3.999E-06	0.000181999	0.11997	5.459971	131.5418	54.65968	29813.8	30000	0.004385	0.001822	0.993793
80	3.6398E-06	0.000165801	0.109193	4.974031	131.4326	49.79484	29818.77	30000	0.004381	0.00166	0.993959
81	3.3131E-06	0.000151044	0.099392	4.531331	131.3332	45.36291	29823.3	30000	0.004378	0.001512	0.99411
82	3.0159E-06	0.000137601	0.090477	4.128024	131.2427	41.32536	29827.43	30000	0.004375	0.001378	0.994248
83	2.7456E-06	0.000125354	0.082367	3.760608	131.1604	37.64712	29831.19	30000	0.004372	0.001255	0.994373
84	2.4996E-06	0.000114196	0.074989	3.425888	131.0854	34.29622	29834.62	30000	0.00437	0.001143	0.994487
85	2.2758E-06	0.000104032	0.068275	3.120956	131.0171	31.24354	29837.74	30000	0.004367	0.001041	0.994591
86	2.0722E-06	9.47721E-05	0.062166	2.843162	130.9549	28.46254	29840.58	30000	0.004365	0.000949	0.994686
87	1.8868E-06	8.63364E-05	0.056605	2.590091	130.8983	25.92906	29843.17	30000	0.004363	0.000864	0.994772
88	1.7182E-06	7.86515E-05	0.051545	2.359544	130.8468	23.62106	29845.53	30000	0.004362	0.000787	0.994851
89	1.5646E-06	7.16505E-05	0.046938	2.149516	130.7998	21.51848	29847.68	30000	0.00436	0.000717	0.994923
90	1.4248E-06	6.52727E-05	0.042745	1.958182	130.7571	19.60304	29849.64	30000	0.004359	0.000653	0.994988
91	1.2976E-06	5.94626E-05	0.038927	1.783877	130.7182	17.85809	29851.42	30000	0.004357	0.000595	0.995047
92	1.1817E-06	5.41695E-05	0.035451	1.625086	130.6827	16.26846	29853.05	30000	0.004356	0.000542	0.995102
93	1.0762E-06	4.93477E-05	0.032287	1.48043	130.6504	14.82031	29854.53	30000	0.004355	0.000494	0.995151
94	9.8019E-07	4.4955E-05	0.029406	1.348649	130.621	13.50107	29855.88	30000	0.004354	0.00045	0.995196
95	8.9273E-07	4.09532E-05	0.026782	1.228597	130.5942	12.29926	29857.11	30000	0.004353	0.00041	0.995237
96	8.131E-07	3.73077E-05	0.024393	1.119232	130.5699	11.20442	29858.23	30000	0.004352	0.000373	0.995274
97	7.4058E-07	3.39867E-05	0.022217	1.019602	130.5476	10.20703	29859.25	30000	0.004352	0.00034	0.995308
98	6.7454E-07	3.09613E-05	0.020236	0.92884	130.5274	9.298428	29860.17	30000	0.004351	0.00031	0.995339
99	6.144E-07	2.82052E-05	0.018432	0.846157	130.509	8.470704	29861.02	30000	0.00435	0.000282	0.995367
100	5.5963E-07	2.56945E-05	0.016789	0.770834	130.4922	7.716658	29861.79	30000	0.00435	0.000257	0.995393
101	5.0975E-07	2.34072E-05	0.015292	0.702216	130.4769	7.029735	29862.49	30000	0.004349	0.000234	0.995416
102	4.6432E-07	2.13235E-05	0.013929	0.639706	130.463	6.403959	29863.13	30000	0.004349	0.000213	0.995438
103	4.2294E-07	1.94253E-05	0.012688	0.58276	130.4503	5.833886	29863.72	30000	0.004348	0.000194	0.995457
104	3.8525E-07	1.76961E-05	0.011558	0.530884	130.4387	5.31456	29864.25	30000	0.004348	0.000177	0.995475
105	3.5093E-07	1.61208E-05	0.010528	0.483625	130.4282	4.841463	29864.73	30000	0.004348	0.000161	0.995491
106	3.1966E-07	1.46858E-05	0.00959	0.440573	130.4186	4.41048	29865.17	30000	0.004347	0.000147	0.995506
107	2.9118E-07	1.33785E-05	0.008736	0.401354	130.4099	4.017862	29865.57	30000	0.004347	0.000134	0.995519
108	2.6524E-07	1.21875E-05	0.007957	0.365625	130.4019	3.660194	29865.94	30000	0.004347	0.000122	0.995531
109	2.4162E-07	1.11026E-05	0.007249	0.333078	130.3947	3.334364	29866.27	30000	0.004346	0.000111	0.995542
110	2.201E-07	1.01142E-05	0.006603	0.303427	130.3881	3.03754	29866.57	30000	0.004346	0.000101	0.995552
111	2.0049E-07	9.21387E-06	0.006015	0.276416	130.382	2.767139	29866.85	30000	0.004346	9.22E-05	0.995562
112	1.8264E-07	8.39365E-06	0.005479	0.25181	130.3766	2.520808	29867.1	30000	0.004346	8.4E-05	0.99557
113	1.6637E-07	7.64645E-06	0.004991	0.229394	130.3716	2.296406	29867.33	30000	0.004346	7.65E-05	0.995578
114	1.5156E-07	6.96576E-06	0.004547	0.208973	130.367	2.09198	29867.54	30000	0.004346	6.97E-05	0.995585
115	1.3806E-07	6.34567E-06	0.004142	0.19037	130.3629	1.905751	29867.73	30000	0.004345	6.35E-05	0.995591
116	1.2577E-07	5.78078E-06	0.003773	0.173423	130.3591	1.736101	29867.9	30000	0.004345	5.79E-05	0.995597
117	1.1457E-07	5.26617E-06	0.003437	0.157985	130.3557	1.581553	29868.06	30000	0.004345	5.27E-05	0.995602
118	1.0436E-07	4.79738E-06	0.003131	0.143921	130.3525	1.440762	29868.21	30000	0.004345	4.8E-05	0.995607
119	9.5072E-08	4.37031E-06	0.002852	0.131109	130.3497	1.312505	29868.34	30000	0.004345	4.38E-05	0.995611
120	8.6607E-08	3.98127E-06	0.002598	0.119438	130.3471	1.195665	29868.46	30000	0.004345	3.99E-05	0.995615

