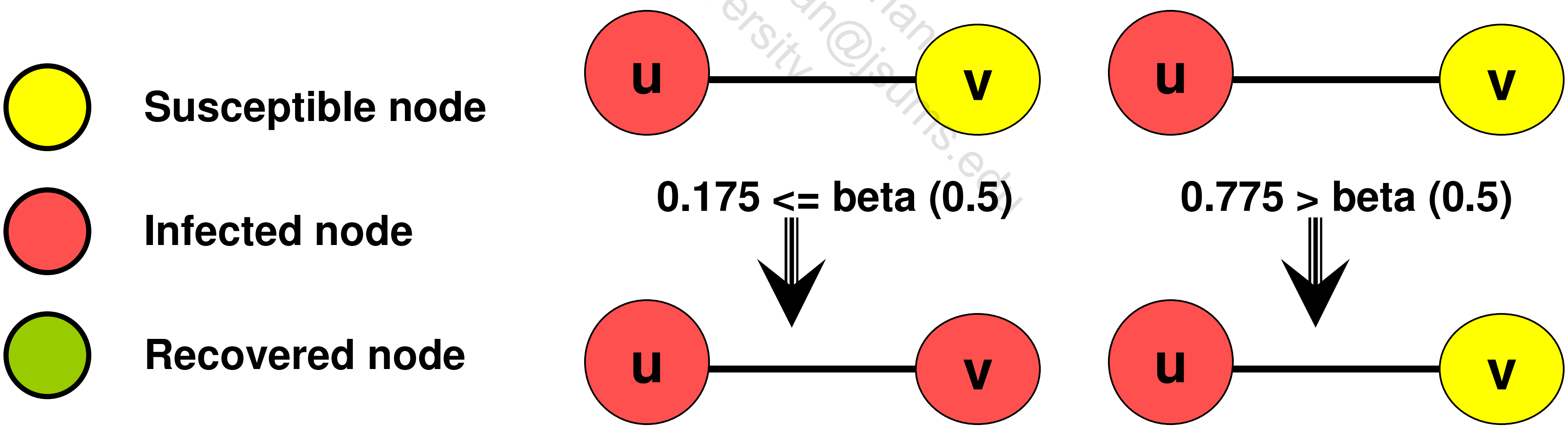
Module 2
Graph-based Simulations and Analytics

2.1 Simulations and Endemic Analysis of Disease Models on Graphs

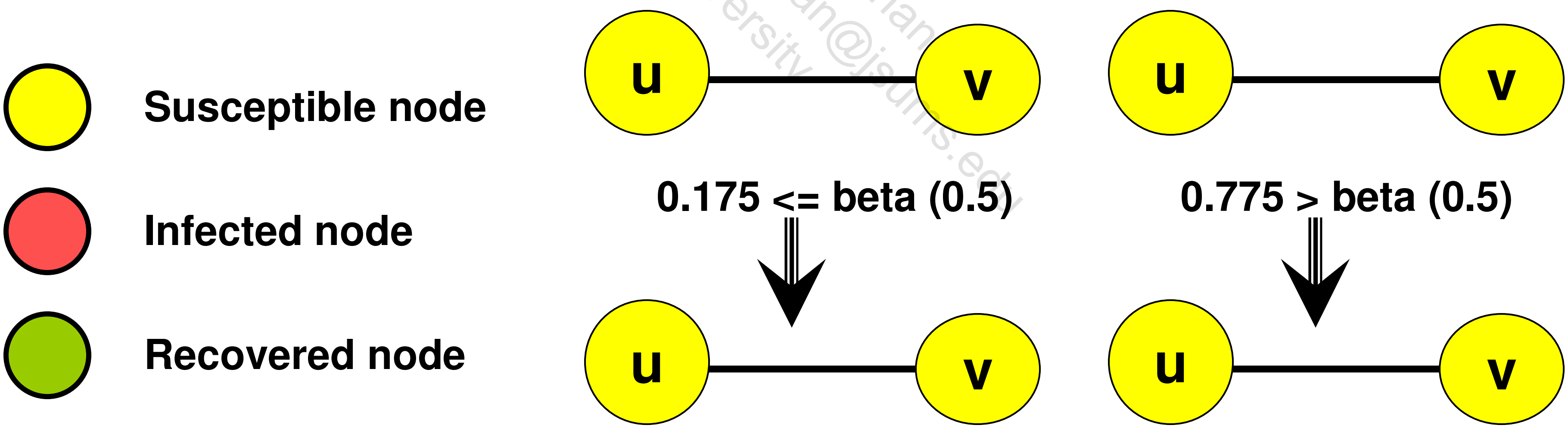
Assumptions and Notations

- We set a node v as part of a link u … v as infected in a particular round if the following are true:
  - Node u is currently in the “infected” status
  - Node v is currently in the “susceptible” status
  - The random number generated for the link is less than or equal to the infection probability beta



Assumptions and Notations

- We set a node v as part of a link u … v as infected in a particular round if the following are true:
  - Node u is currently in the “infected” status
  - Node v is currently in the “susceptible” status
  - The random number generated for the link is less than or equal to the infection probability beta

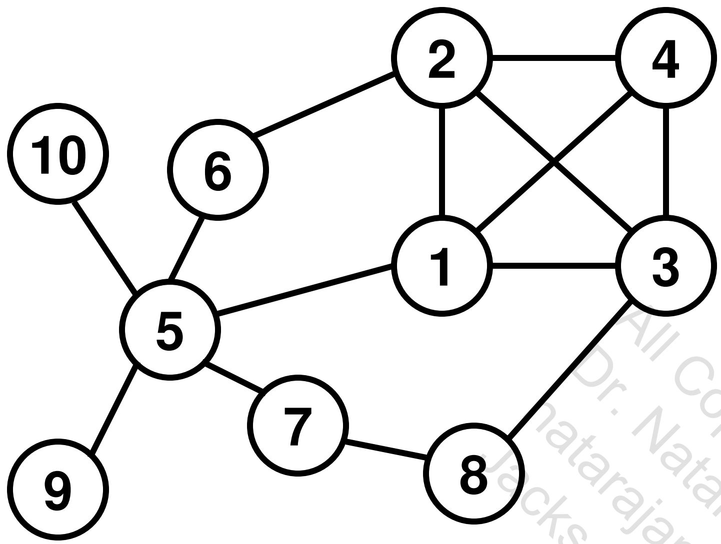


SIR: Example 1

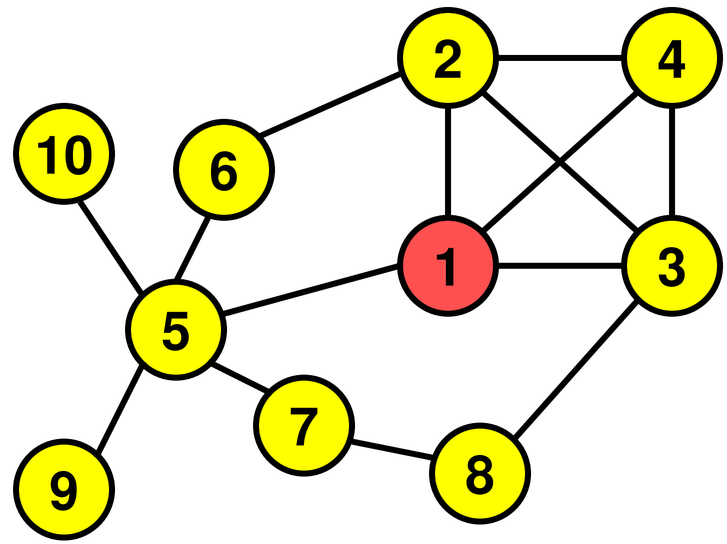
- Start the simulation from a larger degree node that need not be necessarily the largest degree

Simulate until every susceptible node becomes infected

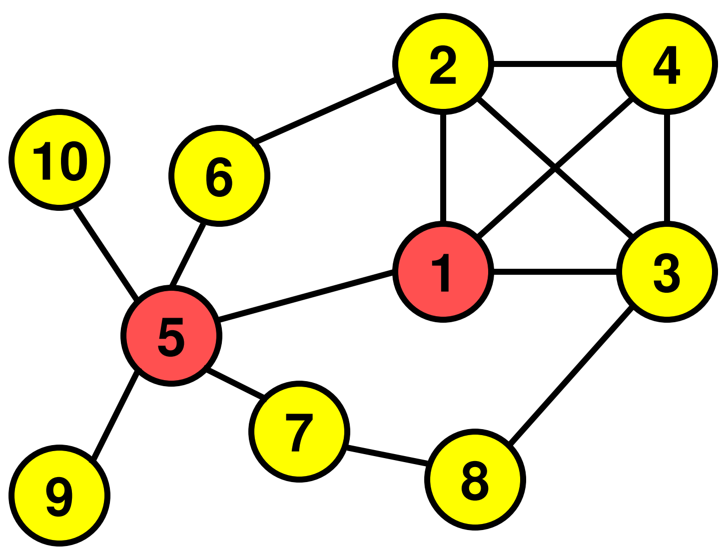
Degree of a node is the number of neighbors for the node (i.e., the number of links incident on the node)



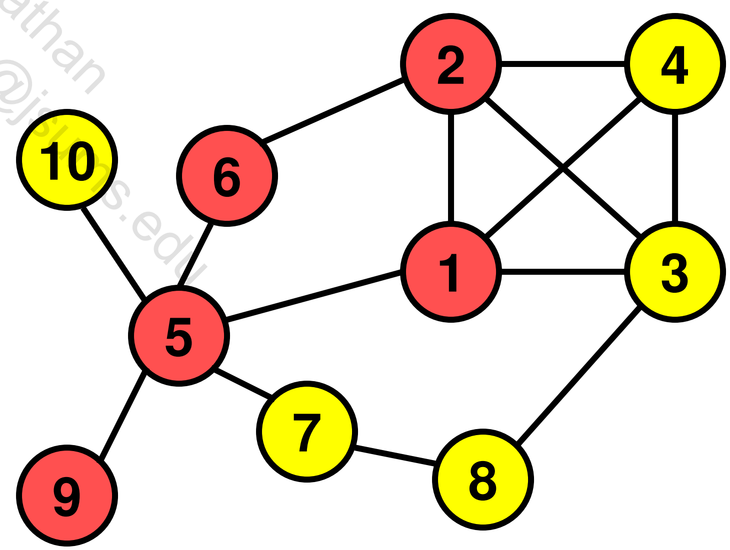
Let infection time be 2 rounds = (1/μ)
Let infection probability beta be 0.5 (β)



Initialization (Round 0): 1



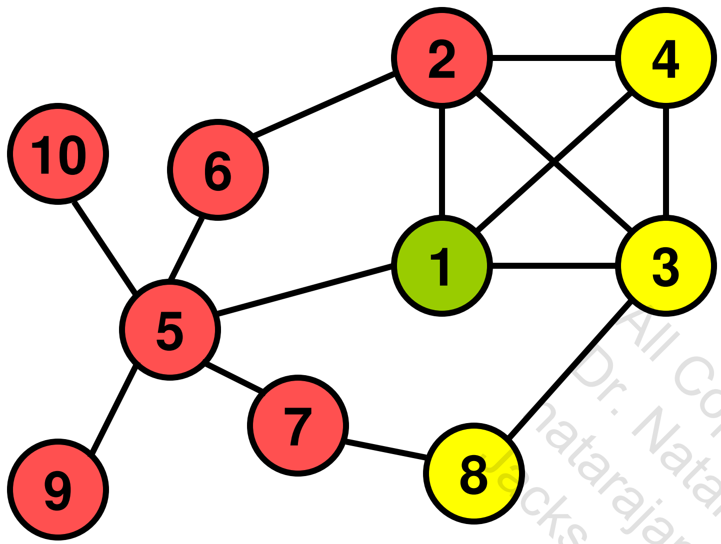
Round 1: 5



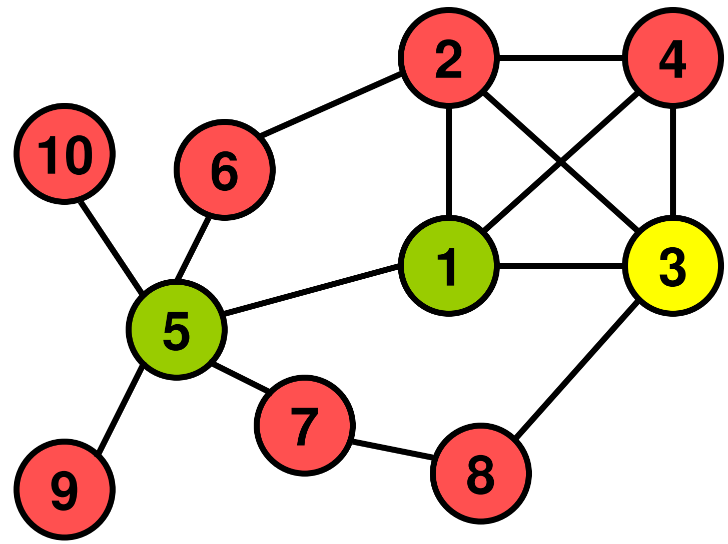
Round 2: 2, 6, 9

Random numbers in the range of 0…1 generated for each link

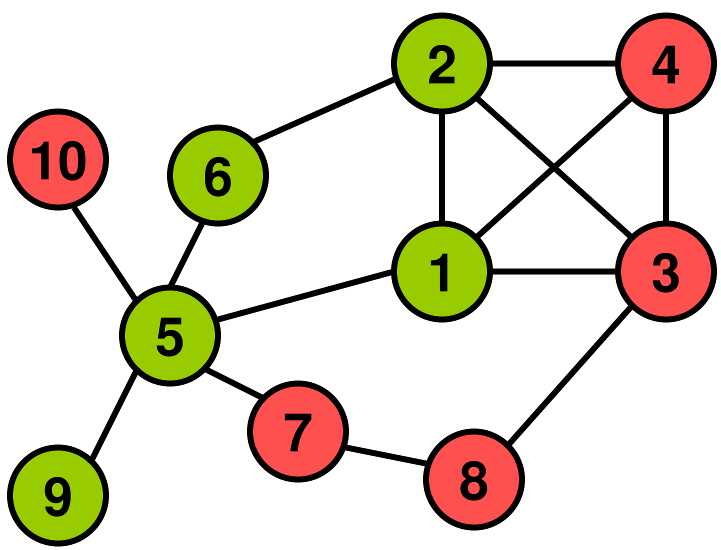
Links	Round 1	Round 2	Round 3	Round 4	Round 5
1…2	0.540248	0.037269	0.539082	0.22241	0.488438
1…3	0.50178	0.537089	0.357936	0.129051	0.957115
1…4	0.733169	0.682445	0.438104	0.487376	0.992214
1…5	0.256378	0.220853	0.075314	0.659316	0.321978
2…3	0.978049	0.701358	0.671188	0.758375	0.286315
2…4	0.57333	0.679456	0.553619	0.170008	0.564355
2…6	0.068427	0.742157	0.28669	0.287777	0.437032
3…4	0.045431	0.069523	0.766988	0.065264	0.970803
3…8	0.420928	0.751161	0.707017	0.946536	0.242076
5…6	0.852163	0.314352	0.280231	0.657614	0.882565
5…7	0.170949	0.542855	0.499618	0.260965	0.813819
5…9	0.678802	0.199807	0.843998	0.643189	0.189297
5…10	0.76882	0.597068	0.026241	0.200732	0.399823
7…8	0.157908	0.823308	0.135346	0.106023	0.900296



