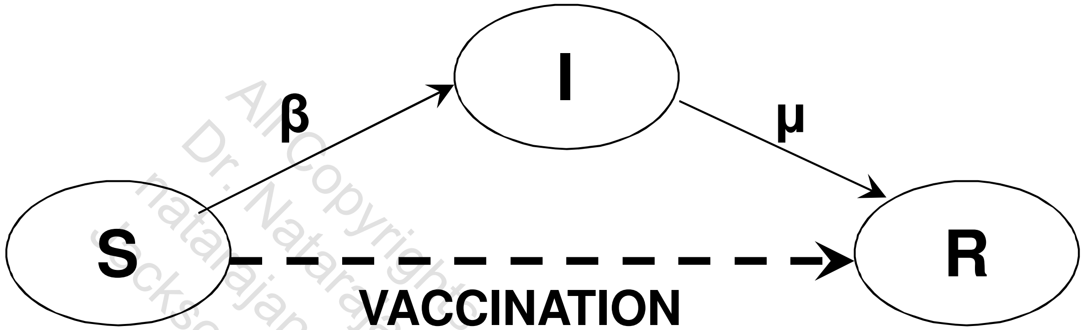
# Module 3
# Vaccination and Herd Immunity

# Effect of Vaccination: SIR Model






**From our earlier discussions on the SIR and SIS models: The rate of Spread of infection can decrease as shown below.**

$$\frac{di}{dt} = i * [< k > \beta * s - \mu] < 0$$

$$s < \frac{1}{< k > \beta / \mu}$$

$$s < \frac{1}{< k > \lambda}$$

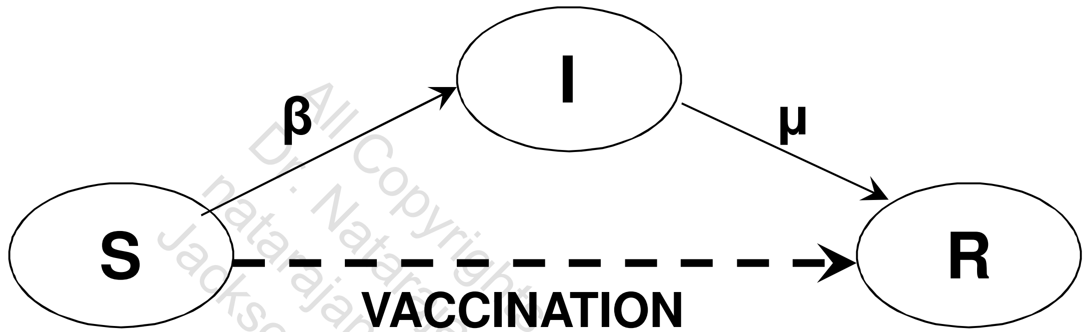
$$[< k > \beta * s - \mu] < 0$$

$$s < \mu / < k > \beta$$

$$s < \frac{1}{R_0}$$

Hence, if at any time, if we keep the Fraction of susceptible individuals (nodes) to be less than 1/R0, we can make the contagion spread to eventually die down.

# Effect of Vaccination: SIR Model






**From our earlier discussions on the SIR and SIS models: The rate of Spread of infection can decrease as shown below.**

$$s < \frac{1}{R_0}$$

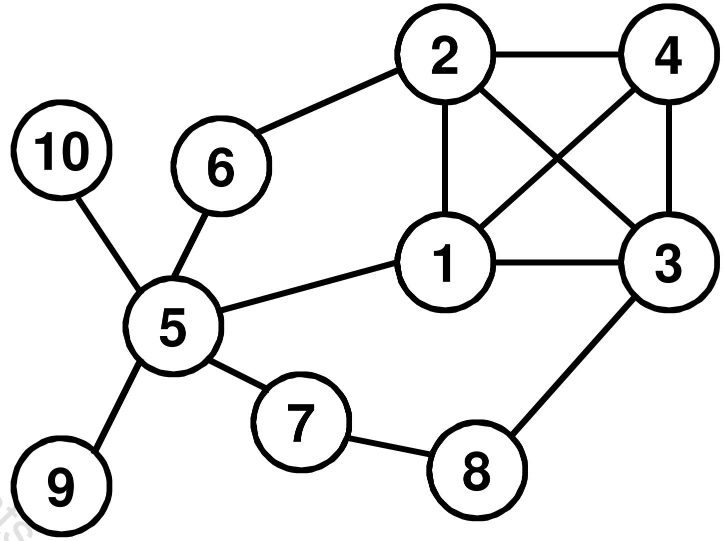
**Fraction of susceptible individuals < 1/R0**
**1 - Fraction of non-susceptible individuals < 1/R0**
**- Fraction of non-susceptible individuals < (1/R0) -1**
**Fraction of non-susceptible individuals ≥ 1 – 1/R0**

$$v \geq 1 - \frac{1}{R_0}$$

**Hence, we need to vaccinate at least 1 – 1/R0 fraction (v) of the population to make the infection spread to eventually die down. The susceptible nodes that are not vaccinated will be protected through the vaccinated nodes: Herd Immunity**
**Note: This fraction holds good for the SIS model as well.**

# Which nodes to vaccinate?

- In a social network, can we just randomly choose the nodes such that if the fraction of nodes vaccinated is $\geq 1-1/R0$, we can obtain the so called "herd immunity" for the infection to not spread and to eventually die down? Or we need to selectively vaccinate certain nodes in the network to obtain herd immunity?



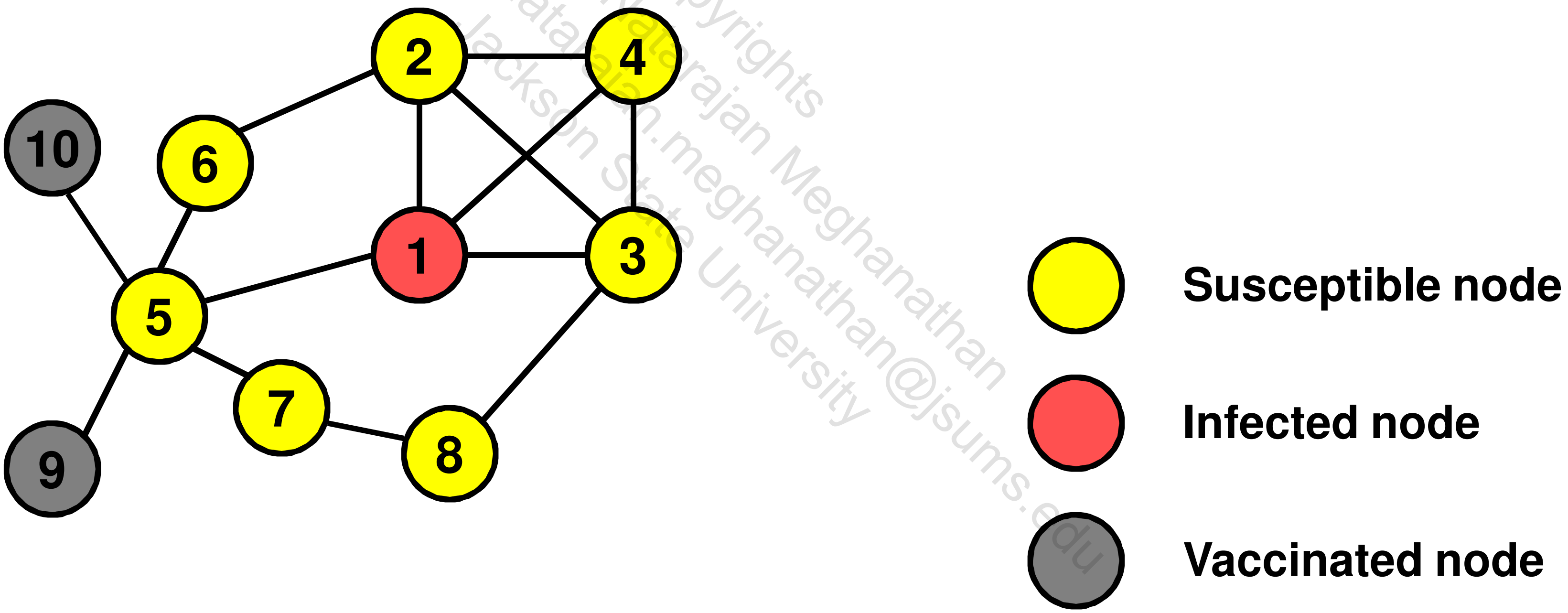

**Let infection time ($1/\mu$) be 2 rounds**
**Let infection probability ($\beta$) be 0.2**

**$\langle k \rangle$ for the above graph is 2.8**

**$R0 = \langle k \rangle \beta/\mu = 2.8 * 0.2 * 2 = 1.12$**

**$1 - 1/R0 = 1 - 1/1.12 = 0.1071$**
**Hence we need to vaccinate at least 10.71% of the 10 nodes**
**$\geq (0.1071)*10$ nodes**
**$\geq 1.071$ nodes**
**➔ We need to vaccinate at least 2 nodes**