Example 5: Consider an epidemic that makes a person stay infected for 6 days. A city of 30,000 people is exposed to the disease wherein a person is on average in contact with 6.8 other people. Let S(0) = 29,000. Find the maximum number of people who will be infected on any day. Assume the disease can spread from an infected individual to anyone with whom he/she comes into contact.

Solution:
Since the disease can spread from an infected individual to anyone with whom he/she comes into contact, the parameter $\beta = 1.0$
The disease makes a person infected for $1/\mu = 6$ days ==> $\mu = 1/6$
$R0 = \langle k \rangle \beta / \mu = 6.8 * 6 = 40.8$

N = 30,000
S(0) = 29,000
s0 = 29000/30000 = 0.967

$$i_{\max} = 1 - \frac{1}{R_0} * \left[1 + \ln(R_0 * s_0)\right]$$

$$i_{\max} = 1 - \frac{1}{40.8} * \left[1 + \ln(40.8 * 0.967)\right] = 0.8854$$

The maximum # people who can be infected in any day = 0.8854 * 30000 = 26562

Example 6: Consider a state of population 3 million and that 60% of this population is initially susceptible to a disease that spreads per the SIR model. The rest of the population are immune to the disease. Determine the basic reproduction number for the disease (which could be anywhere from 1 to 20).

Solution:
Since 60% of the population are only susceptible to the disease; the largest fraction of people who are infected at any time can be at most 0.6. We will use $i_{max} = s_0 = 0.6$.

$$i_{\max} = 1 - \frac{1}{R_0} * \left[1 + \ln(R_0 * s_0)\right]$$

$$0.6 = 1 - \frac{1}{R_0} * \left[1 + \ln(R_0 * 0.6)\right]$$

Simplifying,

$$\frac{1}{R_0} * \left[1 + \ln(R_0 * 0.6)\right] = 1 - 0.6$$

$$\frac{1}{R_0} * \left[1 + \ln(R_0 * 0.6)\right] = 0.4$$

0.4*R0 = 1 + ln(0.6 * R0)

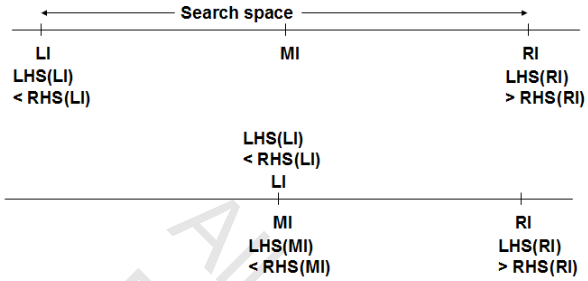
Let LHS = 0.4*R0 and RHS = 1 + ln(0.6 * R0)