Round 3: 7, 10



Round 4: 4, 8

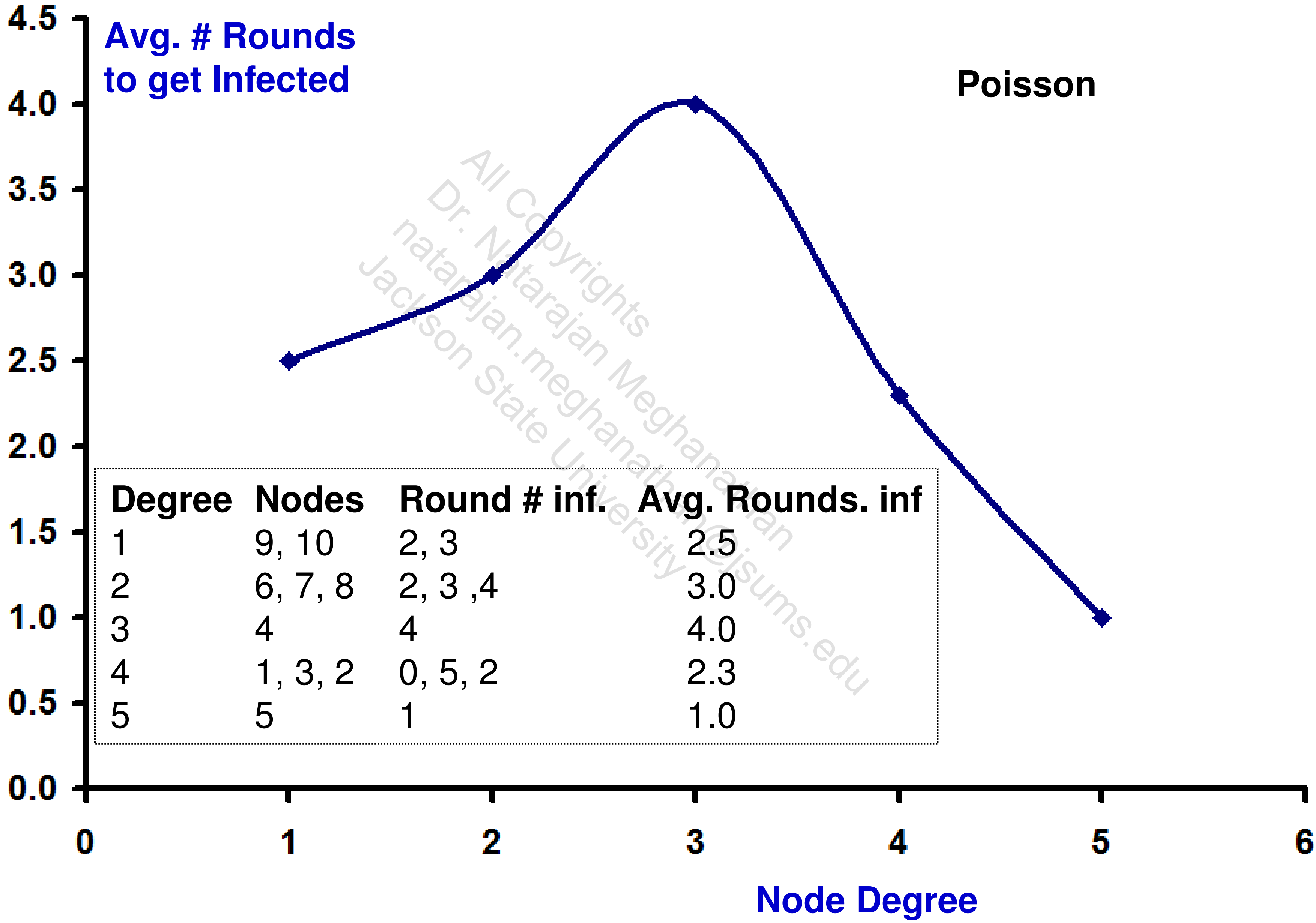


Round 5: 3

Node	Degree
1	4
2	4
3	4
4	3
5	5

Node	Degree
6	2
7	2
8	2
9	1
10	1

Degree	Nodes	Round # inf.	Avg. Rounds. inf
1	9, 10	2, 3	2.5
2	6, 7, 8	2, 3 ,4	3.0
3	4	4	4.0
4	1, 3, 2	0, 5, 2	2.3
5	5	1	1.0



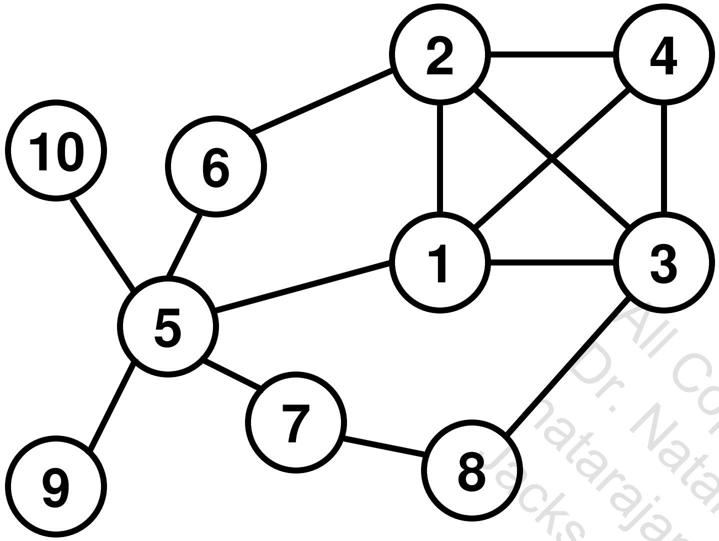
Degree	Nodes	Round # inf.	Avg. Rounds. inf
1	9, 10	2, 3	2.5
2	6, 7, 8	2, 3 ,4	3.0
3	4	4	4.0
4	1, 3, 2	0, 5, 2	2.3
5	5	1	1.0

Inference from the Plot

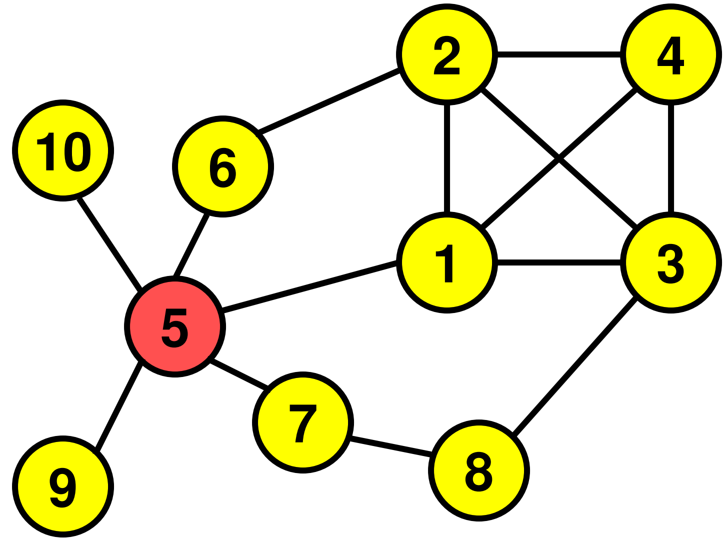
- The distribution of node degree Vs. the average # rounds it takes for a node to get infected appears to be Poisson in nature.
  – Nodes having a larger degree (hub nodes) or lower degree (stub nodes, connected to hub nodes) are more likely to be infected earlier; nodes having moderate degree are infected later.

SIR: Example 2

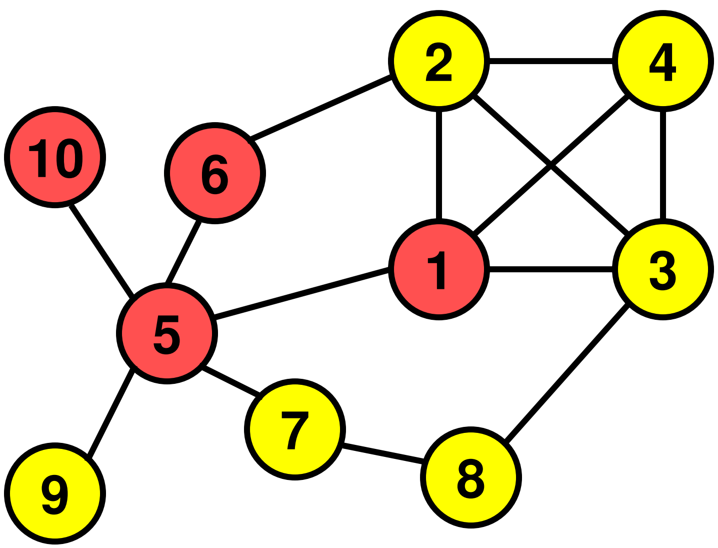
- What if we change the starting node to the node that has the largest degree?



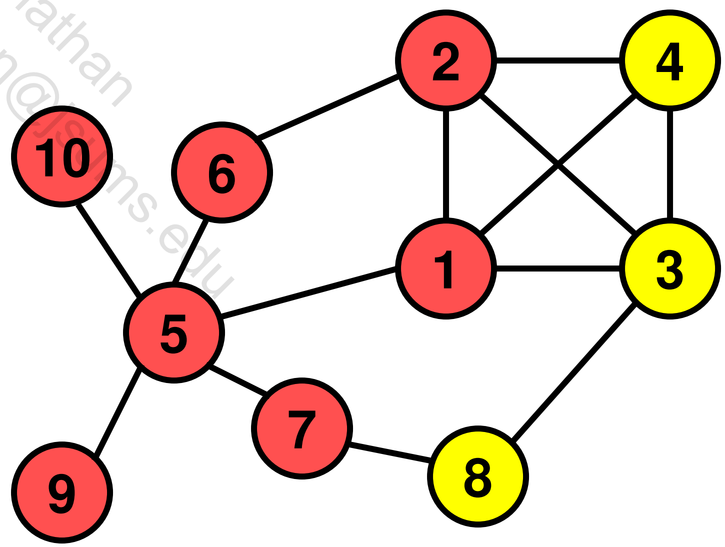
Let infection time be 2 rounds
Let infection probability beta be 0.5