# SIS Model Simulation: Example 1 (with Vaccination)



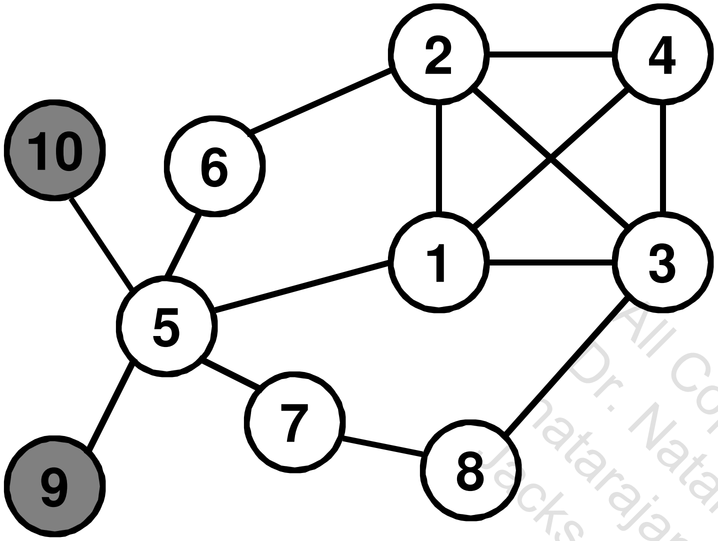



**Assumption:** A vaccinated node remains non-susceptible throughout the simulation

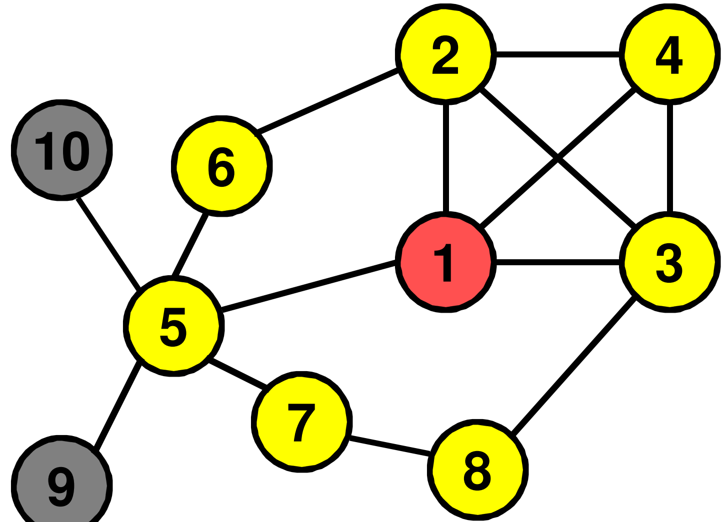
# SIS Model Simulation: Example 1:rand
## (with Vaccination of randomly chosen nodes)

**Random numbers generated**

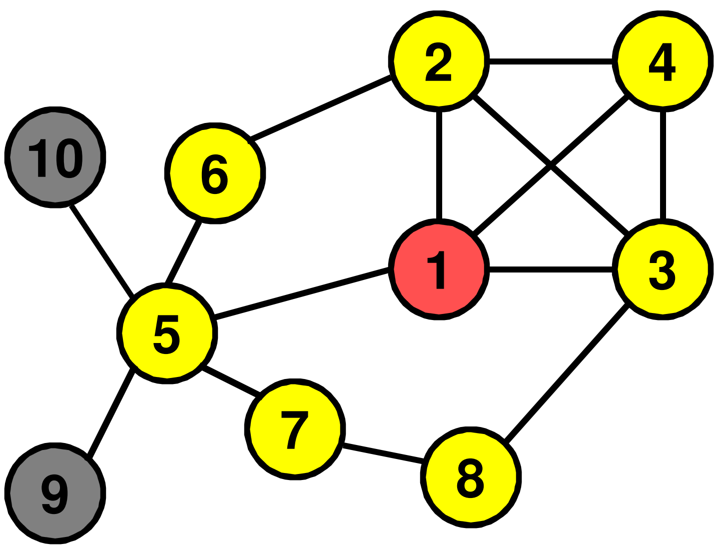
| Links | Round 1 | Round 2 | Round 3 | Round 4 | Round 5 | Round 6 | Round 7 | Round 8 | Round 9 | Round 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1...2 | 0.069727 | 0.327157 | 0.987588 | 0.033982 | 0.397306 | 0.103209 | 0.540245 | 0.214213 | 0.331587 | 0.110146 |
| 1...3 | 0.380447 | 0.785791 | 0.893356 | 0.1377 | 0.939614 | 0.21679 | 0.15035 | 0.823889 | 0.160652 | 0.381949 |
| 1...4 | 0.273639 | 0.2233 | 0.453682 | 0.839541 | 0.423825 | 0.514878 | 0.170527 | 0.10118 | 0.953627 | 0.918737 |
| 1...5 | 0.893866 | 0.362893 | 0.424807 | 0.059336 | 0.900095 | 0.384065 | 0.509922 | 0.013265 | 0.820163 | 0.288808 |
| 2...3 | 0.444439 | 0.77898 | 0.634305 | 0.256588 | 0.131689 | 0.144057 | 0.574832 | 0.375135 | 0.508881 | 0.587843 |
| 2...4 | 0.876286 | 0.512341 | 0.139157 | 0.161313 | 0.163935 | 0.484496 | 0.262967 | 0.499072 | 0.66305 | 0.861896 |
| 2...6 | 0.365708 | 0.410347 | 0.301668 | 0.696285 | 0.314386 | 0.846186 | 0.252213 | 0.840461 | 0.742175 | 0.333662 |
| 3...4 | 0.226968 | 0.642317 | 0.137131 | 0.75405 | 0.497082 | 0.259342 | 0.50987 | 0.368502 | 0.592843 | 0.690716 |
| 3...8 | 0.791793 | 0.461758 | 0.398524 | 0.027178 | 0.07223 | 0.89736 | 0.100144 | 0.163006 | 0.729635 | 0.451381 |
| 5...6 | 0.469923 | 0.316089 | 0.823989 | 0.087085 | 0.972357 | 0.727116 | 0.412238 | 0.333748 | 0.214519 | 0.864602 |
| 5...7 | 0.847914 | 0.738021 | 0.074276 | 0.46388 | 0.362319 | 0.004842 | 0.863546 | 0.849964 | 0.780724 | 0.016044 |
| 5...9 | 0.753155 | 0.915555 | 0.78423 | 0.256046 | 0.876939 | 0.529567 | 0.191222 | 0.388512 | 0.362856 | 0.908738 |
| 5...10 | 0.880366 | 0.579745 | 0.164895 | 0.229082 | 0.234629 | 0.23113 | 0.704058 | 0.853117 | 0.208388 | 0.207667 |
| 7...8 | 0.34678 | 0.103795 | 0.73263 | 0.312683 | 0.269853 | 0.982354 | 0.68226 | 0.228099 | 0.389481 | 0.299793 |