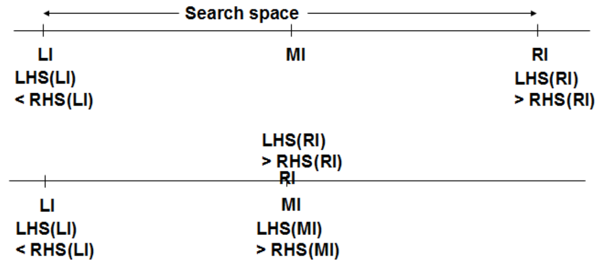
R0 = 1 ==> LHS = 0.4, RHS = 0.4891 LHS < RHS
R0 = 20 ==> LHS = 8.0, RHS = 3.4849 LHS > RHS.
So, the appropriate value for R0 has to be between 1 and 20.

We will run binary search to determine the value of R0 for which LHS ~ RHS, with a threshold difference |RHS - LHS| < 0.01

Invariant: Left Index, LI will correspond to a case wherein LHS < RHS
Right Index, RI will correspond to a case wherein LHS > RHS

if LHS(MI) < RHS(MI) set LI = MI
if LHS(MI) > RHS(MI) set RI = MI





LI	RI	MI = (LI+RI)/2	LHS: 0.4*MI	RHS: 1 + ln(0.6*MI)
1	20	10.5	4.2	2.8405
1	10.5	5.75	2.3	2.2384
1	5.75	3.375	1.35	1.7055
3.375	5.75	4.5625	1.825	2.0070
4.5625	5.75	5.1563	2.0625	2.1294
5.1563	5.75	5.4532	2.1813	2.1854

STOP! as |RHS-LHS| = |2.1854 - 2.1813| < 0.01 for MI = 5.4532.

The value for R0 ~ 5.4532

Example 7: Consider for the following real-time data obtained for the spread of the Influenza virus per the SIR model in a community of 763 people wherein one person was initially infected (on day 0). Assume the remaining 762 people are susceptible to the virus on day 0. Determine the R0 for the virus.

Days	Infected People
3	25
4	75
5	228
6	297
7	259
8	235
9	192
10	126
11	71
12	28
13	9
14	7

Solution:

s0 = 762/763 = 0.998689.
i0 = 1/763 = 0.001311

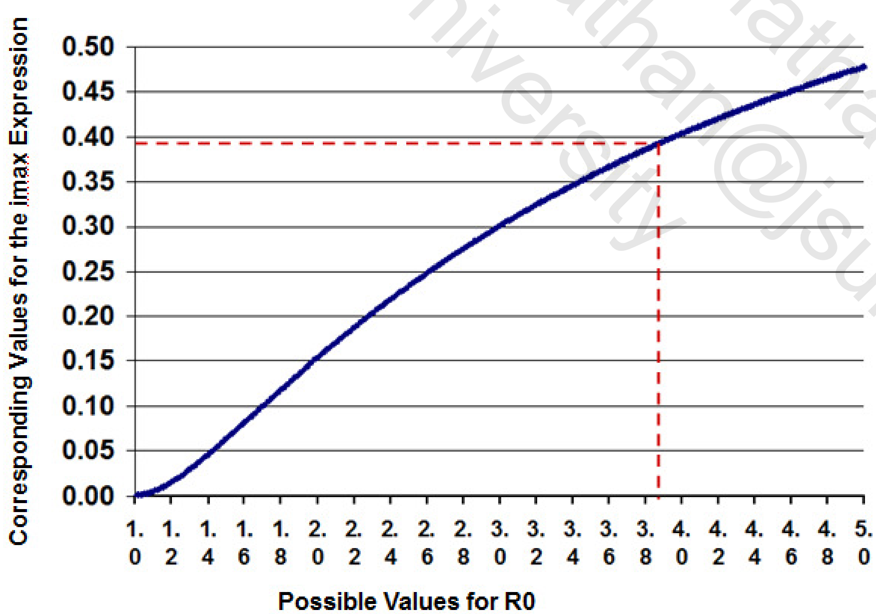
We now compute the fraction i(t) for days 3...14.