Initialization (Round 0): 5

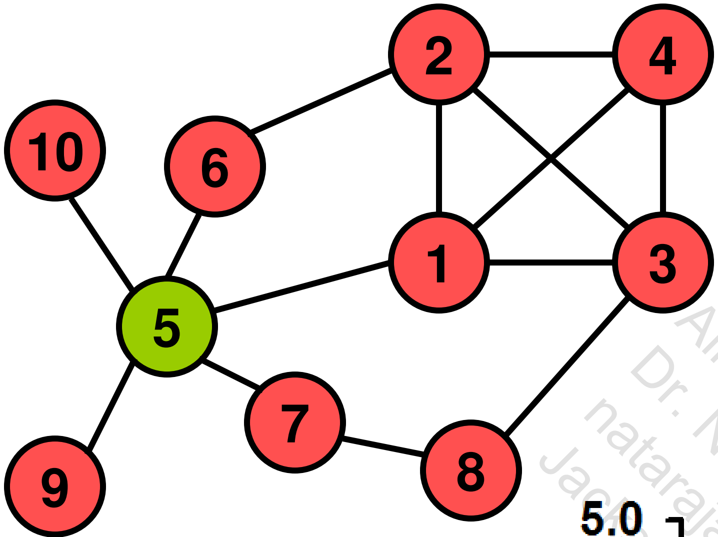


Round 1: 1, 6, 10



Round 2: 2, 7, 9

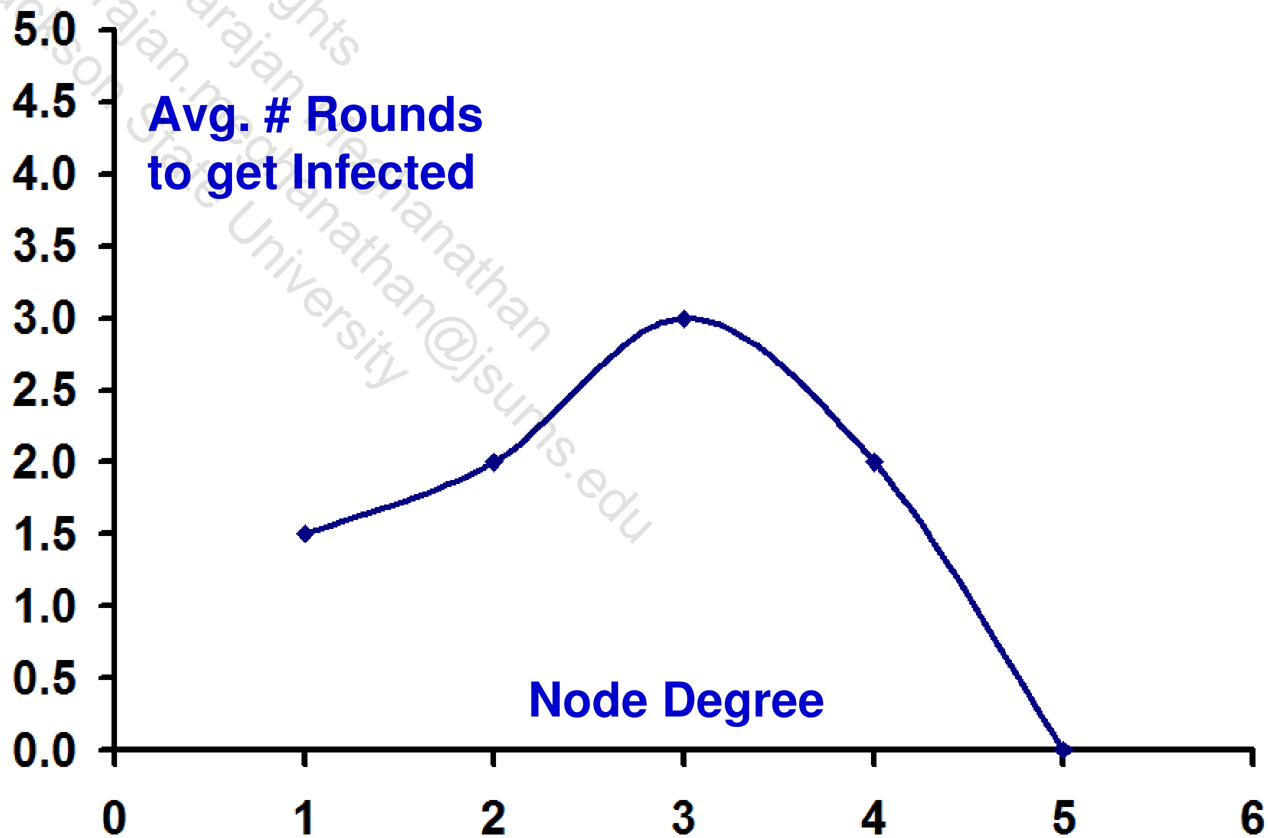
Links	Round 1	Round 2	Round 3
1….2	0.093339	0.4805	0.322803
1….3	0.516579	0.798564	0.503208
1….4	0.487007	0.516201	0.812679
1….5	0.064598	0.022306	0.041765
2….3	0.328752	0.132252	0.386975
2….4	0.563172	0.429722	0.031528
2….6	0.37692	0.075892	0.94097
3….4	0.827337	0.938008	0.365181
3….8	0.387225	0.075615	0.190182
5….6	0.461979	0.386219	0.71502
5….7	0.757855	0.160399	0.720711
5….9	0.91346	0.258447	0.983566
5….10	0.1409	0.072973	0.257606
7….8	0.763996	0.177283	0.286677



Degree	Nodes	Round # inf.	Avg. Rounds. inf
1	9, 10	2, 1	1.5
2	6, 7, 8	1, 2, 3	2.0
3	4	3	3.0
4	1, 3, 2	1, 2, 3	2.0
5	5	0	0.0

Round 3: 3, 4, 8

Node	Degree
1	4
2	4
3	4
4	3
5	5
6	2
7	2
8	2
9	1
10	1



SIS Model

Idea

- An infected node returns to “susceptible” status after staying infected for the infection time (rounds) and could become again infected if any of its neighbors are currently infected and the random number generated for that link is less than or equal to the infection probability, beta.

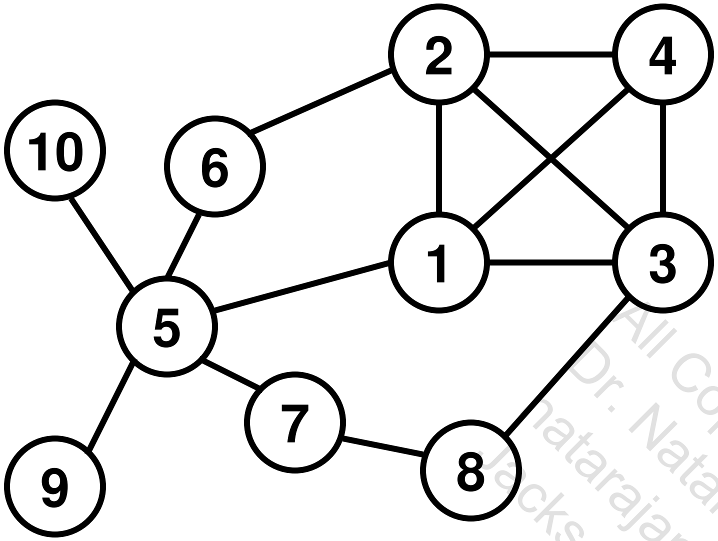
- Proceed for 5-10 rounds and determine the number of rounds each node stays infected and compare this with the degree.

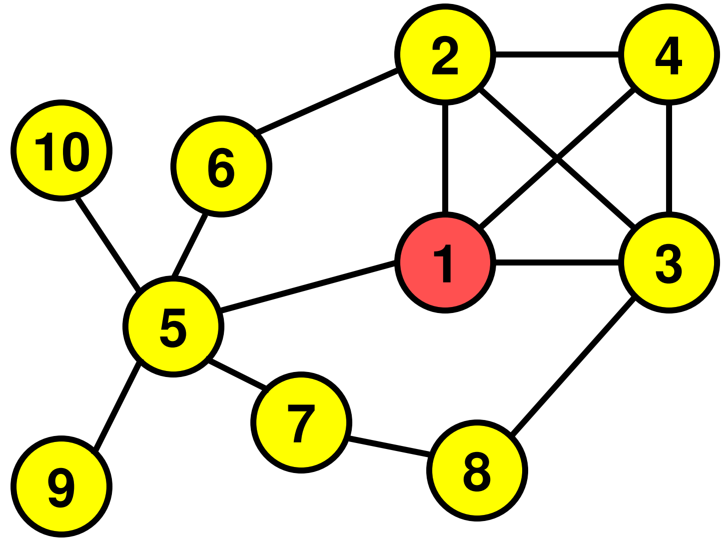
- It is possible that all the nodes (or at least a certain fraction of the nodes) may stay infected at the end of each round after a certain round. We say the network has entered an “endemic” state (at least one node is infected when we stop the simulations).

Example 1: SIS

Random numbers generated

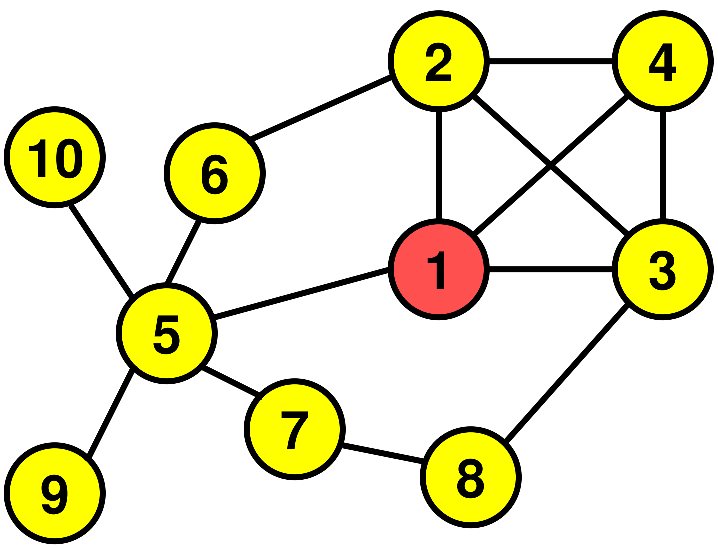
Links	Round 1	Round 2	Round 3	Round 4	Round 5	Round 6	Round 7
1…2	0.069727	0.327157	0.987588	0.033982	0.397306	0.103209	0.540245
1…3	0.380447	0.785791	0.893356	0.1377	0.939614	0.21679	0.15035
1…4	0.273639	0.2233	0.453682	0.839541	0.423825	0.514878	0.170527
1…5	0.893866	0.362893	0.424807	0.059336	0.900095	0.384065	0.509922
2…3	0.444439	0.77898	0.634305	0.256588	0.131689	0.144057	0.574832
2…4	0.876286	0.512341	0.139157	0.161313	0.163935	0.484496	0.262967
2…6	0.365708	0.410347	0.301668	0.696285	0.314386	0.846186	0.252213
3…4	0.226968	0.642317	0.137131	0.75405	0.497082	0.259342	0.50987
3…8	0.791793	0.461758	0.398524	0.027178	0.07223	0.89736	0.100144
5…6	0.469923	0.316089	0.823989	0.087085	0.972357	0.727116	0.412238
5…7	0.847914	0.738021	0.074276	0.46388	0.362319	0.004842	0.863546
5…9	0.753155	0.915555	0.78423	0.256046	0.876939	0.529567	0.191222
5…10	0.880366	0.579745	0.164895	0.229082	0.234629	0.23113	0.704058
7…8	0.34678	0.103795	0.73263	0.312683	0.269853	0.982354	0.68226



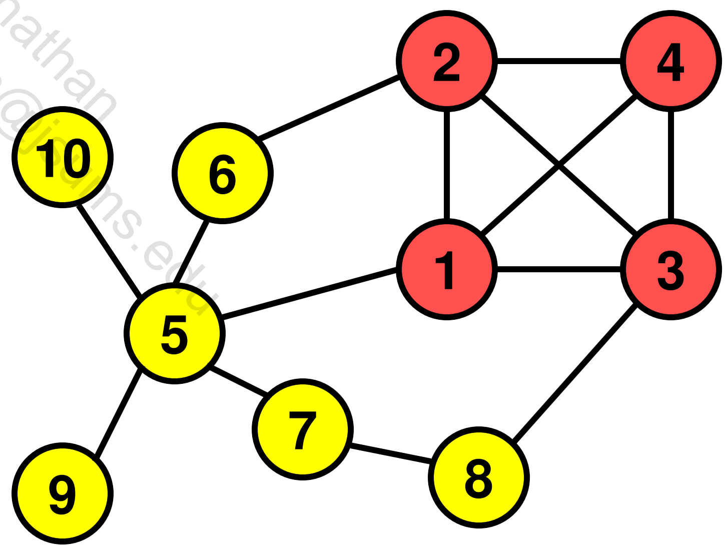


Let infection time be 2 rounds
Let infection probability beta be 0.5

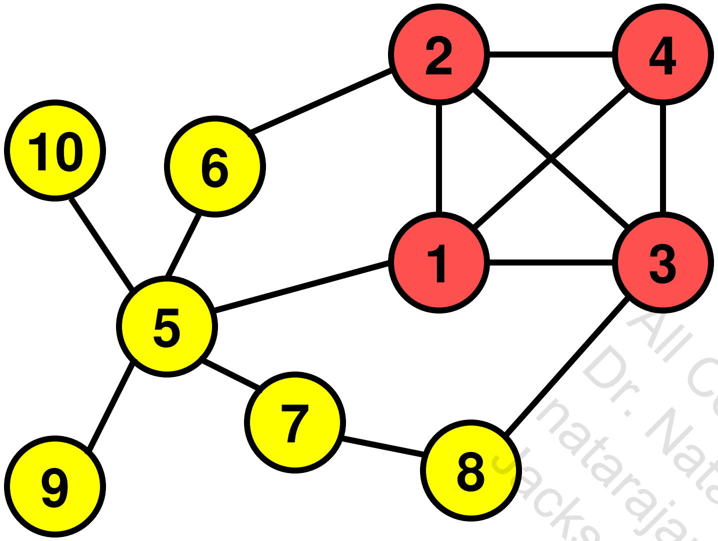
Initialization (Round 0): 1



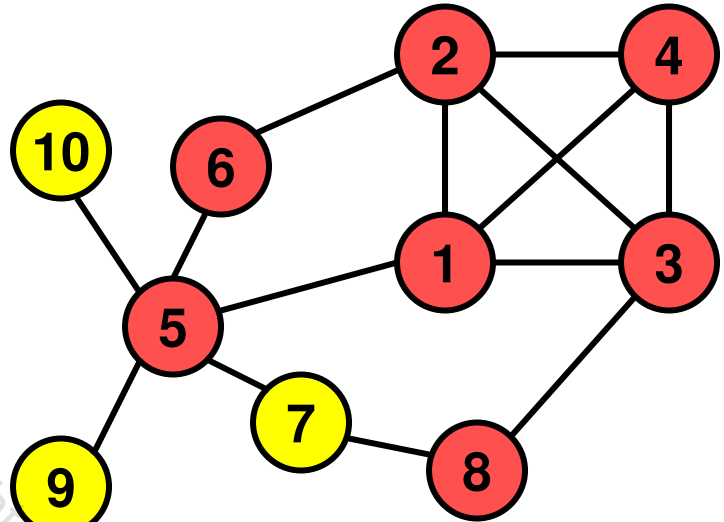
Round 1 (Begin)



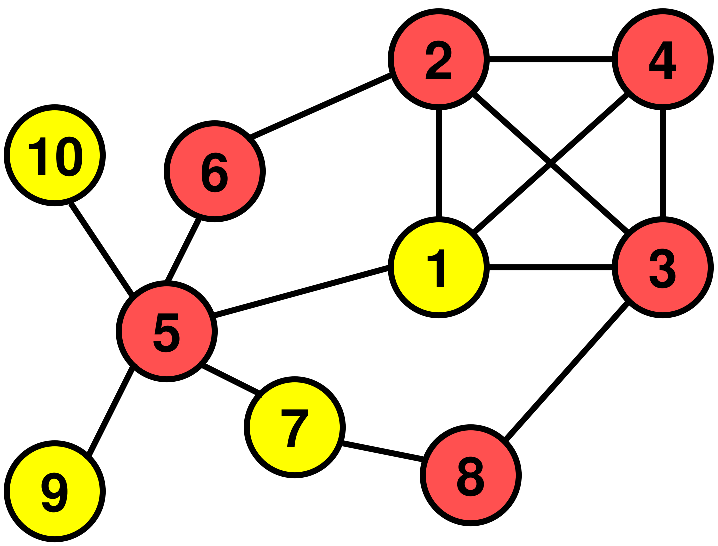
Round 1 (End): 2, 3, 4



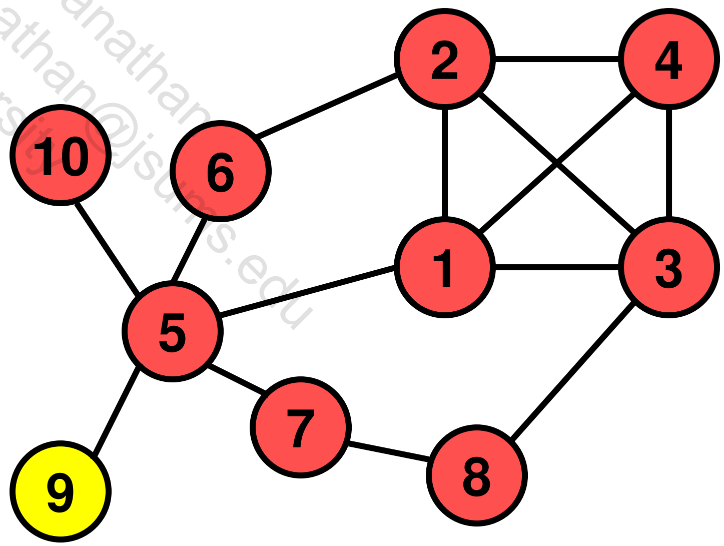
Round 2 (Begin)