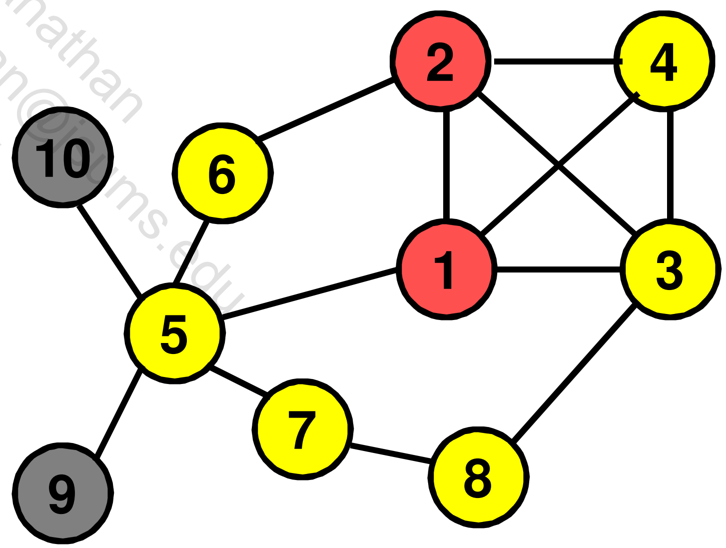

Let infection time be 2 rounds
Let infection probability beta be 0.2



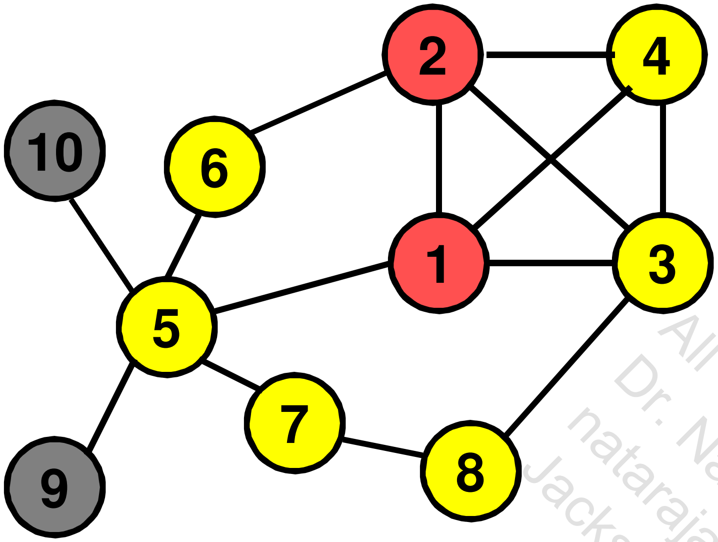

**Initialization (Round 0):** 1



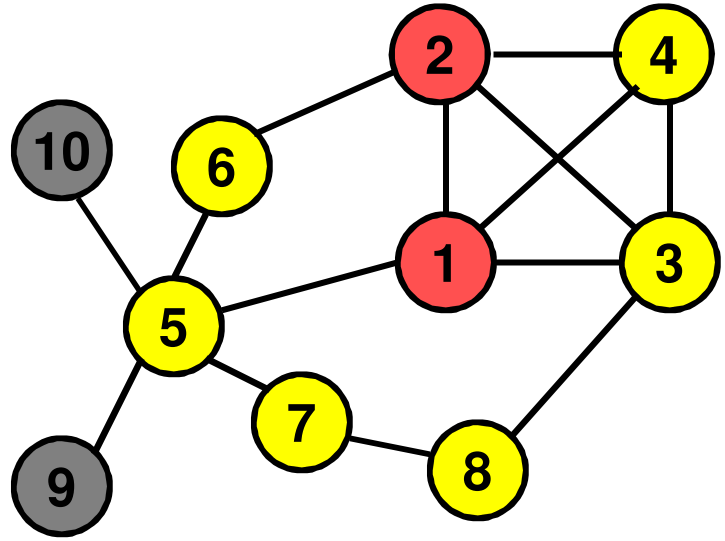

**Round 1 (Begin)**



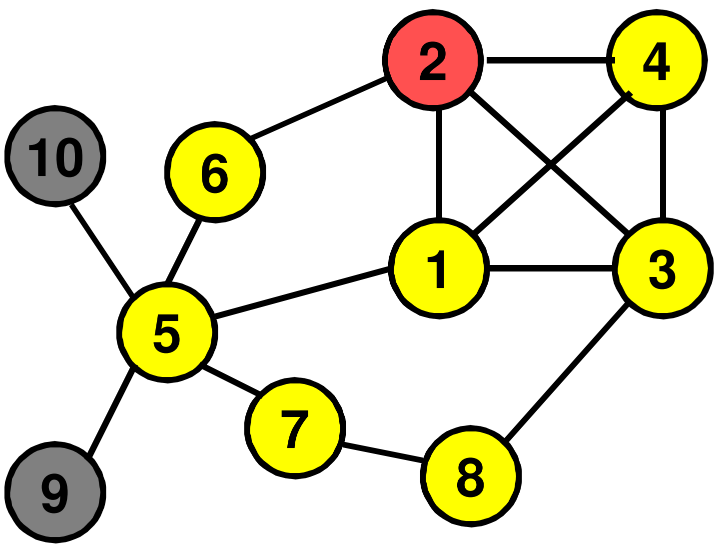

**Round 1 (End):** 2



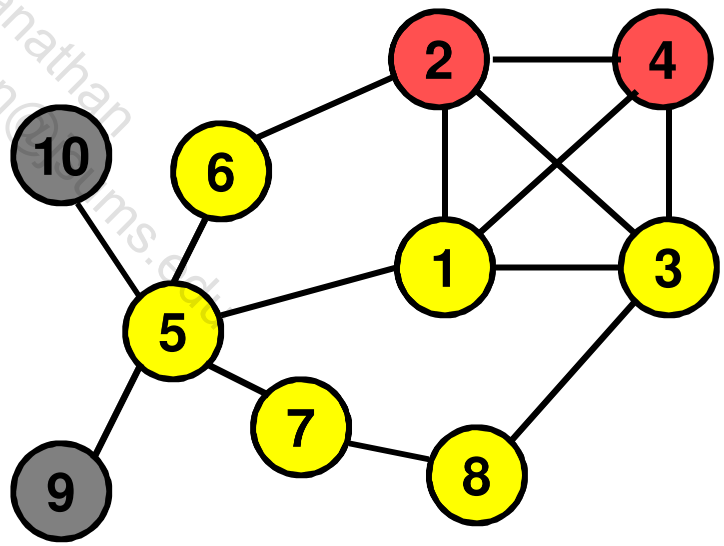

**Round 2 (Begin)**



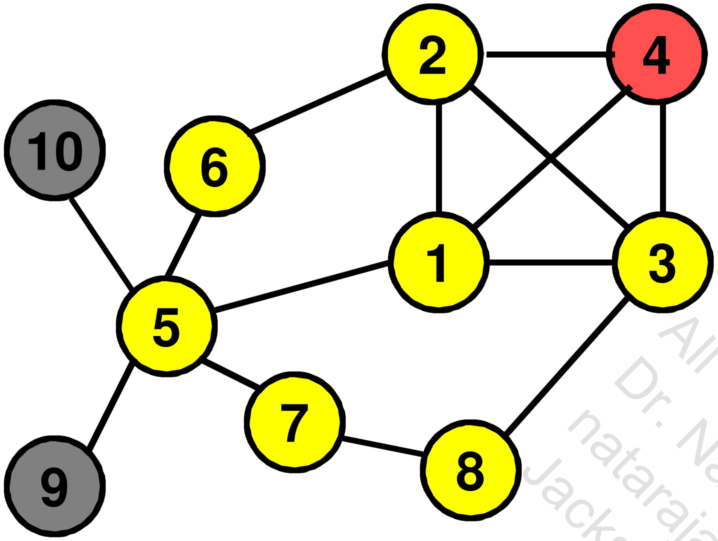

**Round 2 (End):** -



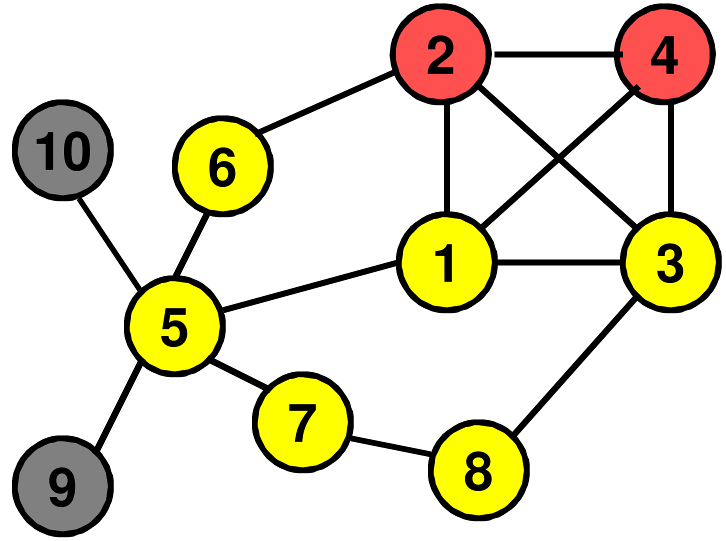

**Round 3 (Begin)**




**Round 3 (End):** 4

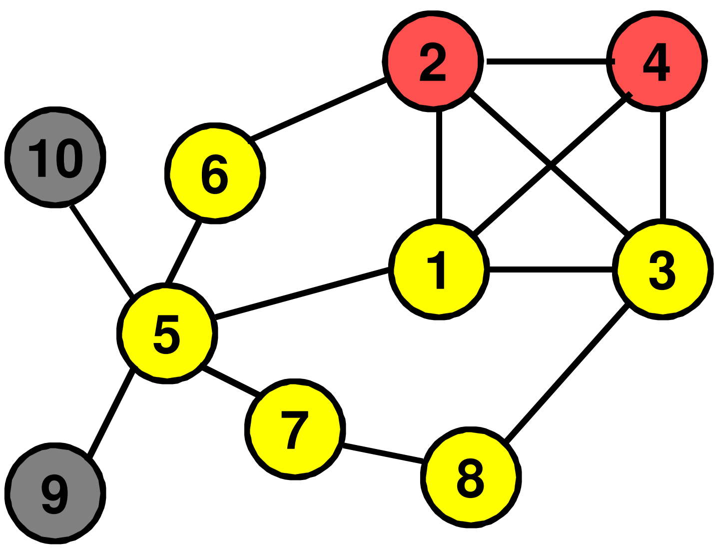



**Round 4 (Begin)**

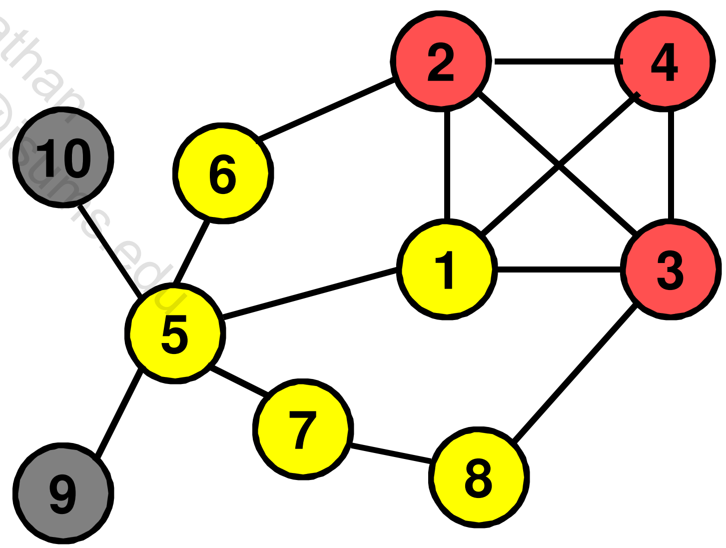



**Round 4 (End):** 2

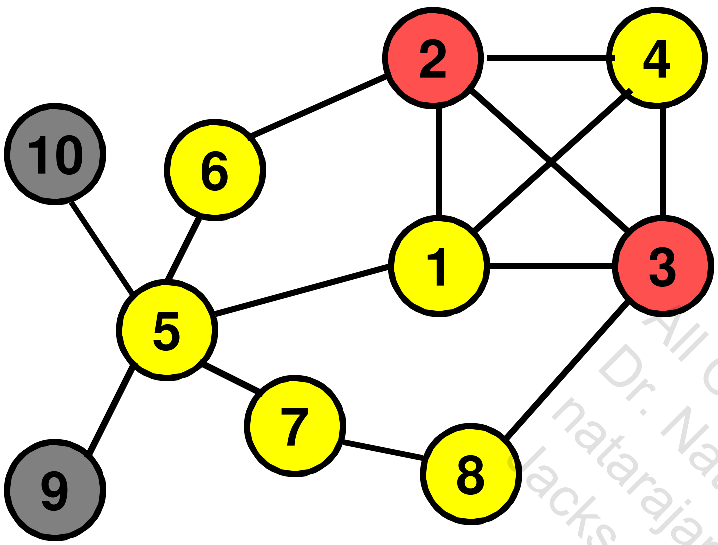




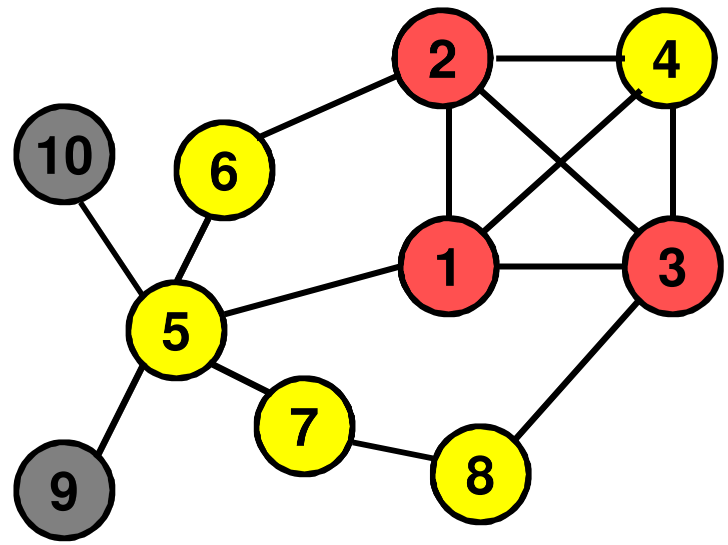

**Round 5 (Begin)**



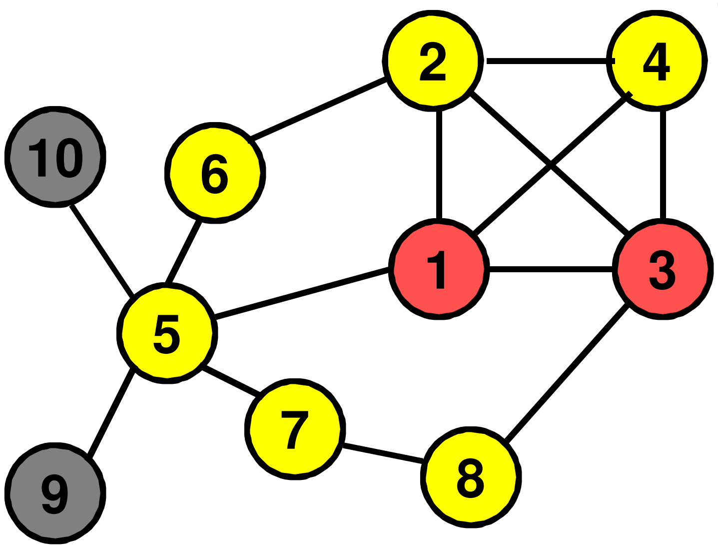

**Round 5 (End):** 3



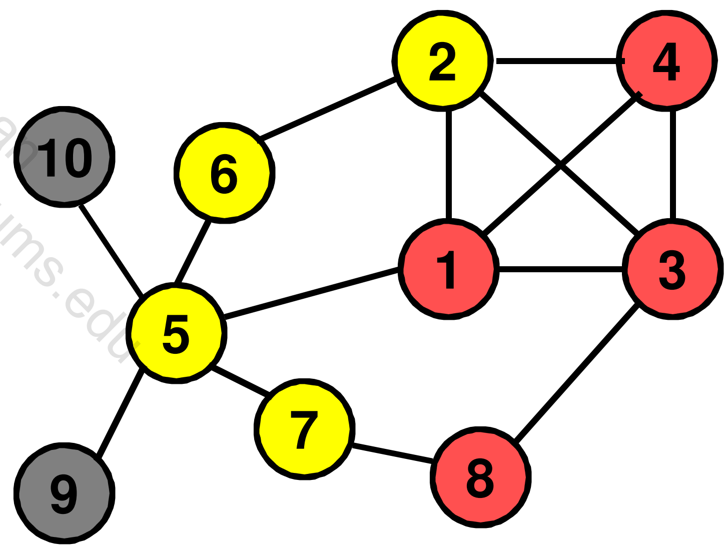

**Round 6 (Begin)**



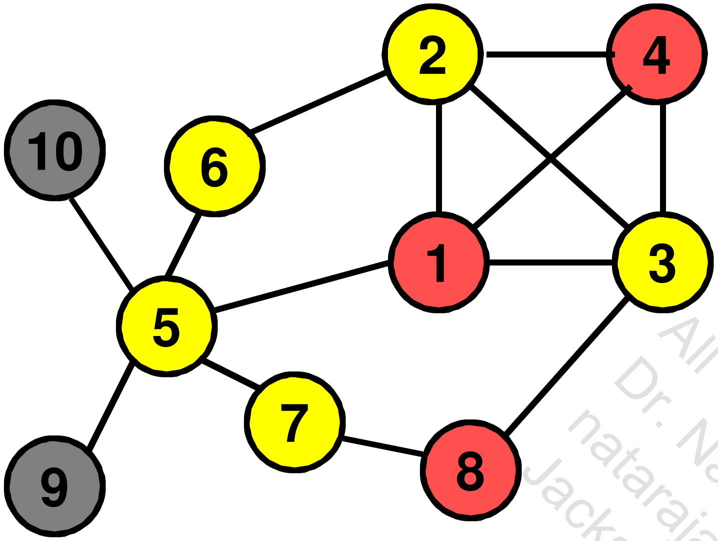

**Round 6 (End):** 1



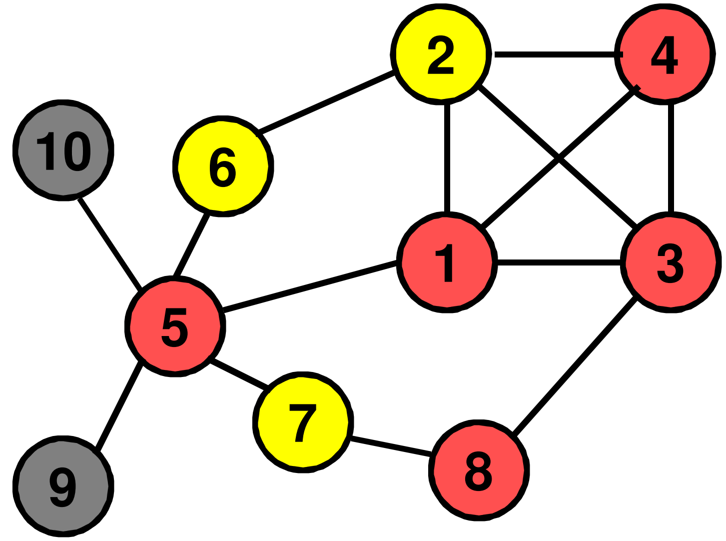

**Round 7 (Begin)**



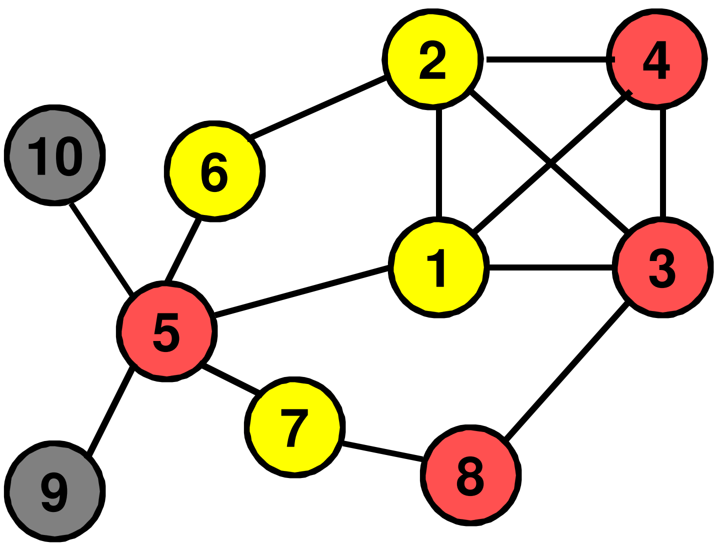

**Round 7 (End):** 4, 8



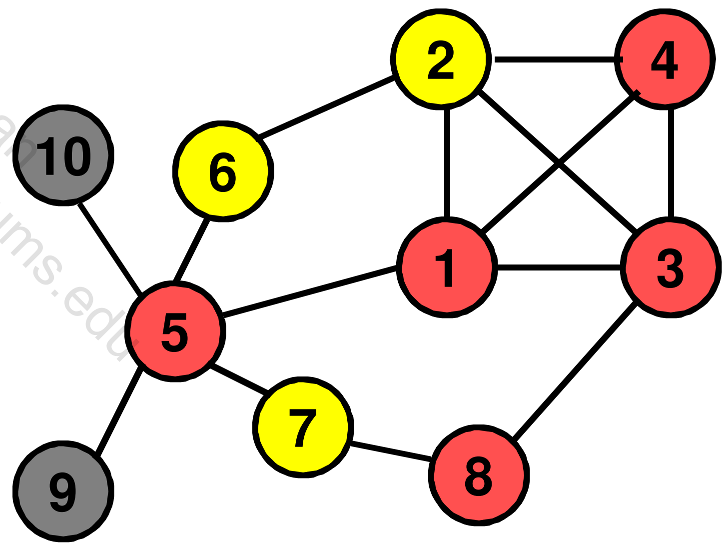

**Round 8 (Begin)**



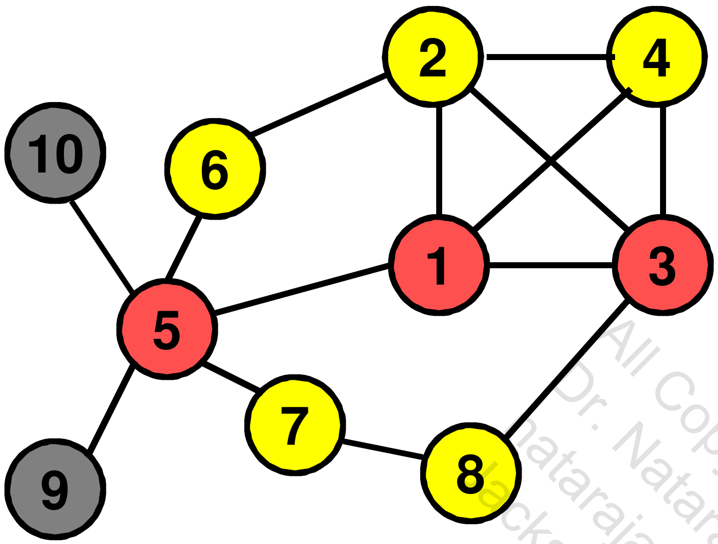