Days	Infected people	fraction, i Infected people / 763
3	25	0.032765
4	75	0.098296
5	228	0.29882
6	297	0.389253
7	259	0.33945
8	235	0.307995
9	192	0.251638
10	126	0.165138
11	71	0.093054
12	28	0.036697
13	9	0.011796
14	7	0.009174

$$i_{\max} = 1 - \frac{1}{R_0} * \left[1 + \ln(R_0 * s_0)\right]$$

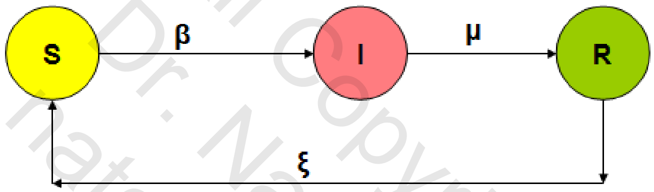
At t = 6, i is maximum.
imax = 0.389253 = 1 - 1/R0 [1+ ln(R0*0.998689)]

We can plot the above expression in Excel for R0 values ranging from 1.01 to 5.0, in increments of 0.01. We can see for what value of R0, the above expression value is close to 0.389253. Using Excel, we observe R0 = 3.84



1.5 SIRS Model and its Endemic State Analysis

The SIR model lets the recovered individual to stay immune to the disease after the first and only infection. Per the SIRS model, the recovered individual returns the susceptible state at the rate (ξ; i.e., the average duration an individual stays in the recovered state is 1/ξ). Accordingly, the differential equations for the SIRS model are as follows:



$$\frac{ds}{dt} = -<k> \beta * i * s + \xi * r \quad \text{.................................. (18)}$$

$$\frac{di}{dt} =<k> \beta * i * s - \mu * i \quad \text{.................................... (11)}$$

$$\frac{dr}{dt} = \mu * i - \xi * r \quad \text{................................ (19)}$$

At any time t, s + i + r = 1

Endemic Analysis

Endemic analysis of the SIRS model is feasible only if R0 > 1. If R0 <= 1, the disease is considered to die down on its own.

In the endemic state, each of the three derivatives $\frac{ds}{dt}, \frac{di}{dt}, \frac{dr}{dt}$ become 0.

We can then evaluate what will be the values for $\bar{s}$, $\bar{i}$ and $\bar{r}$ (representing respectively the values of s, i and r in the endemic state) as a function of the parameters <k>, β, μ and ξ.

Let us first use: $\frac{di}{dt} = 0$

$$\frac{di}{dt} =<k> \beta * i * s - \mu * i = 0$$

$$i * [<k> \beta * s - \mu] = 0$$

So, either i = 0 or $<k> \beta * s - \mu \ = 0$

Note that the SIRS model (due to its looping characteristic) keeps each of the three fractions non-zero in the endemic state. This is because: R0 > 1: there is always a non-zero fraction of the people who remain susceptible and they will get infected, who will then recover and again become susceptible.

So, at the endemic state of a disease for which R0 > 1, i cannot be 0.

Hence, $< k > \beta * s - \mu = 0$

==> $\bar{s} = \dfrac{\mu}{< k > \beta}$

Now, let us use $\dfrac{ds}{dt} = - < k > \beta * i * s + \xi * r = 0$

as well as replace r with 1-i-s

$< k > \beta * i * s = \xi * (1 - i - s) = \xi - \xi * i - \xi * s$

Substituting for s = $\boxed{\bar{s} = \dfrac{\mu}{< k > \beta} = \dfrac{1}{R_0}}$

We get,

$$< k > \beta * i * \frac{\mu}{< k > \beta} = \xi - \xi * i - \xi * s$$

$$i * \mu = \xi - \xi * i - \xi * s$$

Collecting all the *i* terms on the left hand side

$$i * \mu + \xi * i = \xi - \xi * s$$

$$i * [\mu + \xi] = \xi * [1 - \frac{\mu}{< k > \beta}]$$

$$i = \frac{\xi * \left[1 - \dfrac{\mu}{< k > \beta}\right]}{[\mu + \xi]}$$

$$i = \frac{\left[1 - \dfrac{\mu}{< k > \beta}\right]}{\left[\dfrac{\mu + \xi}{\xi}\right]}$$

$$\boxed{\bar{i} = \frac{\left[1 - \dfrac{\mu}{< k > \beta}\right]}{\left[1 + \dfrac{\mu}{\xi}\right]}} \quad \boxed{\bar{i} = \frac{\left[1 - \dfrac{1}{R_0}\right]}{\left[1 + \dfrac{\mu}{\xi}\right]}}$$

Substituting for s = $\bar{s}$ and i = $\bar{i}$ as derived above in s + i + r = 1