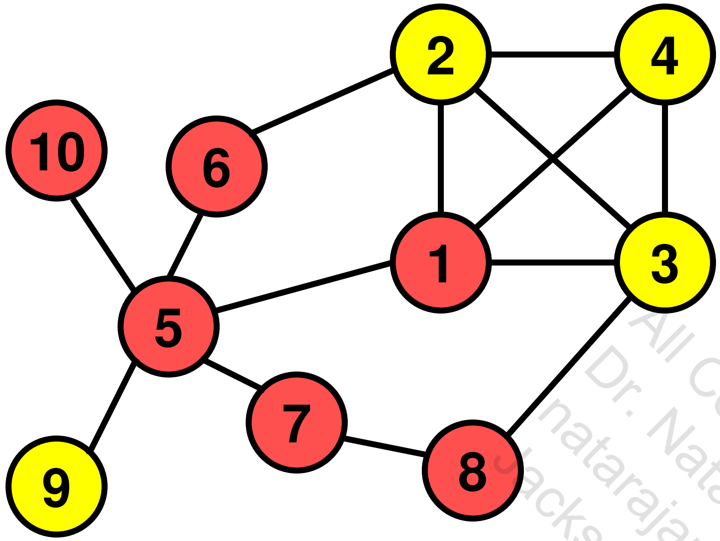
Round 2 (End): 5, 6, 8



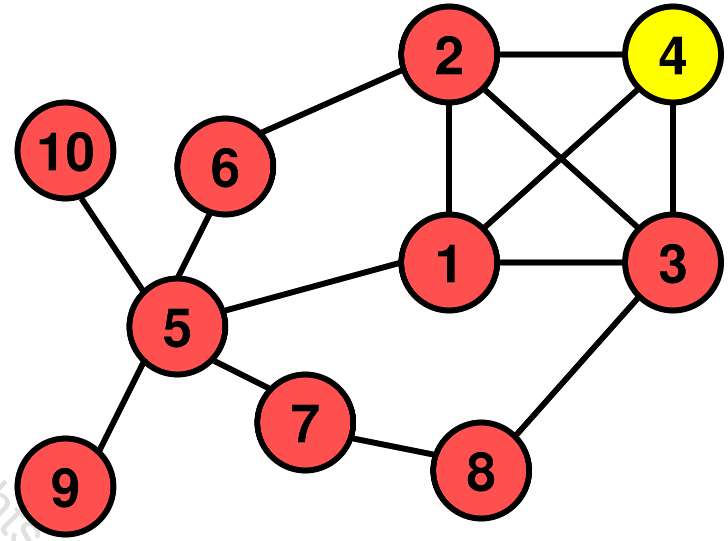
Round 3 (Begin)



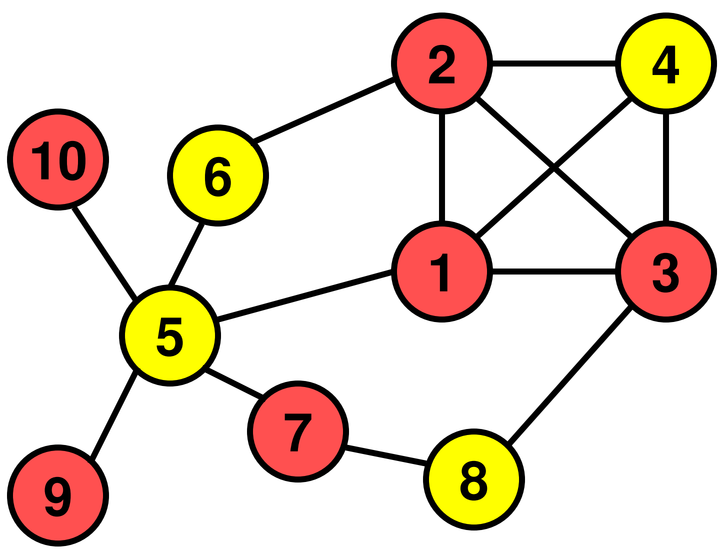
Round 3 (End): 1, 7, 10



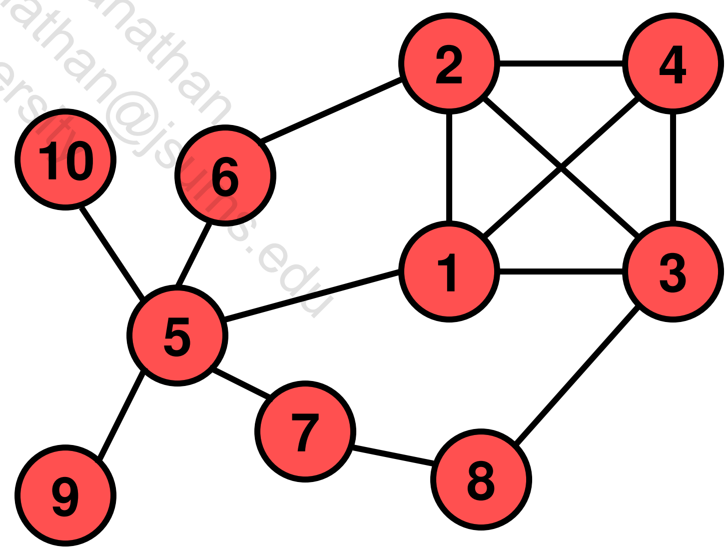
Round 4 (Begin)



Round 4 (End): 2, 3, 9



Round 5 (Begin)



Round 5 (End): 4, 6, 8, 5

Example 1: Analysis

Degree	Nodes	# rounds stays infected	Avg. # rounds stays infected
1	9, 10	2, 3	2.5
2	6, 7, 8	4, 3, 4	3.67
3	4	4	4.0
4	1, 2, 3	5, 5, 5	5.0
5	5	4	4.0

Inference: Nodes with smaller degree are more likely to stay infected for a fewer number of rounds compared to nodes with larger degree.

Round #	Total # Inf. Nodes
1	4
2	7
3	9
4	9
5	10
6	10
7	10

Inference:

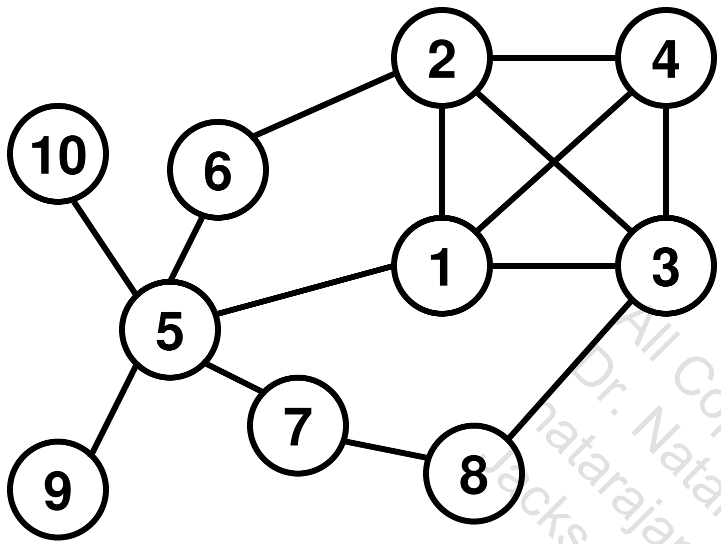
(1) The total # infected nodes tends to increase quickly with time and eventually all nodes are infected.

(2) The disease could easily transition from one cluster to another.

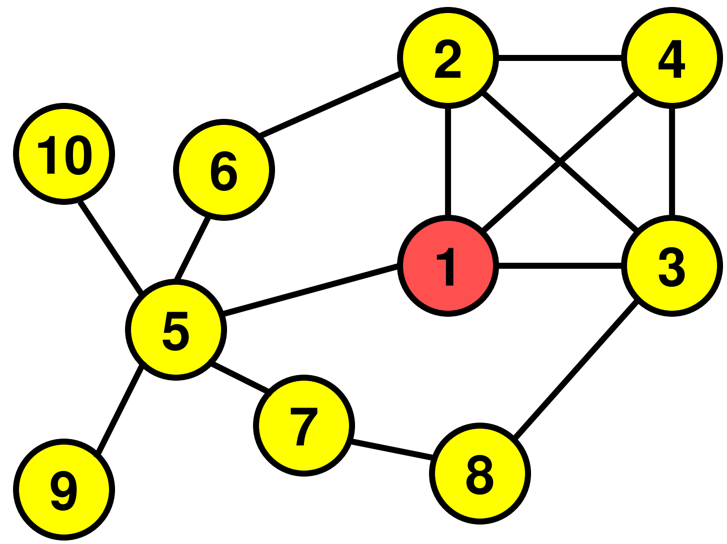
Example 2

Random numbers generated

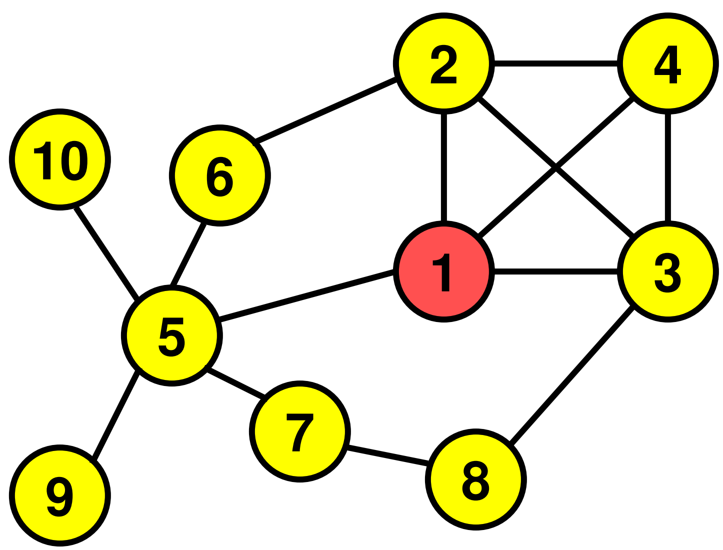
Links	Round 1	Round 2	Round 3	Round 4	Round 5	Round 6	Round 7	Round 8	Round 9	Round 10
1…2	0.069727	0.327157	0.987588	0.033982	0.397306	0.103209	0.540245	0.214213	0.331587	0.110146
1…3	0.380447	0.785791	0.893356	0.1377	0.939614	0.21679	0.15035	0.823889	0.160652	0.381949
1…4	0.273639	0.2233	0.453682	0.839541	0.423825	0.514878	0.170527	0.10118	0.953627	0.918737
1…5	0.893866	0.362893	0.424807	0.059336	0.900095	0.384065	0.509922	0.013265	0.820163	0.288808
2…3	0.444439	0.77898	0.634305	0.256588	0.131689	0.144057	0.574832	0.375135	0.508881	0.587843
2…4	0.876286	0.512341	0.139157	0.161313	0.163935	0.484496	0.262967	0.499072	0.66305	0.861896
2…6	0.365708	0.410347	0.301668	0.696285	0.314386	0.846186	0.252213	0.840461	0.742175	0.333662
3…4	0.226968	0.642317	0.137131	0.75405	0.497082	0.259342	0.50987	0.368502	0.592843	0.690716
3…8	0.791793	0.461758	0.398524	0.027178	0.07223	0.89736	0.100144	0.163006	0.729635	0.451381
5…6	0.469923	0.316089	0.823989	0.087085	0.972357	0.727116	0.412238	0.333748	0.214519	0.864602
5…7	0.847914	0.738021	0.074276	0.46388	0.362319	0.004842	0.863546	0.849964	0.780724	0.016044
5…9	0.753155	0.915555	0.78423	0.256046	0.876939	0.529567	0.191222	0.388512	0.362856	0.908738
5…10	0.880366	0.579745	0.164895	0.229082	0.234629	0.23113	0.704058	0.853117	0.208388	0.207667
7…8	0.34678	0.103795	0.73263	0.312683	0.269853	0.982354	0.68226	0.228099	0.389481	0.299793



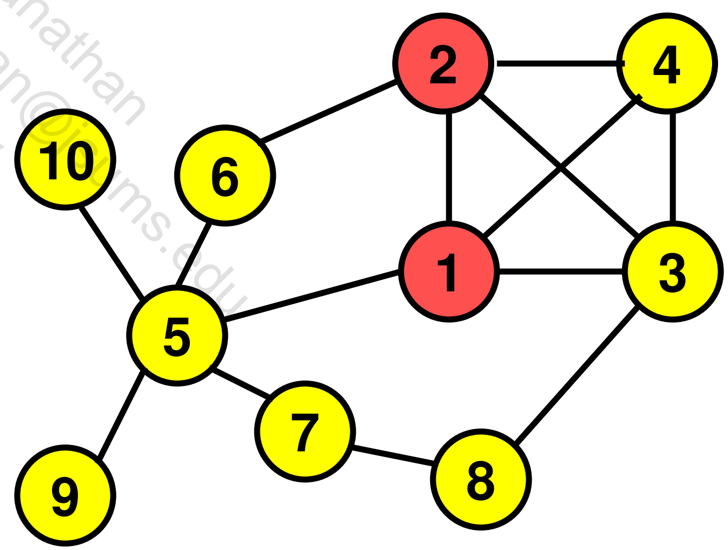
Let infection time be 2 rounds
Let infection probability beta be 0.2