**Round 8 (End):** 3, 5



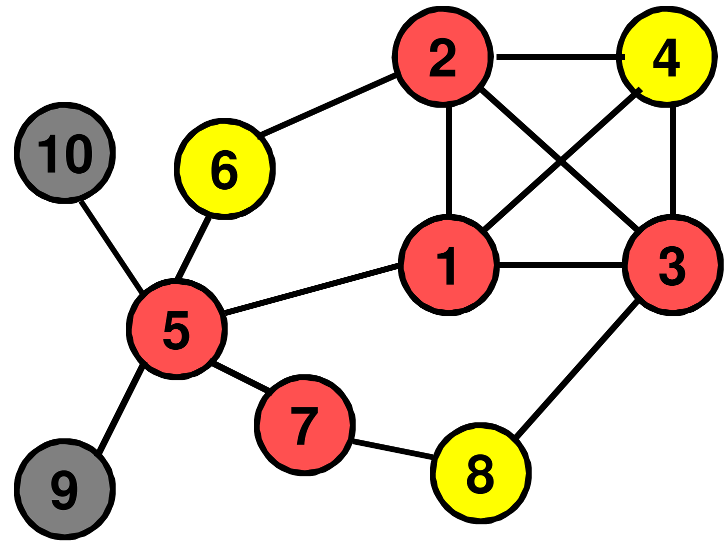

**Round 9 (Begin)**




**Round 9 (End):** 1




Round 10 (Begin)




Round 10 (End): 2, 7

| Round # | Total # Inf. Nodes |
|---|---|
| 1 | 2 |
| 2 | 2 |
| 3 | 2 |
| 4 | 2 |
| 5 | 3 |
| 6 | 3 |
| 7 | 4 |
| 8 | 5 |
| 9 | 5 |
| 10 | 5 |

**Inference:**

The rate of infection does not appear to die down. It slowly, but steadily keeps Increasing.

Once the infection reaches the “bridge” nodes 5 and 3, it starts spreading outside the cluster (1, 2, 3, 4).

# SIS Model Simulation: Example 1:NBNC (with Vaccination of NBNC-based chosen nodes)

**Random numbers generated**

| Links | Round 1 | Round 2 | Round 3 | Round 4 | Round 5 | Round 6 | Round 7 | Round 8 | Round 9 | Round 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1...2 | 0.069727 | 0.327157 | 0.987588 | 0.033982 | 0.397306 | 0.103209 | 0.540245 | 0.214213 | 0.331587 | 0.110146 |
| 1...3 | 0.380447 | 0.785791 | 0.893356 | 0.1377 | 0.939614 | 0.21679 | 0.15035 | 0.823889 | 0.160652 | 0.381949 |
| 1...4 | 0.273639 | 0.2233 | 0.453682 | 0.839541 | 0.423825 | 0.514878 | 0.170527 | 0.10118 | 0.953627 | 0.918737 |
| 1...5 | 0.893866 | 0.362893 | 0.424807 | 0.059336 | 0.900095 | 0.384065 | 0.509922 | 0.013265 | 0.820163 | 0.288808 |