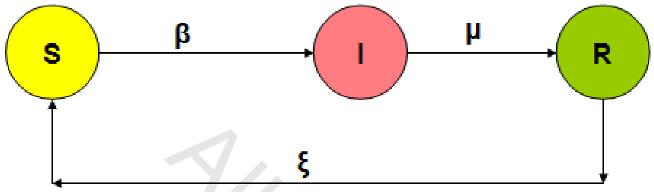
$$\bar{r} = 1 - \bar{s} - \bar{i}$$

$$\bar{r} = 1 - \frac{\mu}{<k>\beta} - \frac{\left[1 - \frac{\mu}{<k>\beta}\right]}{\left[1 + \frac{\mu}{\xi}\right]}$$

$$\boxed{\bar{r} = \left(1 - \frac{\mu}{<k>\beta}\right) * \left[1 - \frac{1}{\left[1 + \frac{\mu}{\xi}\right]}\right]}$$

$$\boxed{\bar{r} = \left(1 - \frac{1}{R_0}\right) * \left[1 - \frac{1}{\left[1 + \frac{\mu}{\xi}\right]}\right]}$$

Discrete Event Simulations of the SIRS Model and the Differential Equations

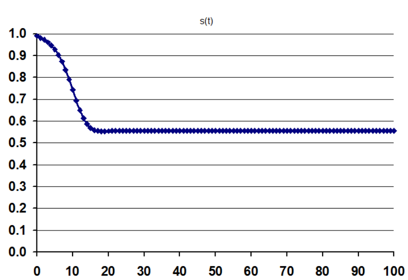


$$\left.\begin{aligned} &s(t+1) = s(t) - \langle k\rangle\beta*s(t)*i(t) + r(t)*\xi \\ &i(t+1) = i(t) + \langle k\rangle\beta*s(t)*i(t) - i(t)*\mu \\ &r(t+1) = r(t) + i(t)*\mu - r(t)*\xi \\ &\text{At any time } t,\ s(t) + i(t) + r(t) = 1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(20)}$$

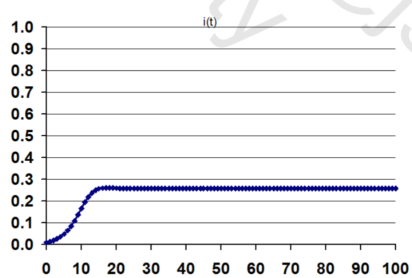
$$\left.\begin{aligned} \frac{ds(t)}{dt} &= -\langle k\rangle\beta*s*i + r*\xi \\ \frac{di(t)}{dt} &= \langle k\rangle\beta*s*i - i*\mu \\ \frac{dr(t)}{dt} &= i*\mu - r*\xi \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(21)}$$

Example 1:
Let the parameter values for the SIRS model be β = 0.9, μ = 0.5, ξ = 0.7 and <k> = 1.0. Simulate the execution of the SIRS model with these parameter values for 100 time units (by which time the endemic state is expected to have reached) and compare the fractions for the susceptible, infected and recovered nodes in the endemic state obtained in the simulations with those using the theoretical formulae.

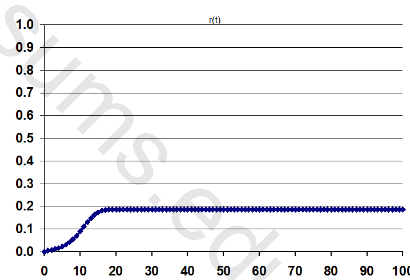
$R0 = \langle k\rangle * \beta / \mu = 1*0.9/0.5 = 1.8$



Susceptible



Infected



Recovered

$$\boxed{\bar{s} = \frac{\mu}{\langle k\rangle\beta} = \frac{0.5}{1.0*0.9} = 0.5556}$$

$$\bar{i} = \frac{\left[1 - \dfrac{\mu}{<k>\beta}\right]}{\left[1 + \dfrac{\mu}{\xi}\right]} = \frac{1 - 0.5556}{1 + \dfrac{0.5}{0.7}} = 0.2592$$

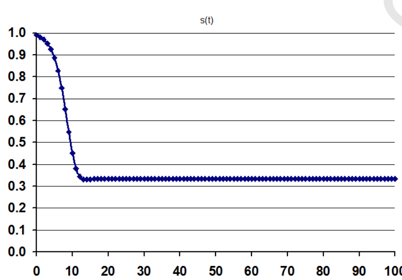
$$\bar{r} = \left(1 - \frac{\mu}{<k>\beta}\right) * \left[1 - \frac{1}{\left[1 + \dfrac{\mu}{\xi}\right]}\right] = (1 - 0.5556) * \left[1 - \frac{1}{1 + \dfrac{0.5}{0.7}}\right] = 0.1852$$