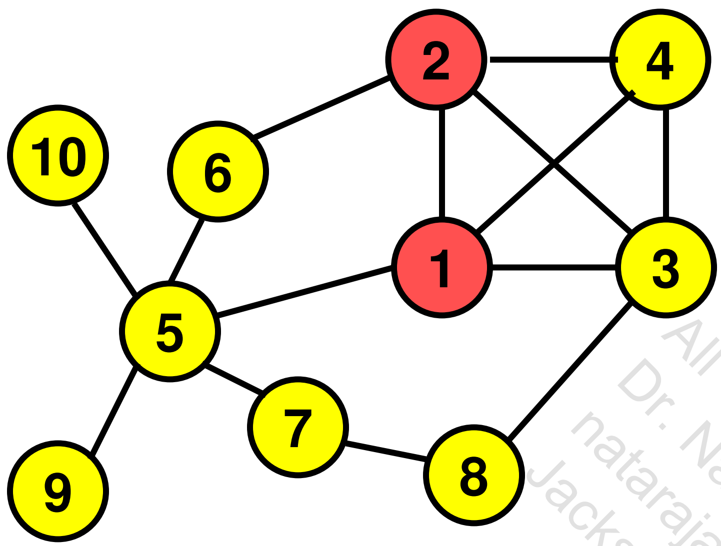
Initialization (Round 0): 1



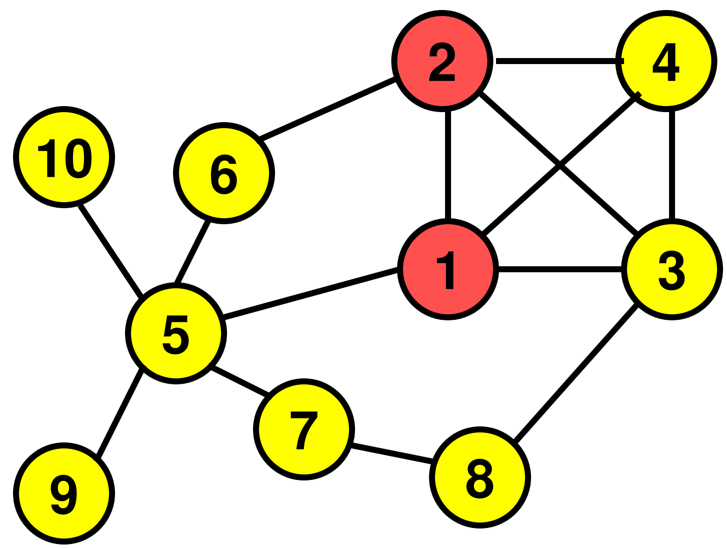
Round 1 (Begin)



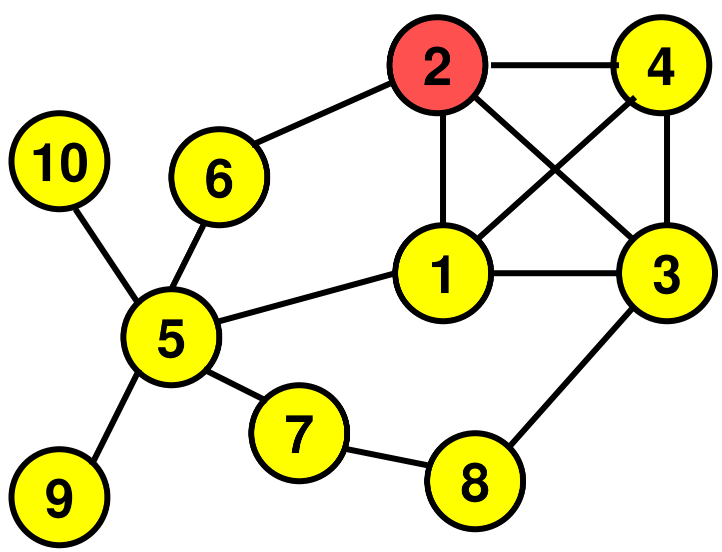
Round 1 (End): 2



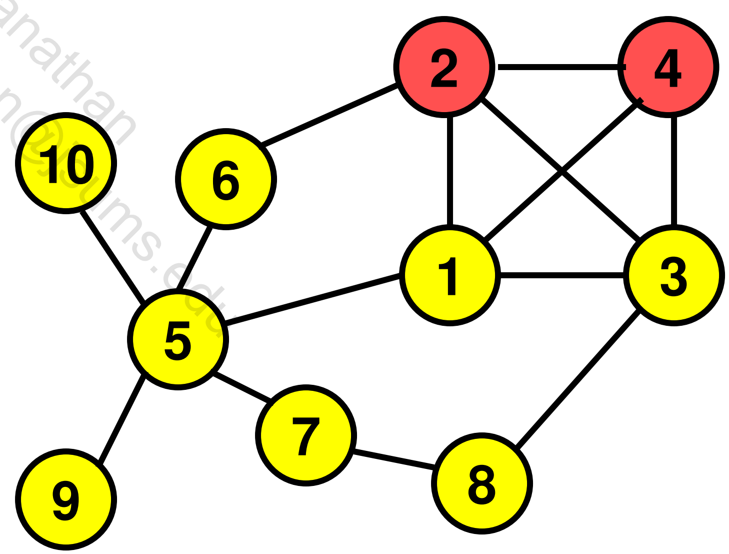
Round 2 (Begin)



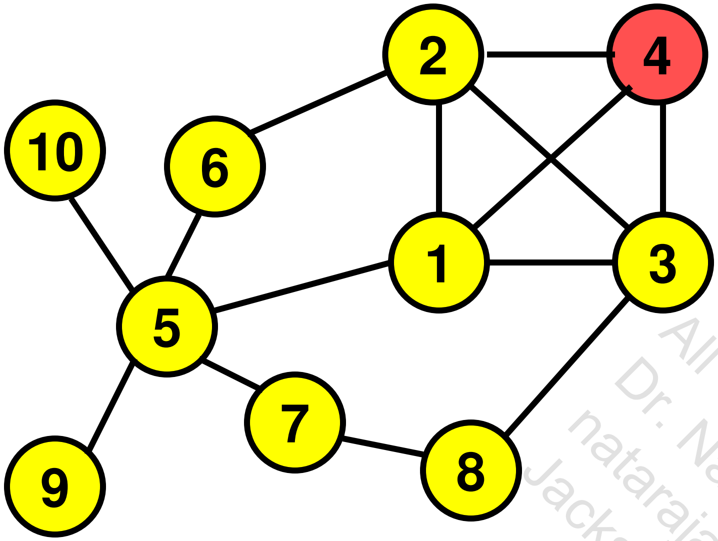
Round 2 (End): -



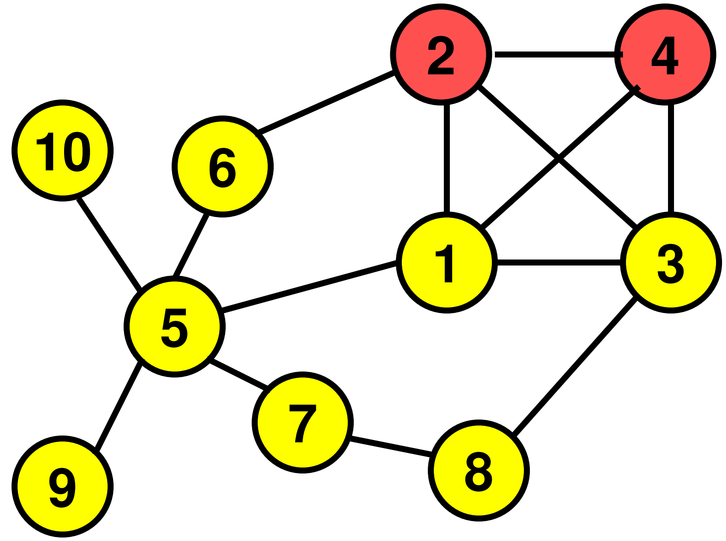
Round 3 (Begin)



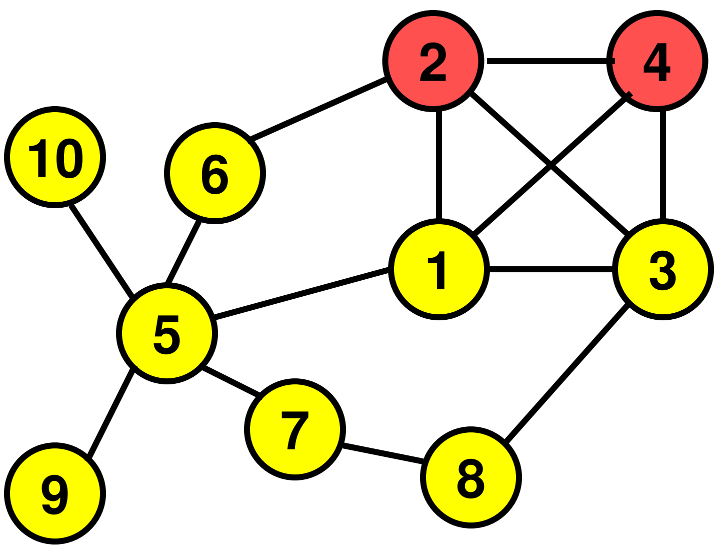
Round 3 (End): 4



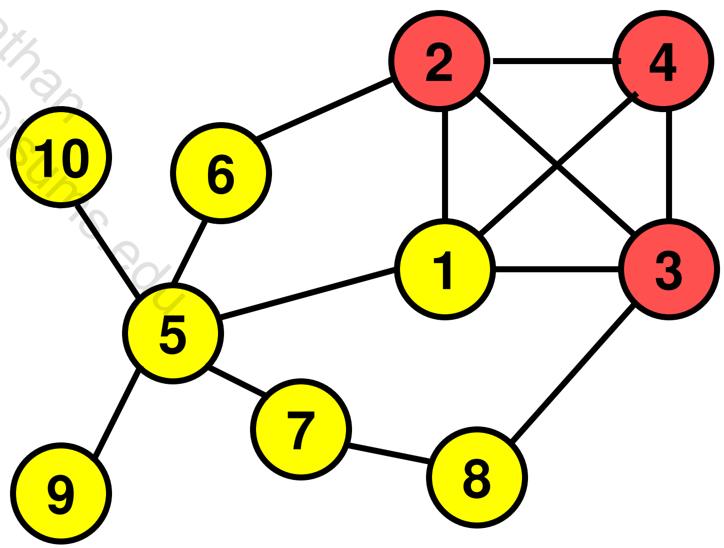
Round 4 (Begin)



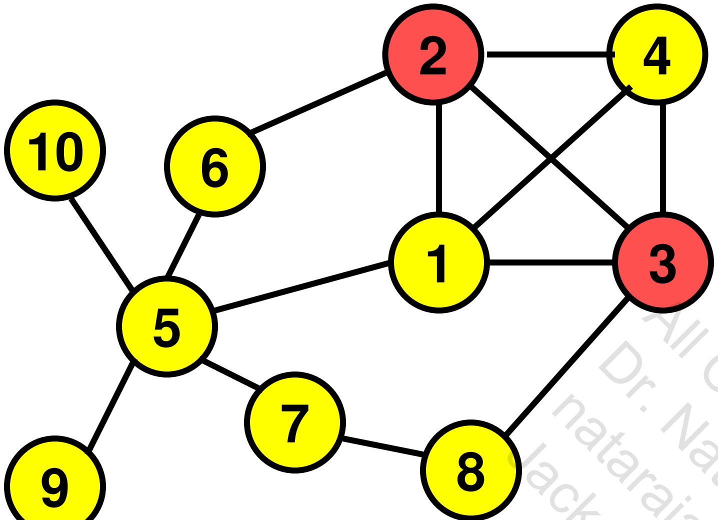
Round 4 (End): 2



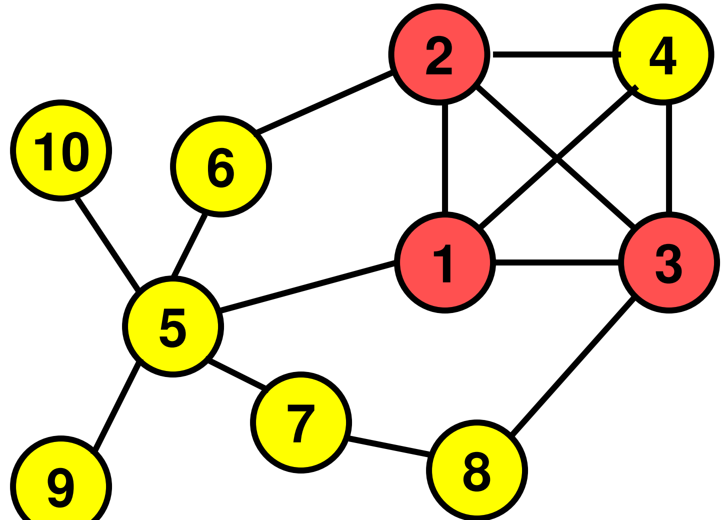
Round 5 (Begin)