| 2...3 | 0.444439 | 0.77898 | 0.634305 | 0.256588 | 0.131689 | 0.144057 | 0.574832 | 0.375135 | 0.508881 | 0.587843 |
| 2...4 | 0.876286 | 0.512341 | 0.139157 | 0.161313 | 0.163935 | 0.484496 | 0.262967 | 0.499072 | 0.66305 | 0.861896 |
| 2...6 | 0.365708 | 0.410347 | 0.301668 | 0.696285 | 0.314386 | 0.846186 | 0.252213 | 0.840461 | 0.742175 | 0.333662 |
| 3...4 | 0.226968 | 0.642317 | 0.137131 | 0.75405 | 0.497082 | 0.259342 | 0.50987 | 0.368502 | 0.592843 | 0.690716 |
| 3...8 | 0.791793 | 0.461758 | 0.398524 | 0.027178 | 0.07223 | 0.89736 | 0.100144 | 0.163006 | 0.729635 | 0.451381 |
| 5...6 | 0.469923 | 0.316089 | 0.823989 | 0.087085 | 0.972357 | 0.727116 | 0.412238 | 0.333748 | 0.214519 | 0.864602 |
| 5...7 | 0.847914 | 0.738021 | 0.074276 | 0.46388 | 0.362319 | 0.004842 | 0.863546 | 0.849964 | 0.780724 | 0.016044 |
| 5...9 | 0.753155 | 0.915555 | 0.78423 | 0.256046 | 0.876939 | 0.529567 | 0.191222 | 0.388512 | 0.362856 | 0.908738 |
| 5...10 | 0.880366 | 0.579745 | 0.164895 | 0.229082 | 0.234629 | 0.23113 | 0.704058 | 0.853117 | 0.208388 | 0.207667 |
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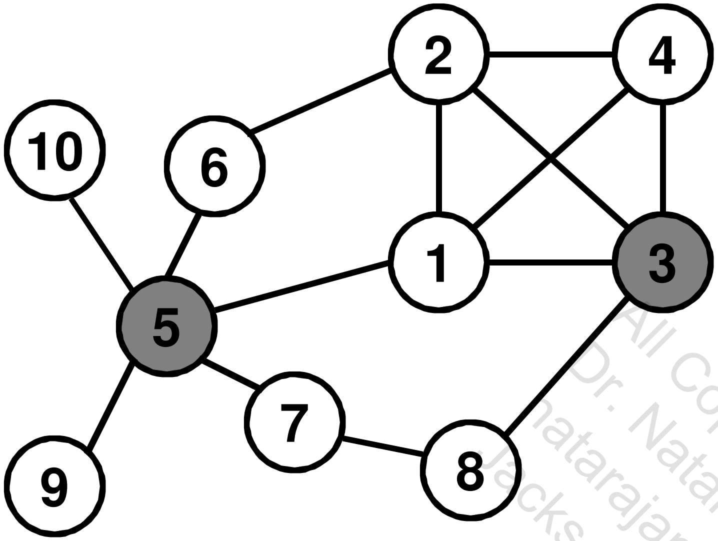
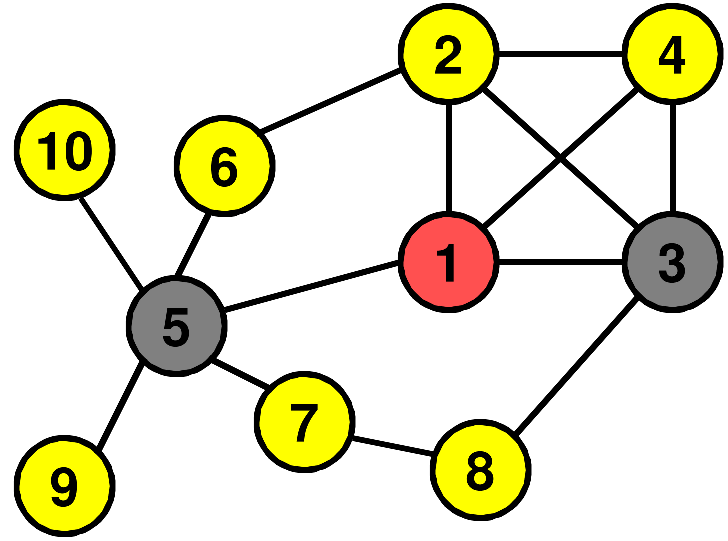
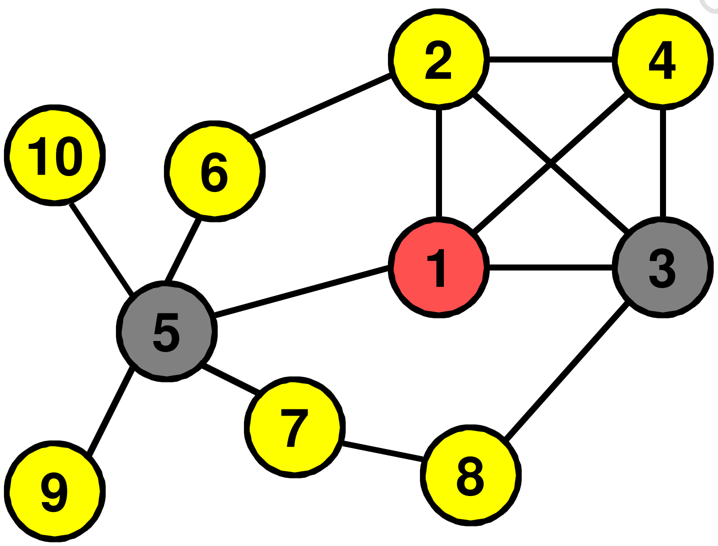