Comparison of Results in the Endemic State

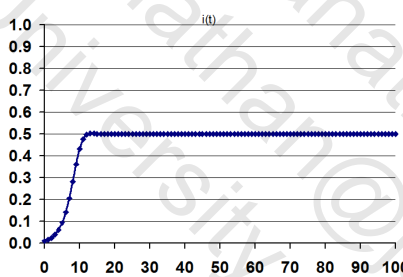
Fraction of Nodes	Theory	Simulations
Susceptible	0.5556	0.5556
Infected	0.2592	0.2592
Recovered	0.1852	0.1852

Example 2:
Let β = 0.9, μ = 0.3, ξ = 0.9 and <k> = 1.0. Compute the fraction of susceptible, infected and recovered nodes in the endemic state per the SIRS model.

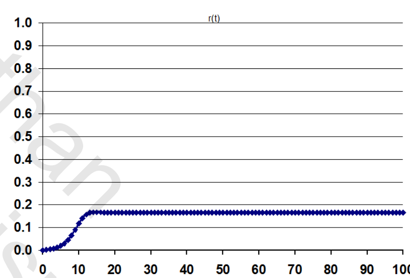
R0 = <k>* β / μ = 1.0*0.9/0.3 = 3.0



Susceptible



Infected



Recovered

$$\bar{s} = \frac{\mu}{<k>\beta} = \frac{0.3}{1.0 * 0.9} = 0.3333$$

$$\bar{i} = \frac{\left[1 - \dfrac{\mu}{<k>\beta}\right]}{\left[1 + \dfrac{\mu}{\xi}\right]} = \frac{1 - 0.3333}{1 + \dfrac{0.3}{0.9}} = 0.50$$

$$\bar{r} = \left(1-\frac{\mu}{<k>\beta}\right) * \left[1-\frac{1}{\left[1+\frac{\mu}{\xi}\right]}\right] = (1-0.3333) * \left[1-\frac{1}{1+\frac{0.3}{0.9}}\right] = 0.1667$$

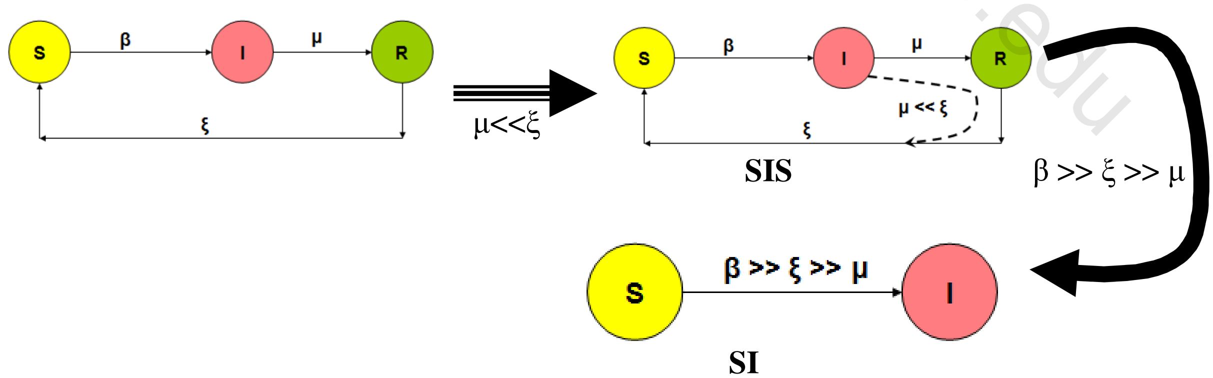
Comparison of Results in the Endemic State

Fraction of Nodes	Theory	Simulations
Susceptible	0.3333	0.3333
Infected	0.5000	0.5000
Recovered	0.1667	0.1667

Comparison of Examples 1 and 2: As R0 (3.0) for Example 2 is greater than the R0 (1.8) for Example 1, a larger fraction of the susceptible nodes get infected. Also, with 1 + μ/ξ of Example 2 (1.3333) being lower than that of Example 1 (1.7143) < ξ, a larger fraction of the recovered nodes return to susceptible state and get infected as well. We thus observe a large fraction of the nodes to be eventually (in the endemic state) infected for larger values of R0.

What can be inferred? For a fixed R0, the more smaller is μ compared to ξ (i.e., more smaller the value for 1 + μ/ξ), the value for $\frac{1}{\left(1+\frac{\mu}{\xi}\right)}$ increases (resulting in the eventual increase of $\bar{i}$) and the value for $1-\frac{1}{\left(1+\frac{\mu}{\xi}\right)}$ decreases (resulting in the eventual decrease of $\bar{r}$). The magnitude of the increase in $\bar{i}$ and the decrease in $\bar{r}$ influences the relative magnitudes of the values for $\bar{s}$ in the two scenarios.