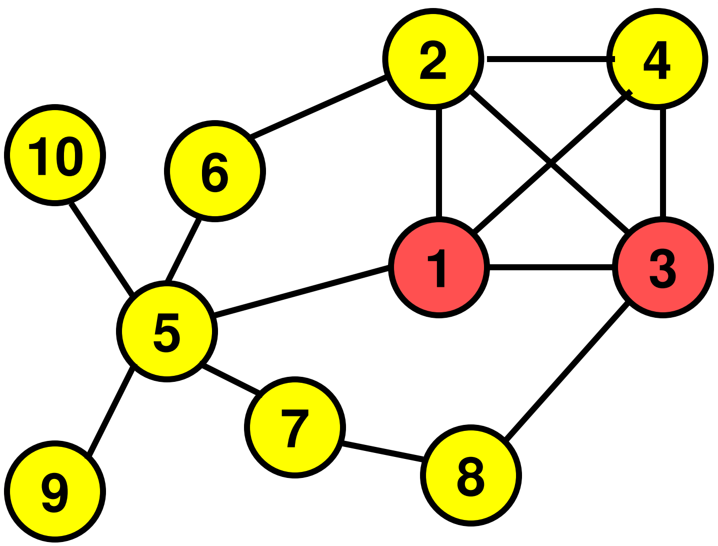
Round 5 (End): 3



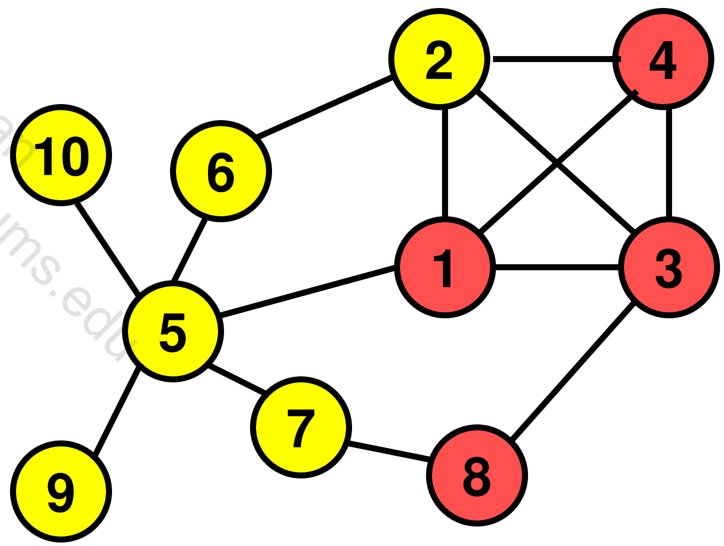
Round 6 (Begin)



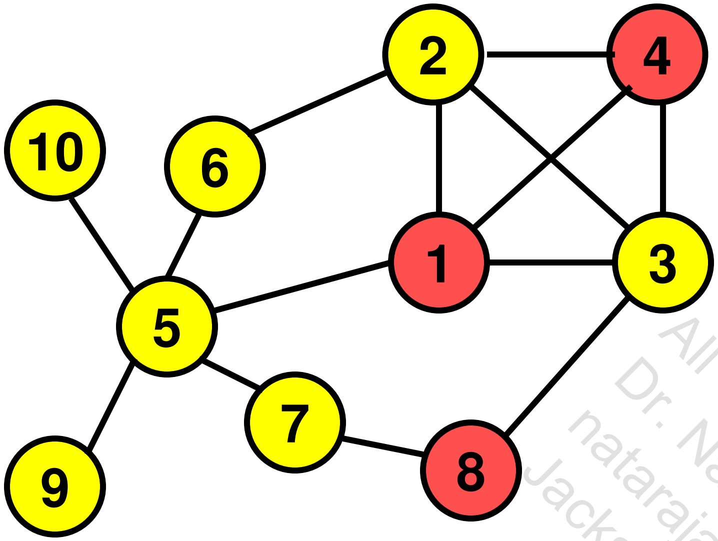
Round 6 (End): 1



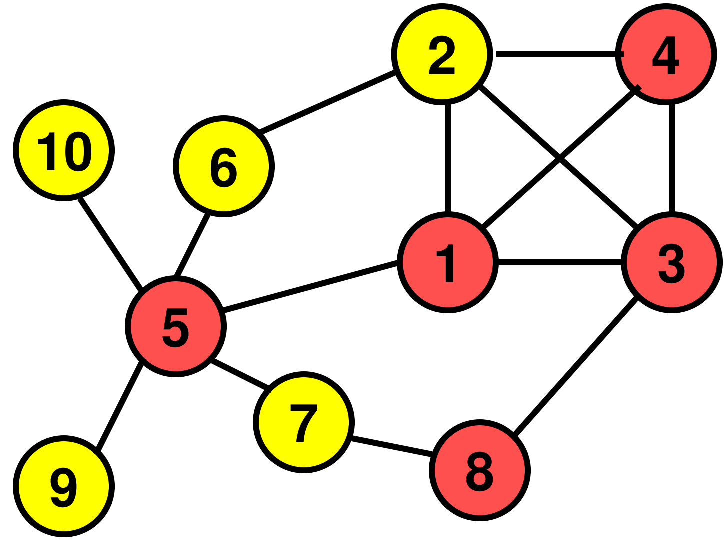
Round 7 (Begin)



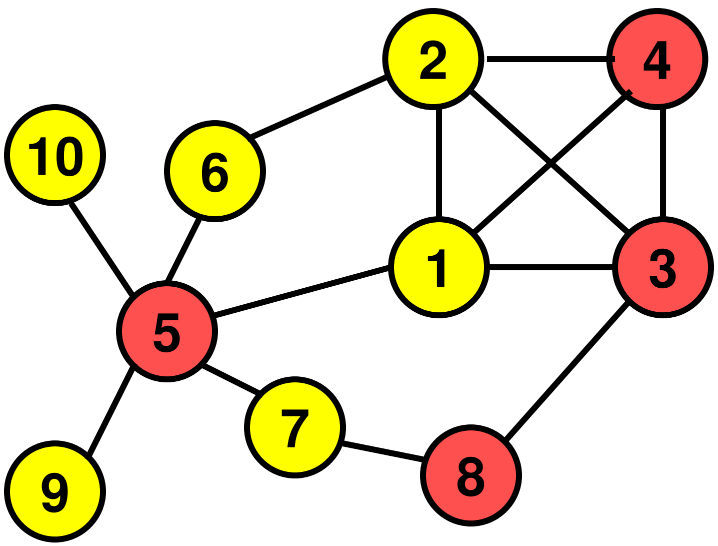
Round 7 (End): 4, 8



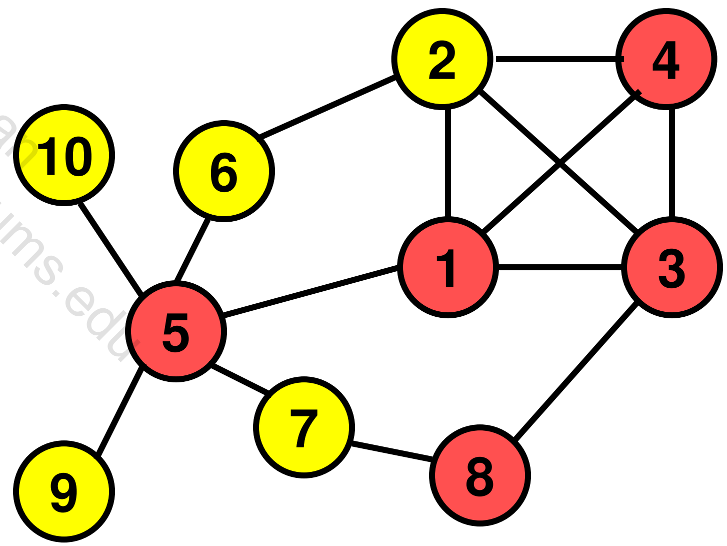
Round 8 (Begin)



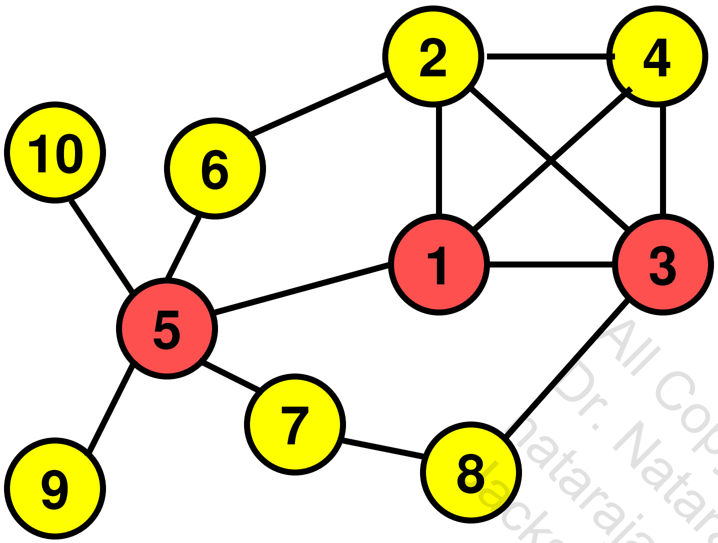
Round 8 (End): 3, 5



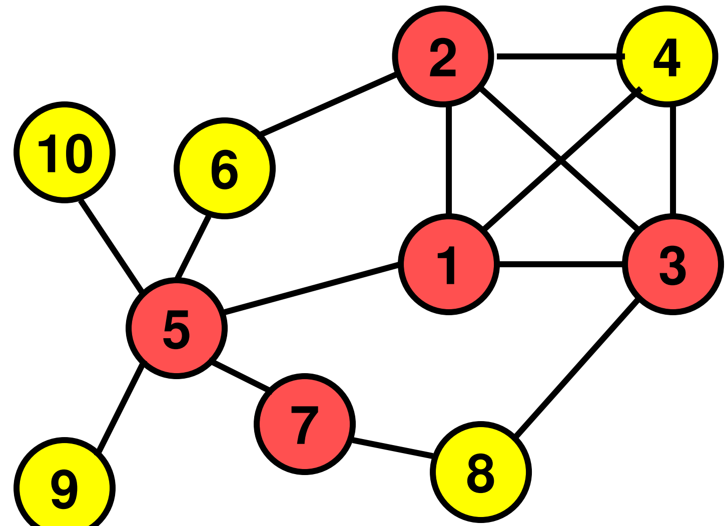
Round 9 (Begin)



Round 9 (End): 1



Round 10 (Begin)



Round 10 (End): 2, 7

Round #	Total # Inf. Nodes
1	2
2	2
3	2
4	2
5	3
6	3
7	4
8	5
9	5
10	5

Inference:

(1) The total # infected nodes across the time instants tends to increase very slowly and may tend to even remain the same.

(2) It is difficult for the infections to leave a cluster, especially if the starting node is not a "bridge" node that connects two clusters.

Basic Reproduction Number R0

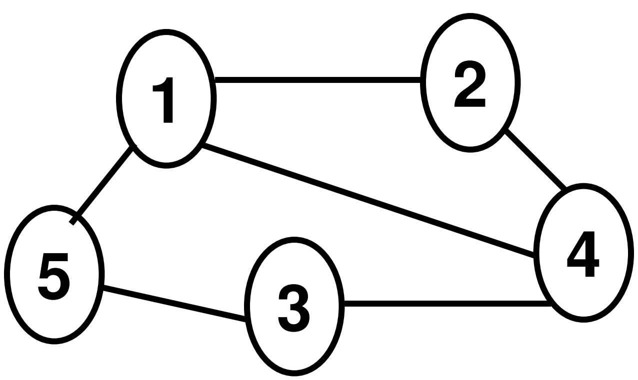
- For the SIR, SIS and SIRS models simulated on a network, the basic reproduction number $R0 = <k>*\beta/\mu = <k>*\lambda$, where $<k>$ is the average degree of the nodes in the network and β/μ is called the spreading rate (λ) of the epidemic.
- We also know that an infection can spread and make the network to reach an endemic state when $R0 \geq 1$.
  - i.e., $<k>*\beta/\mu \geq 1$
  - $\beta/\mu \geq 1/<k>$, for the epidemic to not die out and reach an endemic state.

Example 1 (SIS Model Simulation): $\beta = 0.5$, $\mu = 0.5$, $\beta/\mu = 1 \geq 1/<k>$ for any connected network graph. Hence, a finite % of the nodes (which is all the nodes in Example 1) stay infected starting from Round 5.

Example 2 (SIS Model Simulation): $\beta = 0.2$, $\mu = 0.5$, $\beta/\mu = 0.4$; $<k> = 2.8$ (i.e., $1/<k> = 0.36$) for the graph in Ex. 2. As $\beta/\mu = 0.4 > 0.36$, we again see the % of the nodes to be infected to only slowly increase, and not decrease.

Average Degree of Nodes

- The average degree of the nodes in a network can be calculated with just the number of nodes (N) and number of links (L) in the network and the actual network is not needed.
- This is because, for an undirected network, when we count the sum of the degrees of the nodes, each link in the network is counted twice.
  - Hence, <k> = 2L/N.



Average degree = (3 + 2 + 2 + 3 + 2) / 5 = 2.4
Average degree = 2 * 6 / 5 = 2.4

Example 3

- Consider a social network of 100 nodes that is vulnerable for an epidemic spread under the SIS model. If the spreading rate of the epidemic is 0.2, what is the maximum possible number of links you can have among the nodes in the network such that the epidemic will die down and not become an endemic?
- For the epidemic to die down,
  - Spreading rate $\lambda < 1/<k>$.
- i.e., $<k> < 1/ \lambda$
- $<k> < 1/0.2$
- $<k> < 5$
- $2L/N < 5$
- $L < 5*100/2$
- $L < 250$ links ➔ max. possible number of links = 249

SIS Endemic Analysis

- From the basic SIS analysis, the fraction of infected nodes is

$$i(time = \infty) = 1 - \frac{1}{\left(\beta / \mu\right) * \langle k \rangle}$$

- For the above equation to make sense for a network of N nodes, at least one among the nodes has to be infected in the endemic state (for the state to be called an endemic state).

$$i(time = \infty) = 1 - \frac{1}{\left(\beta / \mu\right) * \langle k \rangle} \geq \frac{1}{N}$$

SIS Endemic Analysis

$$1-\frac{1}{N} \geq \frac{1}{\left(\beta/\mu\right) * <k>}$$

$$1-\frac{1}{N} \geq \frac{1}{\lambda * <k>}$$

$$\frac{N-1}{N} \geq \frac{1}{\lambda * <k>}$$

$$\boxed{\lambda \geq \frac{N}{(N-1) * <k>}}$$

$$\boxed{\frac{\beta}{\mu} \geq \frac{N}{(N-1) * <k>}}$$

Example 4

- Consider a social network of 50 nodes with 200 links. Consider the spread of a contagion under the SIS model in this social network such that the average time it takes for an infection to recover is 5 days. What should be the minimum probability for infection across a link such that at least one node will remain “infected” in the endemic state?

$$\frac{\beta}{\mu} \geq \frac{N}{(N-1)* < k >}$$

<k> = 2*L/N = 2*200/50 = 8.0

$$\frac{1}{\mu} = 5; \mu = 0.2$$

$$\beta \geq \frac{N * \mu}{(N-1)* < k >}$$

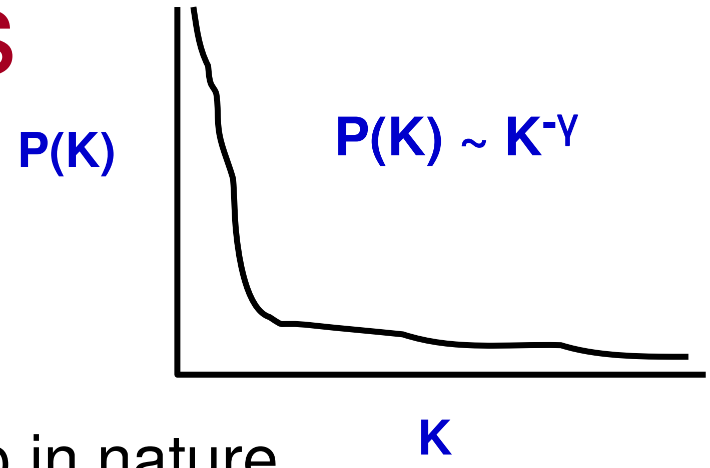
β ≥ 0.0255

$$\beta \geq \frac{50 * 0.2}{49 * 8}$$

In other words, if **β < 0.0255** for every link, then the contagion will eventually die down and no node would have been infected as time tends to infinity.