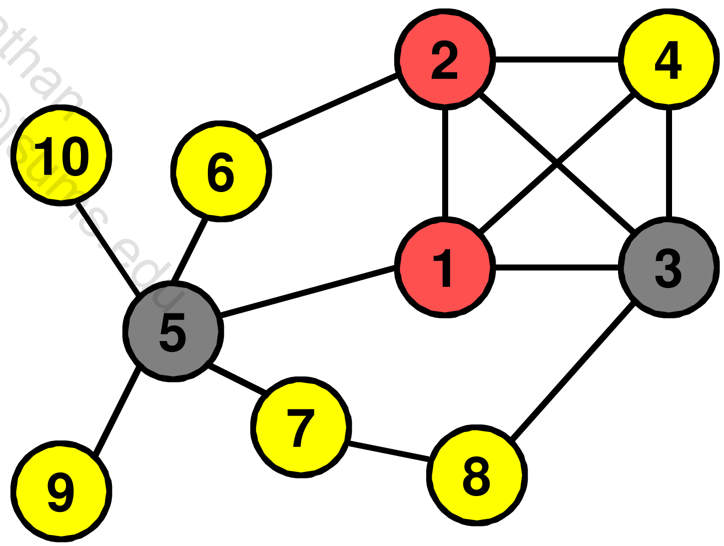

Let infection time be 2 rounds
Let infection probability beta be 0.2



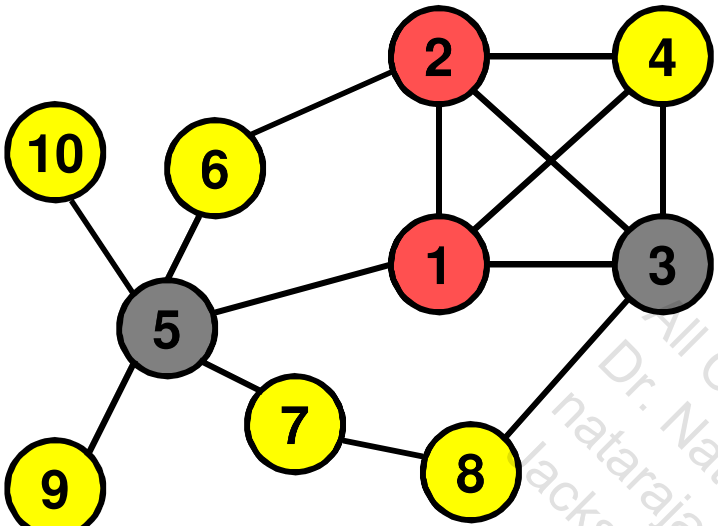

**Initialization (Round 0):** 1



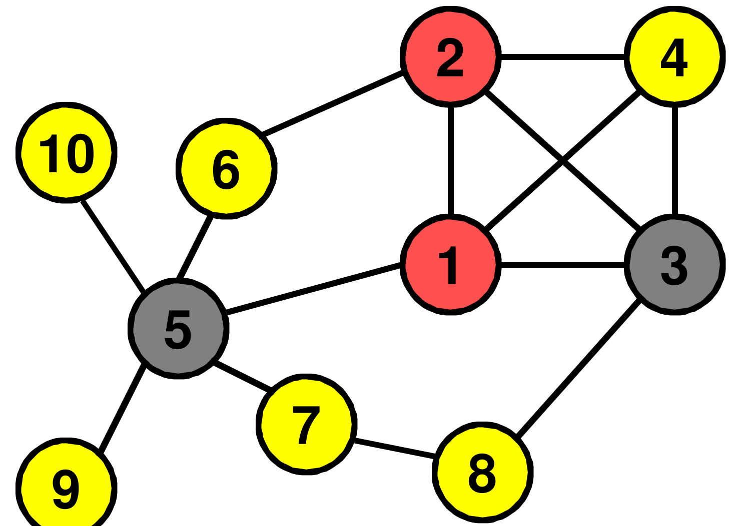

**Round 1 (Begin)**



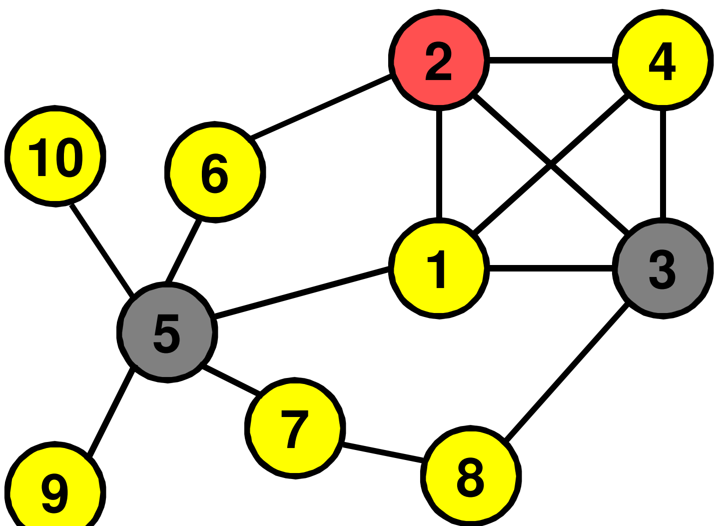

**Round 1 (End):** 2



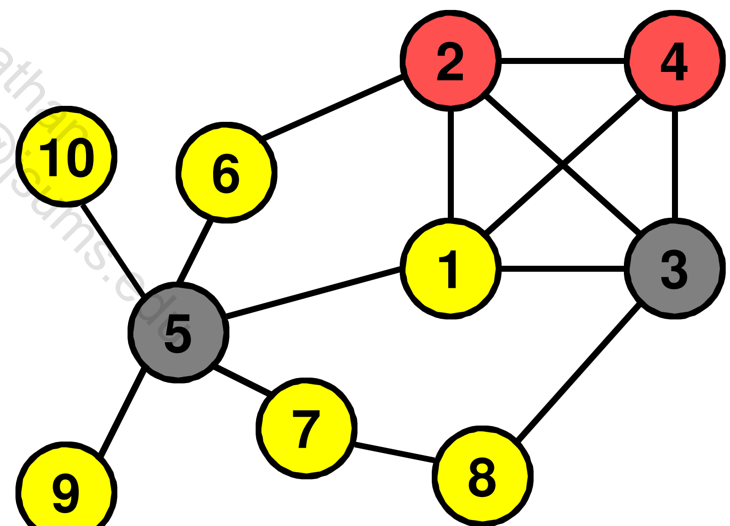

**Round 2 (Begin)**



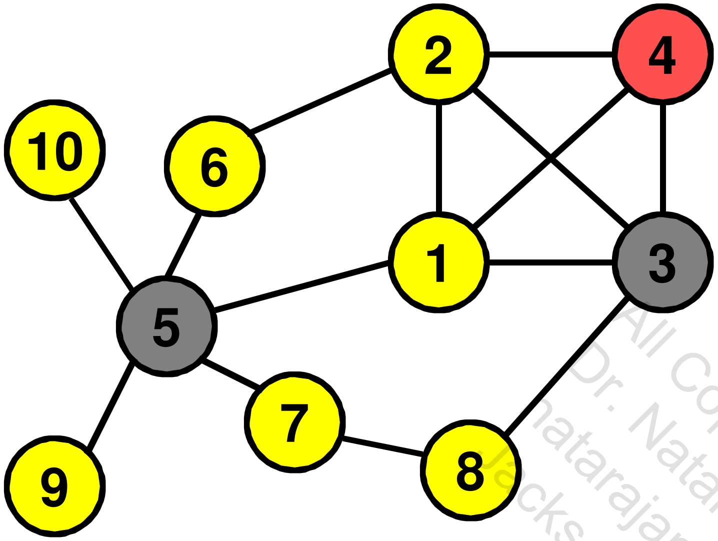