In summary, as μ gets increasingly lower than ξ, relatively fewer fraction of infected nodes recover from infected state, and whoever has recovered are more likely to reach the susceptible state (and may enter the infected state if β is high).
If μ << ξ (i.e., μ/ξ << 1 ==> 1+ μ/ξ ~ 1), the rate at which people enter the Recovered state is lower than the rate at which people leave the recovered state to become susceptible again. There is bound to be eventually nobody in the Recovered state. ==> SIRS model becomes the SIS model **and eventually to the SI model as β gets increasingly larger than ξ as well.**



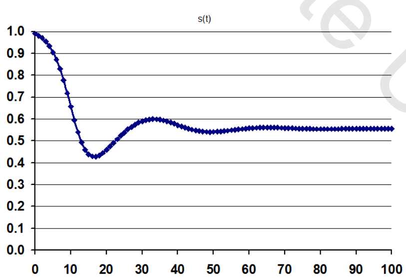
Example 3:
Let β = 0.9, μ = 0.5, ξ = 0.1 and <k> = 1.0. Compute the fraction of susceptible, infected and recovered nodes in the endemic state per the SIRS model (using theoretical formulations and simulations).

R0 = <k>* β / μ = 1.0*0.9/0.5 = 1.8

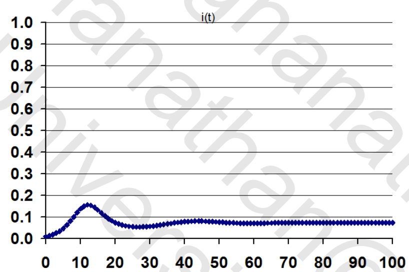
$$\bar{s} = \frac{\mu}{<k>\beta} = \frac{0.5}{1.0*0.9} = 0.5556$$

$$\bar{i} = \frac{\left[1 - \frac{\mu}{<k>\beta}\right]}{\left[1 + \frac{\mu}{\xi}\right]} = \frac{1 - 0.5556}{1 + \frac{0.5}{0.1}} = 0.0741$$

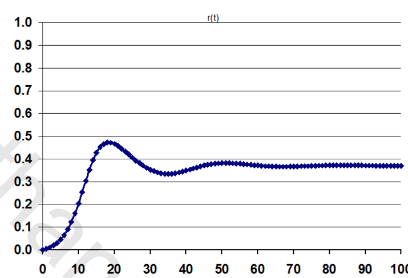
$$\bar{r} = \left(1 - \frac{\mu}{<k>\beta}\right) * \left[1 - \frac{1}{\left[1 + \frac{\mu}{\xi}\right]}\right] = (1 - 0.5556) * \left[1 - \frac{1}{1 + \frac{0.5}{0.1}}\right] = 0.3703$$



Susceptible



Infected



Recovered

<u>Comparison of Results in the Endemic State</u>

Fraction of Nodes	Theory	Simulations
Susceptible	0.5556	0.5560
Infected	0.0741	0.0740
Recovered	0.3703	0.3699

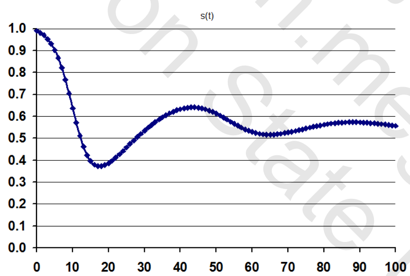
Example 4:
Let β = 0.9, μ = 0.5, ξ = 0.05 and <k> = 1.0. Compute the fraction of susceptible, infected and recovered nodes in the endemic state per the SIRS model (using theoretical formulations and simulations).

R0 = <k>* β / μ = 1.0*0.9/0.5 = 1.8

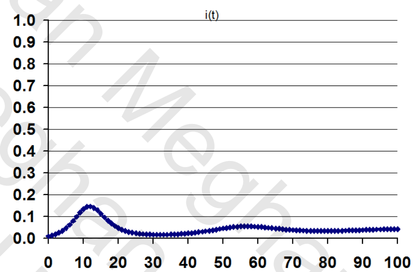
$$\bar{s} = \frac{\mu}{<k>\beta} = \frac{0.5}{1.0*0.9} = 0.5556$$

$$\bar{i} = \frac{\left[1 - \frac{\mu}{<k>\beta}\right]}{\left[1 + \frac{\mu}{\xi}\right]} = \frac{1 - 0.5556}{1 + \frac{0.5}{0.05}} = 0.0404$$