2.2 Epidemic Analysis for Scale-Free Networks

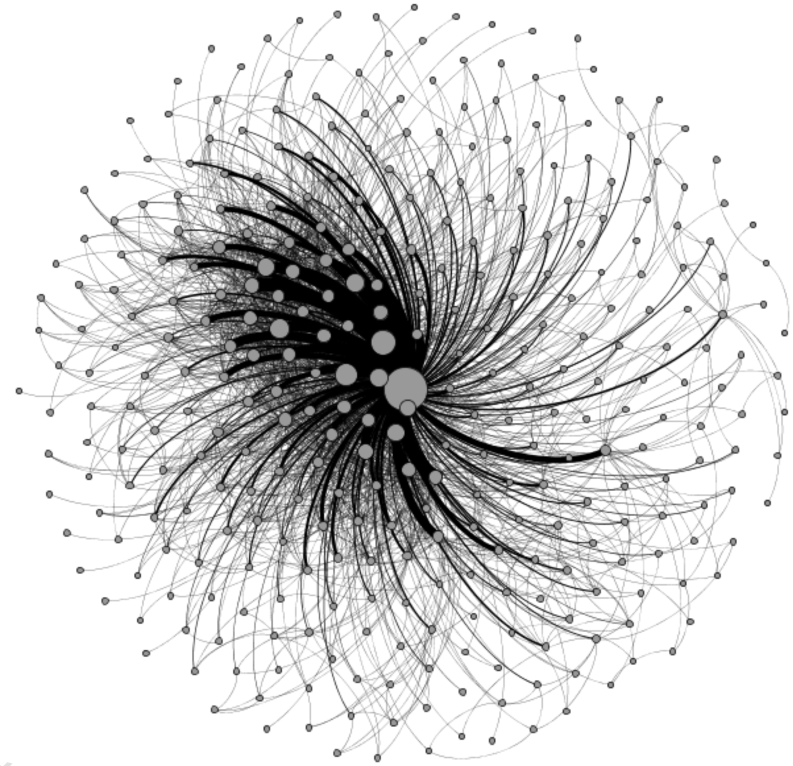
Scale-Free Networks

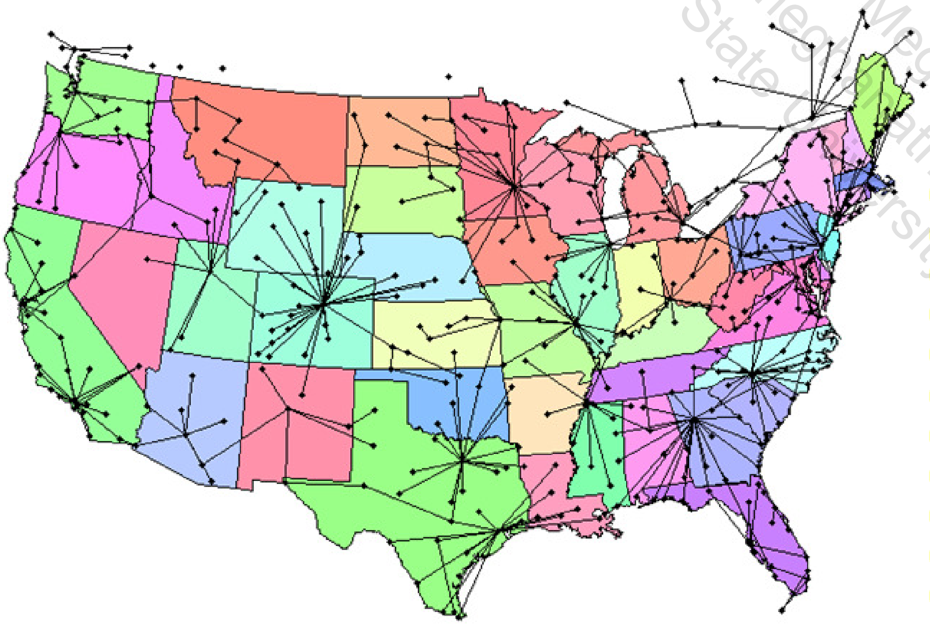


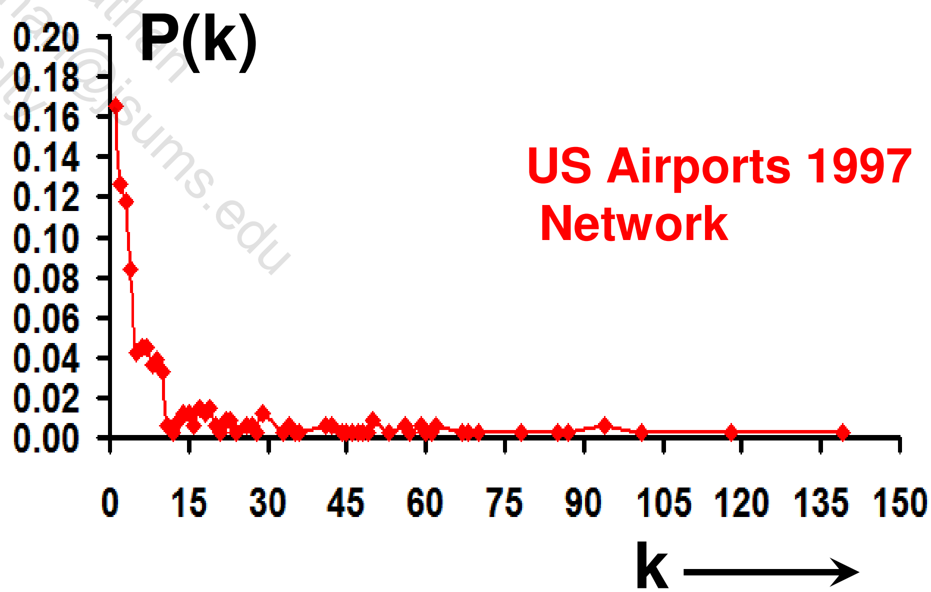
- Most real-world networks are scale-free in nature.

  In a scale-free network:
  - The degree distribution is Pareto in nature
    - Heavy tailed
      - There are few, but appreciable number of *hub* nodes with a very large degree, and several smaller degree nodes connected to one or more hub nodes.
  - The standard deviation of node degree is much greater than the average node degree
  - Probability for finding a node with degree k is proportional to $k^{-\gamma}$.
  - The degree exponent γ is a critical parameter that decides the extent of scale-freeness. Lower the value of γ, more scale-free is the network and vice-versa.

Scale-Free Networks Example: US Airports Network







SIS Model Behavior for Scale-Free Networks (2 < γ < 3)

- Θ(λ) is the fraction of infected neighbors of the susceptible nodes in the endemic state (i.e., the number of rounds t -> ∞)

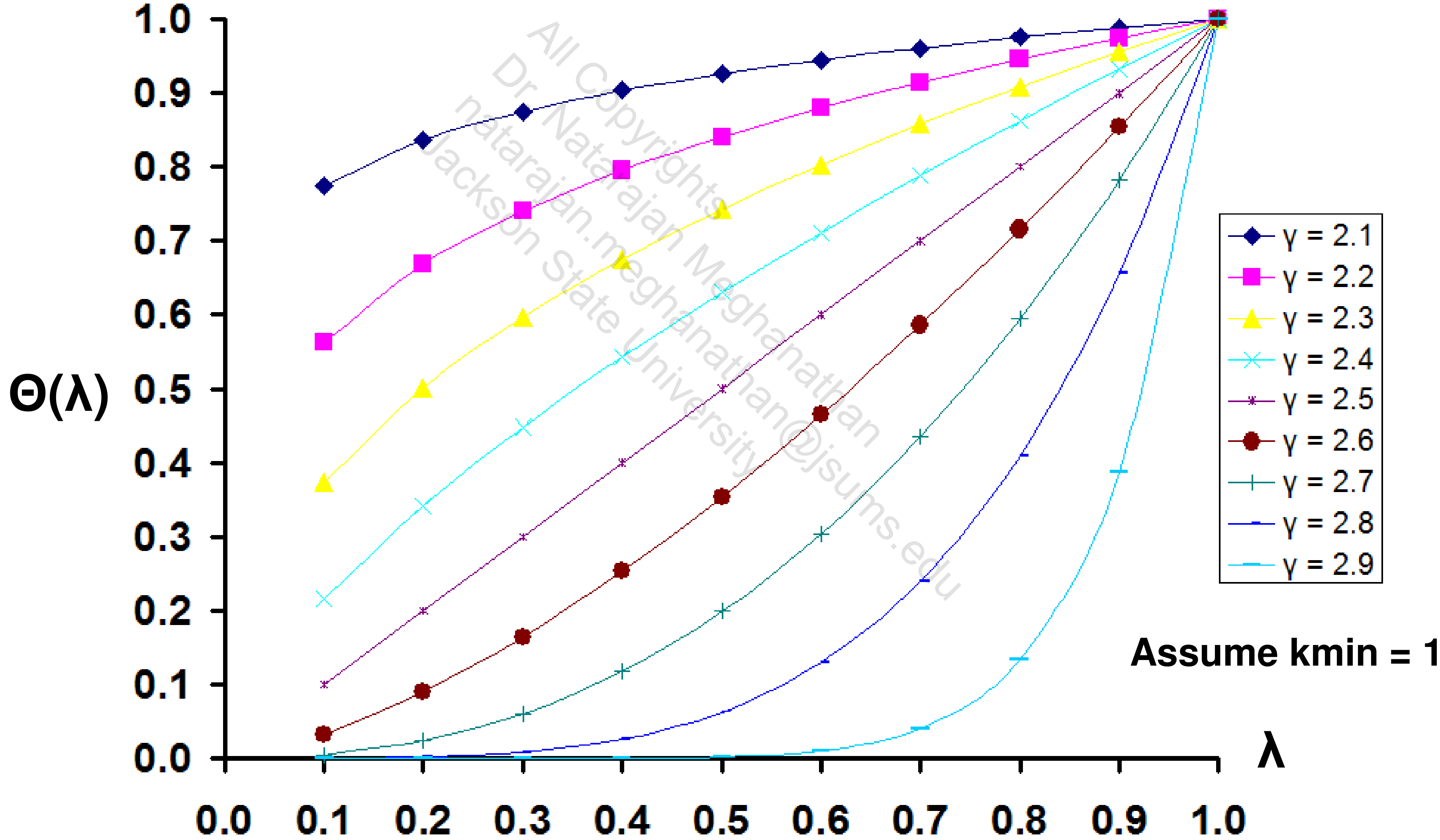
λ = β/μ

$$\Theta(\lambda) \sim \left(k_{\min}\lambda\right)^{(\gamma-2)/(3-\gamma)}$$

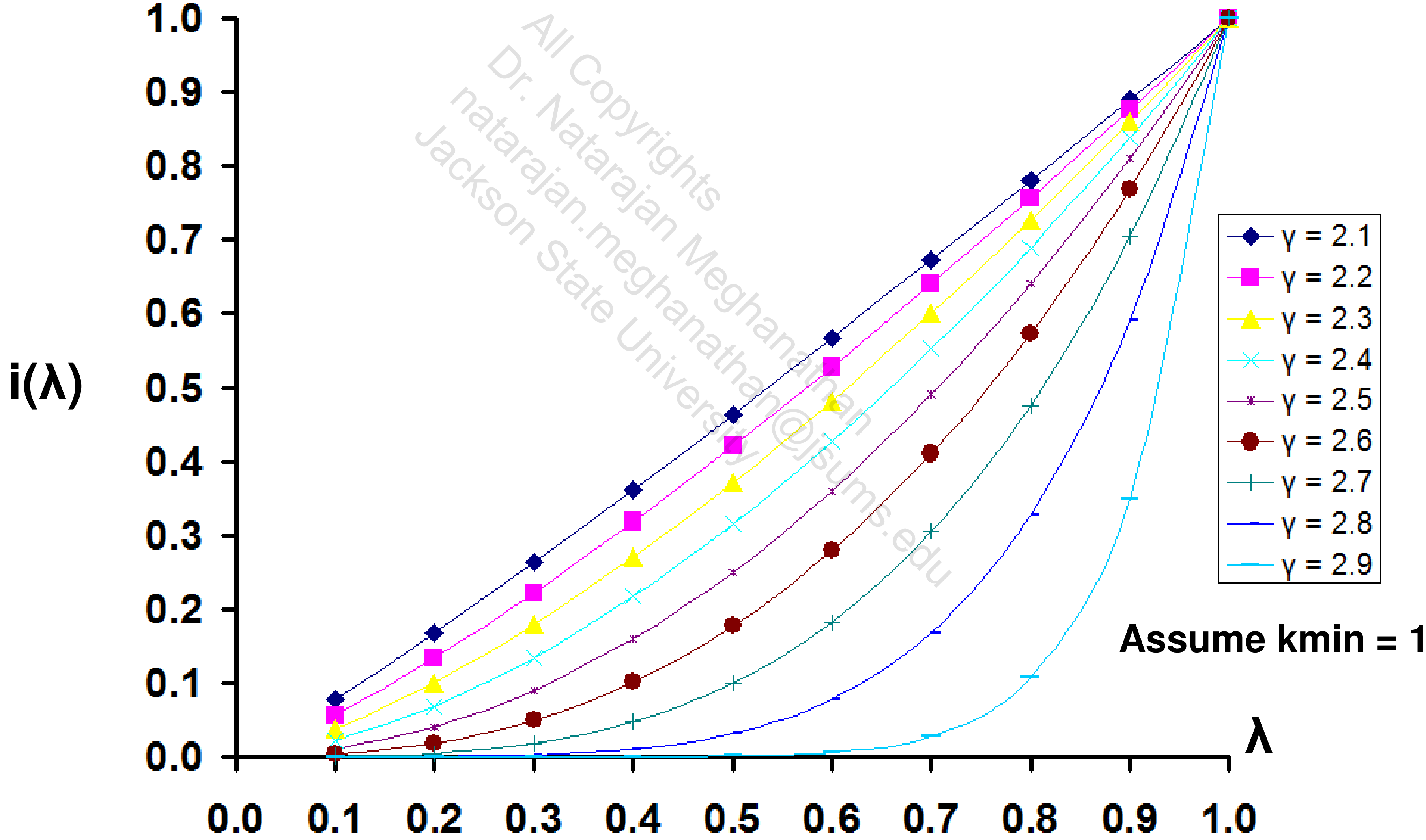
- i(λ) is the fraction of infected nodes in the endemic state.