**Round 2 (End):** -



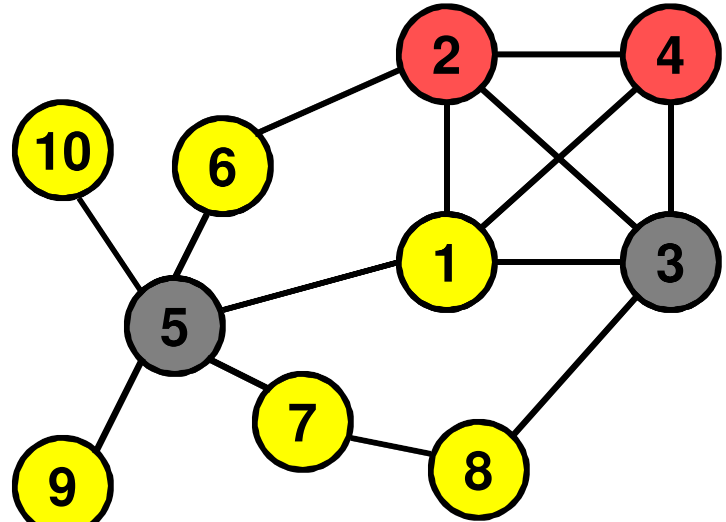

**Round 3 (Begin)**



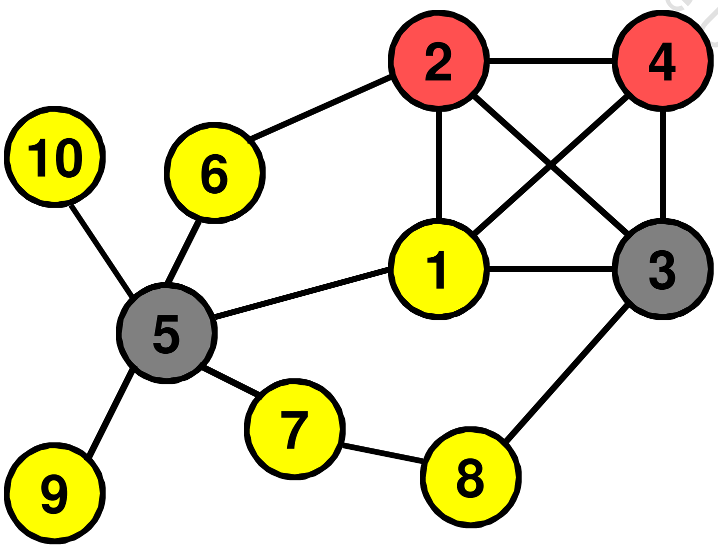

**Round 3 (End):** 4



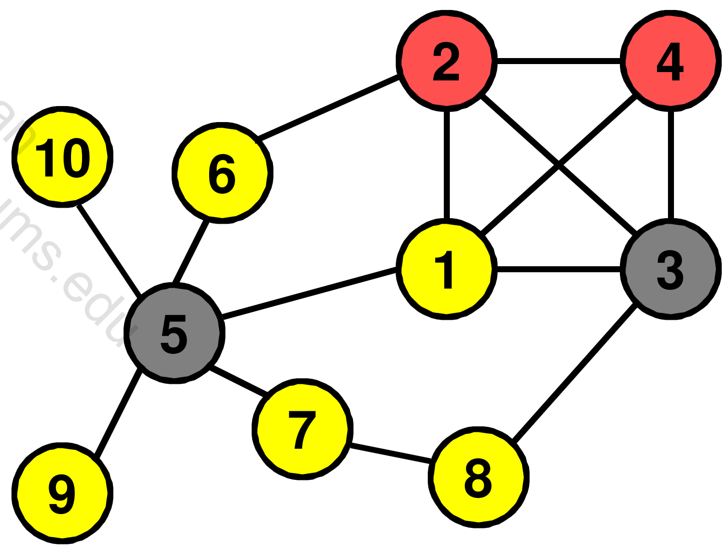

**Round 4 (Begin)**



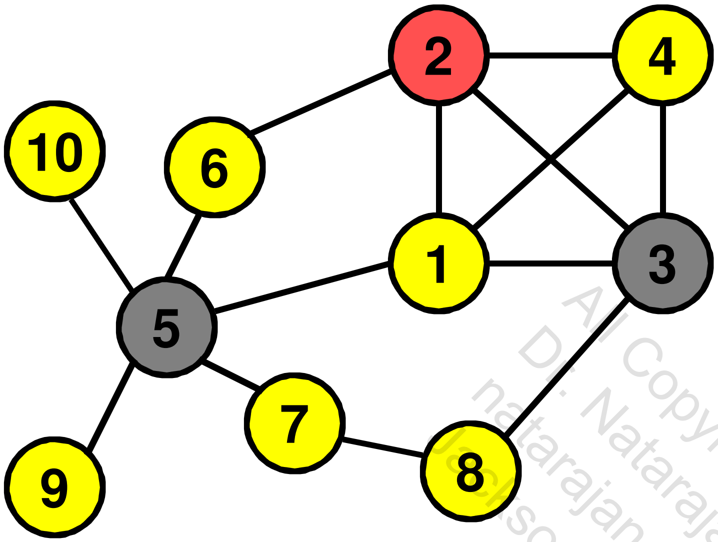

**Round 4 (End):** 2



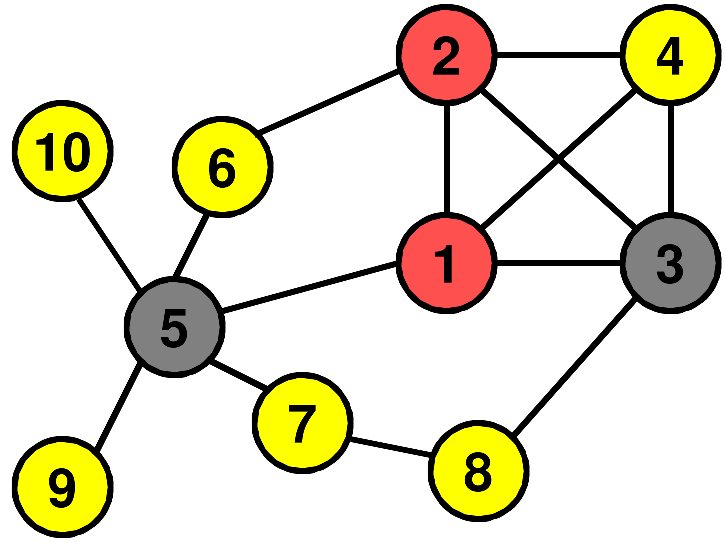

**Round 5 (Begin)**



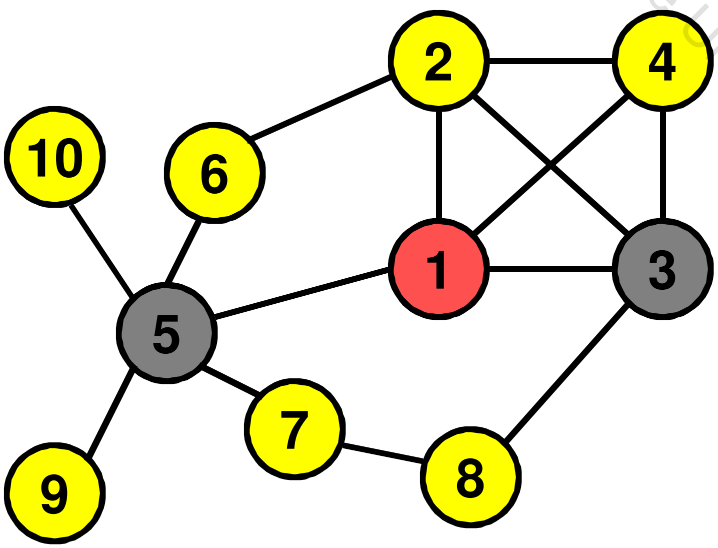

**Round 5 (End):** -



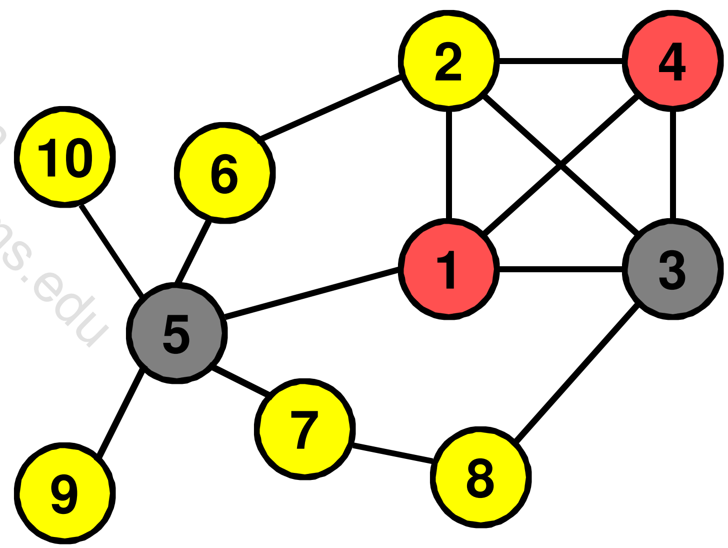

Round 6 (Begin)