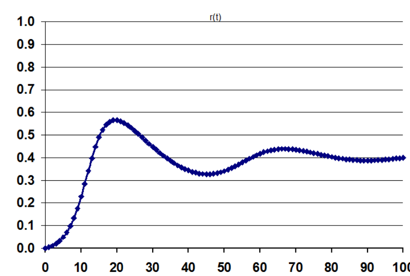
$$\bar{r} = \left(1 - \frac{\mu}{<k>\beta}\right) * \left[1 - \frac{1}{\left[1 + \frac{\mu}{\xi}\right]}\right] = (1 - 0.5556) * \left[1 - \frac{1}{1 + \frac{0.5}{0.05}}\right] = 0.4040$$



Susceptible



Infected



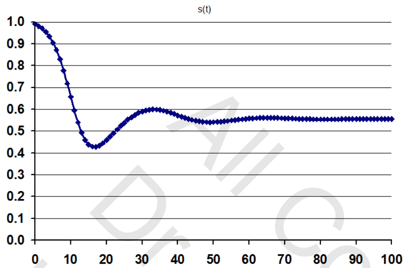
Recovered

<u>Comparison of Results in the Endemic State</u>

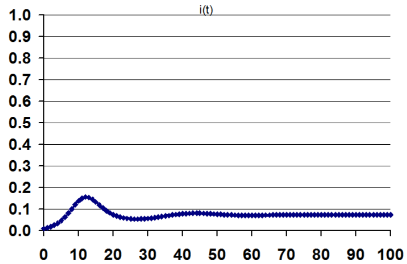
Fraction of Nodes	Theory	Simulations
Susceptible	0.5556	0.5570
Infected	0.0404	0.0433
Recovered	0.4040	0.3997

Comparison of Examples 3 and 4

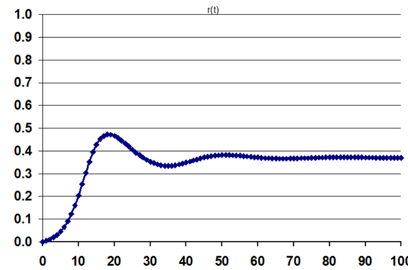
Example 3: $\beta = 0.9$, $\mu = 0.5$, $\xi = 0.1$ and $<k> = 1.0$



Susceptible

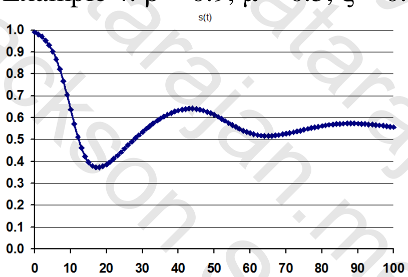


Infected

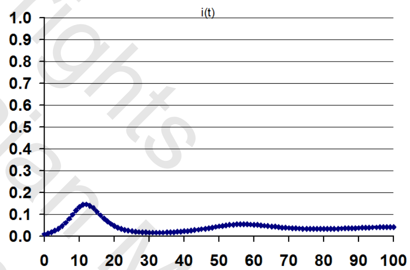


Recovered

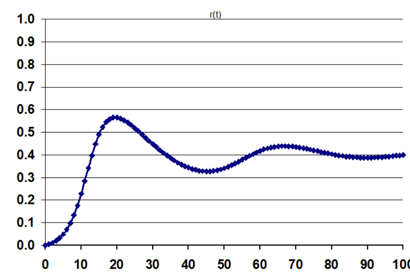
Example 4: $\beta = 0.9$, $\mu = 0.5$, $\xi = 0.05$ and $<k> = 1.0$



Susceptible



Infected



Recovered

As the μ/ξ ratio gets larger than 1, there are more instances of the epidemic showing a noticeable increase followed by a decrease in the fraction of infected nodes as well as in the fraction of recovered nodes, albeit with a reduced peak (and likewise a decrease followed by an increase in the fraction of susceptible nodes, with ups an downs in the peak). Nevertheless, the epidemic eventually reaches an endemic state with the same value for the fraction of nodes in the susceptible state (for a fixed R0, irrespective of the value of the μ/ξ ratio). However, if $\mu >> \xi$ (i.e., $\mu/\xi >> 1$ ==> $1+ \mu/\xi \sim \mu/\xi$), the rate at which people enter the Recovered state is greater than the rate at which people leave the recovered state to become susceptible again. There is bound to be people accumulated in the R state over time. ==> SIRS model becomes the SIR model.