$$i(\lambda) \sim \left(\lambda\right)^{1/(3-\gamma)}$$

SIS Model for Scale-Free Networks (2 < γ < 3): Θ(λ)



SIS Model for Scale-Free Networks (2 < γ < 3): i(λ)

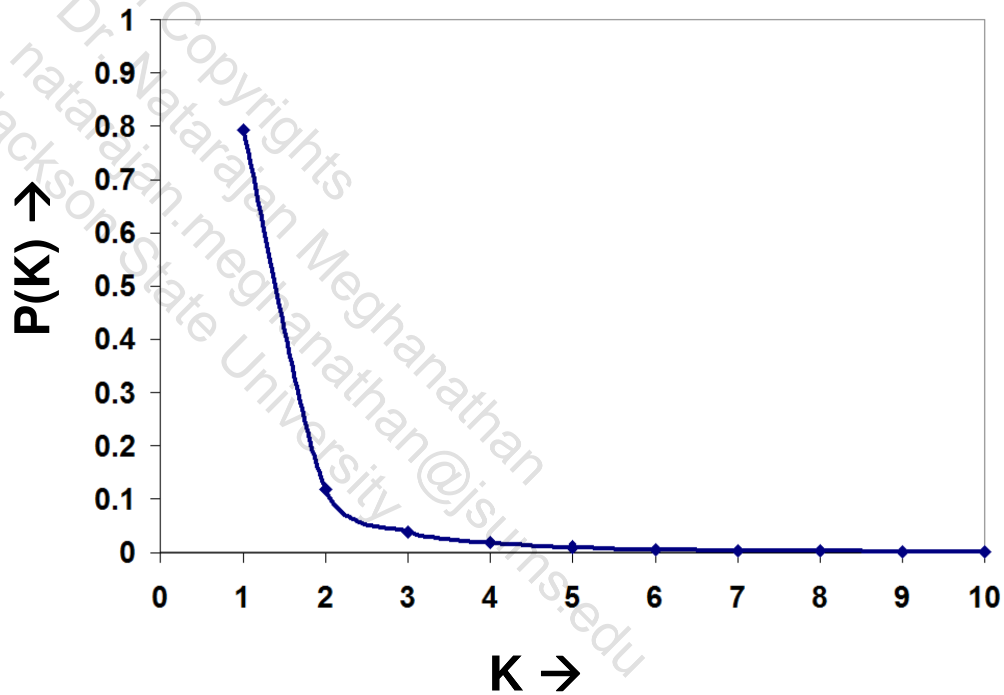


Example 1: Scale-Free Networks, SIS Model, Endemic State Analysis

- Consider the following degree distribution for a scale-free network and a simulation of the SIS model on this network with a spreading rate of 0.5. Determine the following:
  - The degree exponent, γ
  - The fraction of infected neighbors of the susceptible nodes in the endemic state.
  - The fraction of infected nodes in the endemic state.

Example 1(1): Scale-Free Networks, SIS Model, Endemic State Analysis

K	P(K)
1	0.794
2	0.119
3	0.039
4	0.018
5	0.010
6	0.006
7	0.004
8	0.003
9	0.002
10	0.001



$P(K) \sim K^{-\gamma}$
$P(K) = C^*K^{-\gamma}$

where C is a proportionality constant

Example 1(1): Scale-Free Networks, SIS Model, Endemic State Analysis

$P(K) = C^*K^{-\gamma}$

$\ln P(K) = \ln C + (-\gamma^*\ln K)$: Compared to $Y = \text{intercept} + (\text{slope})^*X$;

$Y = C + m^*X$ slope = $-\gamma$; intercept = $\ln C$;

We will use curve (line) fitting in Excel to find the slope and constant

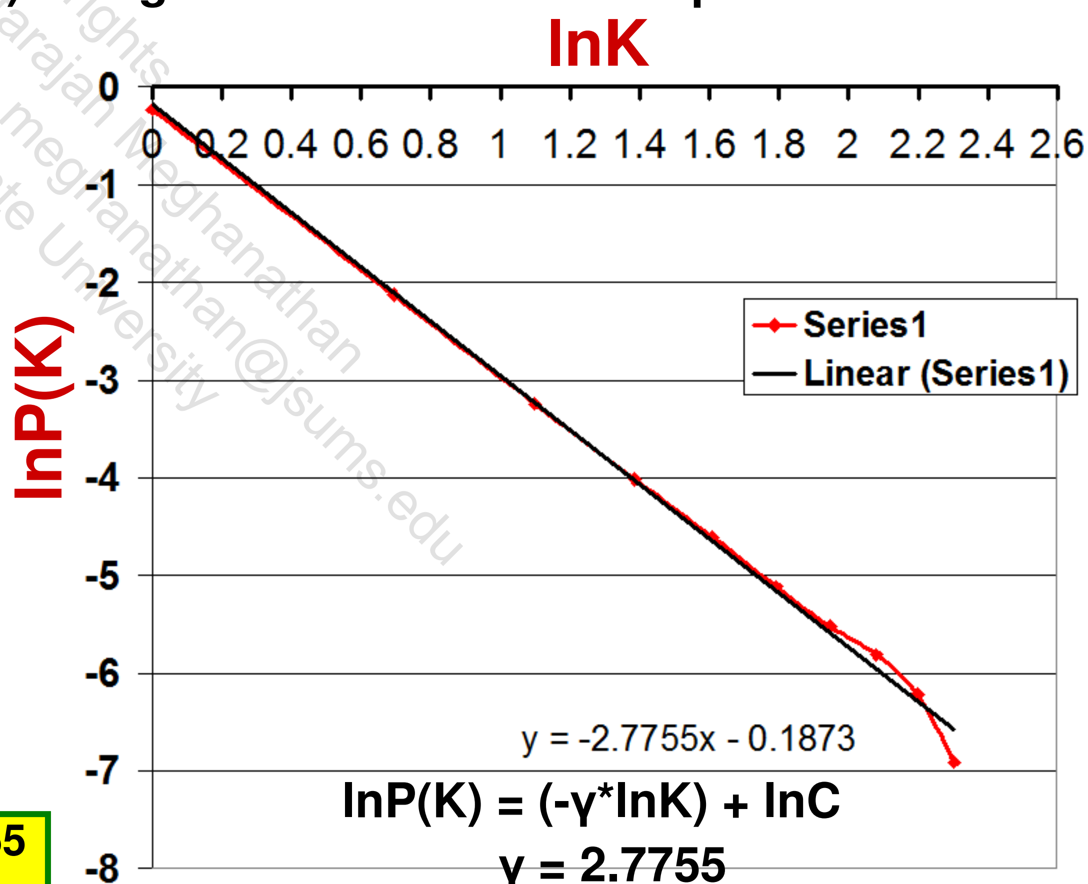
K	P(K)	lnK	lnP(K)
1	0.794	0	-0.23
2	0.119	0.69	-2.13
3	0.039	1.10	-3.24
4	0.018	1.39	-4.02
5	0.010	1.61	-4.61
6	0.006	1.79	-5.12
7	0.004	1.95	-5.52
8	0.003	2.08	-5.81
9	0.002	2.20	-6.21
10	0.001	2.30	-6.91

Intercept = $\ln C = -0.1873$

$C = e^{-0.1873} = 2.718^{-0.1873}$

$C = 0.829$

$\boxed{P(K) = 0.829^*K^{-2.7755}}$



Example 1(1): Scale-Free Networks, SIS Model, Endemic State Analysis

$$\Theta(\lambda) \sim (k_{\min}\lambda)^{(\gamma-2)/(3-\gamma)} \qquad i(\lambda) \sim (\lambda)^{1/(3-\gamma)}$$

Since for the given degree distribution, P(K = 1) > 0, kmin = 1
Given the spreading rate for the SIS model is λ = 0.5.
We just found the degree exponent γ to be 2.7755.

$$\frac{\gamma-2}{3-\gamma} = \frac{2.7755-2}{3-2.7755} = 3.4543$$

$$\Theta(\lambda) = 0.5^{3.4543} = 0.0912$$

$$\frac{1}{(3-\gamma)} = \frac{1}{3-2.7755} = 4.4543$$

$$i(\lambda) = 0.5^{4.4543} = 0.0456$$

Example 2: Scale-Free Network, SIS Model in Endemic State

- Consider a scale-free network modeled using the power-law, $P(K) = CK^{-\gamma}$. Upon an analysis of the degree distribution, it was observed that approximately 4% of the nodes are with degree 4 and 8% of the nodes are with degree 3. Using the above information, determine the degree exponent γ and the proportionality constant C.

- Let the kmin value for the network be the smallest K value for which $P(K) \leq 1$. Using the above results, determine the Kmin value for the above network.

- Assume the SIS model is simulated on the above scale-free network and the network has reached an endemic state with 5% of the nodes observed to have been infected. Determine the spreading rate λ and the fraction Θ of infected neighbor nodes for the susceptible nodes.

- Using the above information and the results obtained, what can you say about the average degree of the nodes in the scale-free network?

Example 2(1): Scale-Free Network, SIS Model in Endemic State

- Consider a scale-free network modeled using the power-law, $P(K) = CK^{-\gamma}$. Upon an analysis of the degree distribution, it was observed that approximately 4% of the nodes are with degree 4 and 8% of the nodes are with degree 3. Using the above information, determine the degree exponent γ.

Solution:

ln(A*B) = lnA + lnB

ln(C*k^-y) = lnC + ln(K^-y)

- $P(K) = CK^{-\gamma} \Rightarrow \ln P(K) = \ln C + (-\gamma)\ln K$
- Given that $P(3) = 0.08$ and $P(4) = 0.04$

$\ln(0.08) = \ln C + (-\gamma)\ln(3) \Rightarrow -2.526 = \ln C + (-\gamma)*1.098$

$\ln(0.04) = \ln C + (-\gamma)\ln(4) \Rightarrow -3.219 = \ln C + (-\gamma)*1.386$

Solving for γ, we get $\gamma = (3.219 – 2.526)/(1.386 – 1.098) = 2.40$

Solving for $C = P(K) / K^{-\gamma} \Rightarrow C = P(4) / 4^{-2.4} = 0.04 / 0.03589 = 1.1143$

Example 2(2): Scale-Free Network, SIS Model in Endemic State

- Let the kmin value for the network be the smallest K value for which P(K) ≤ 1. Using the above results, determine the Kmin value for the above network.

Solution:

- $P(K) = CK^{-\gamma}$
- We have P(K=3) = 0.08 < 1.
- Let us try for K = 2; P(2) = $1.1143 * 2^{-2.4} = 0.2111 < 1$.
- Let us try for K = 1; P(1) = $1.1143 * 1^{-2.4} = 1.1143 > 1$. Hence, K = 1 is not a possible value for Kmin.
- Hence, Kmin = 2.

Example 2(3): Scale-Free Network, SIS Model in Endemic State

- Assume the SIS model is simulated on the above scale-free network and the network has reached an endemic state with 5% of the nodes observed to have been infected. Determine the spreading rate λ and the fraction Θ of infected neighbor nodes for the susceptible nodes.

$$i(\lambda) \sim (\lambda)^{1/(3-\gamma)} = 0.05 \qquad \frac{1}{(3-\gamma)} = \frac{1}{3-2.4} = 1.6667$$

$$0.05 = \lambda^{1.6667}$$

ln(0.05) = 1.6667 ln(λ)
ln(λ) = ln(0.05) / 1.6667
ln(λ) = -2.9957 / 1.6667 = -1.7974

$$\lambda = e^{-1.7974} = 2.7183^{-1.7974} = 0.1657$$

Example 2(4): Scale-Free Network, SIS Model in Endemic State

- Assume the SIS model is simulated on the above scale-free network and the network has reached an endemic state with 5% of the nodes observed to have been infected. Determine the spreading rate λ and the fraction Θ of infected neighbor nodes for the susceptible nodes.

$$\Theta(\lambda) \sim \left(k_{\min}\lambda\right)^{(\gamma-2)/(3-\gamma)}$$

$$\frac{\gamma-2}{3-\gamma} = \frac{2.4-2}{3-2.4} = 0.6667$$

$$\Theta\left(\lambda = 0.1657\right) = (2 * 0.1657)^{0.6667} = 0.4789$$

Example 2(5): Scale-Free Network, SIS Model in Endemic State

- Using the above information and the results obtained, what can you say about the average degree of the nodes in the scale-free network?

We know that for a network to reach an endemic state (fraction of Infected nodes > 0 as time -> ∞) under the SIS model, the spreading rate λ ≥ 1/<k>,
where <k> is the average degree of the nodes

We have λ = 0.1657
Hence, <k> ≥ 1/λ ➔ <k> ≥ 1/0.1657
i.e., average degree of the nodes <k> ≥ 6.035.