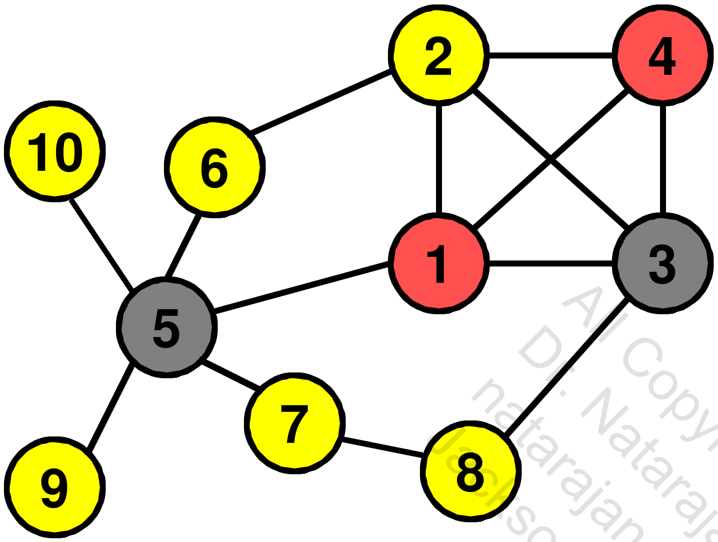

Round 6 (End): 1



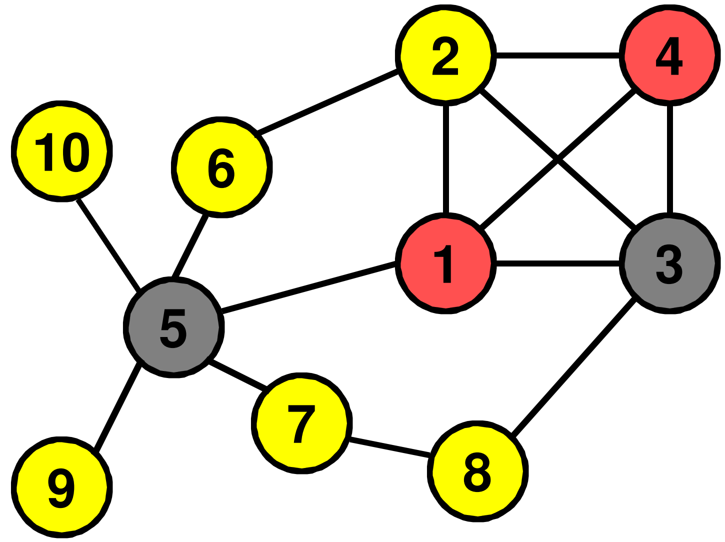

Round 7 (Begin)




Round 7 (End): 4

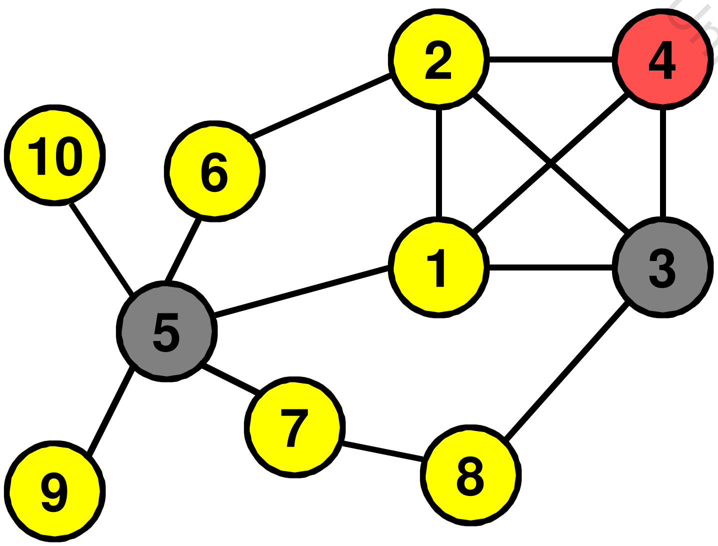



**Round 8 (Begin)**

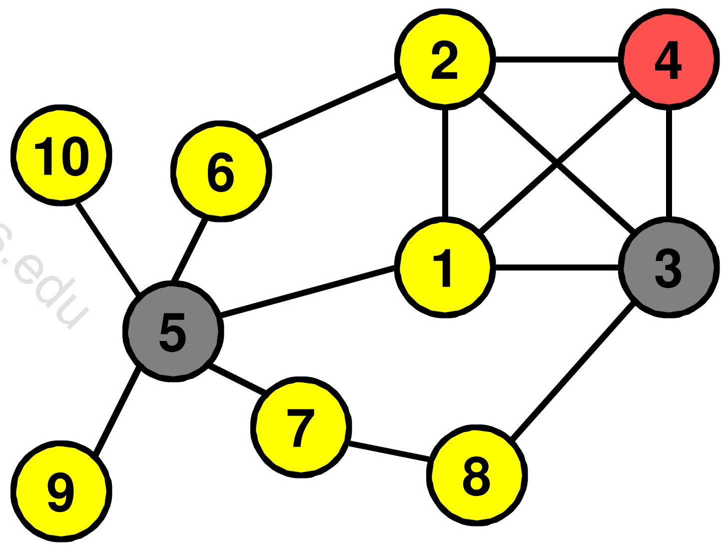



**Round 8 (End): -**

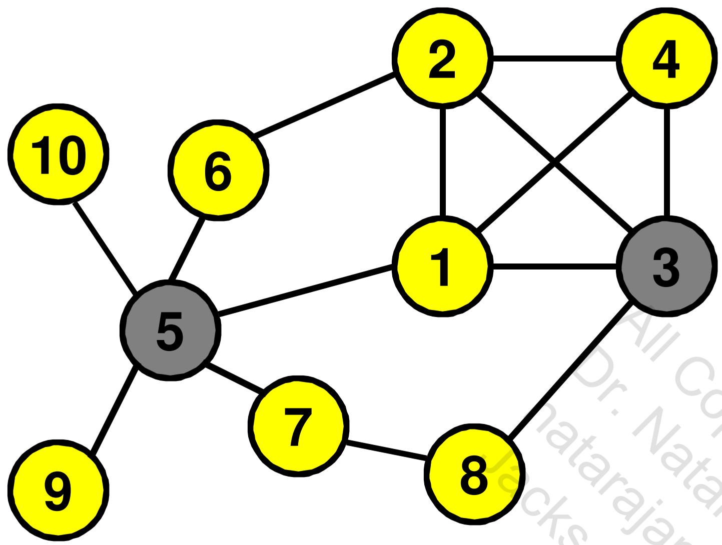




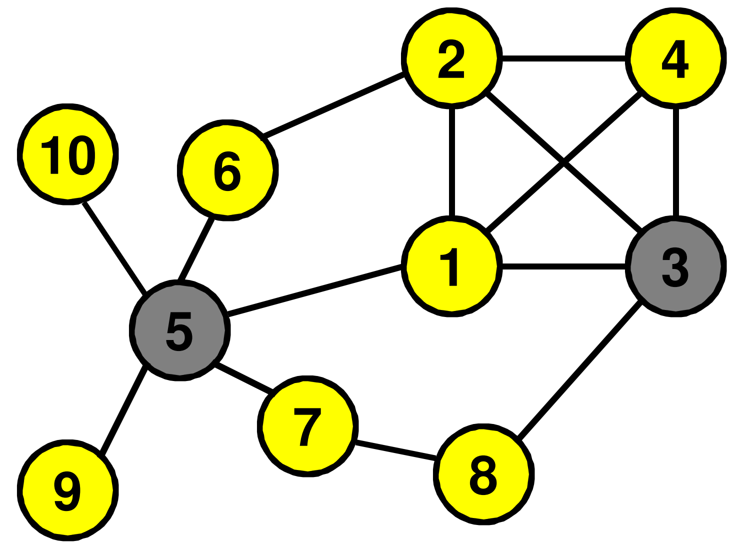

**Round 9 (Begin)**




**Round 9 (End): -**




Round 10 (Begin)




Round 10 (End): -

| Round # | Total # Inf. Nodes |
|---|---|
| 1 | 2 |
| 2 | 2 |
| 3 | 2 |
| 4 | 2 |
| 5 | 2 |
| 6 | 2 |
| 7 | 2 |
| 8 | 2 |
| 9 | 1 |
| 10 | 0 |

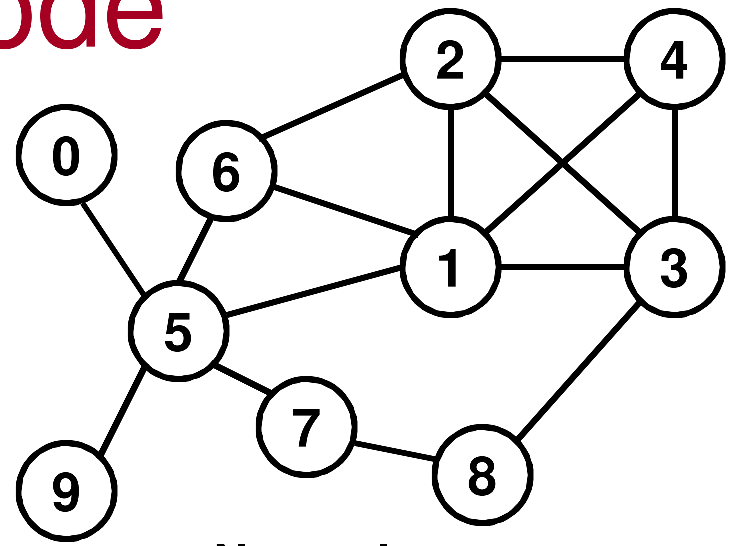
**Inference:** We show that by carefully selecting at least 1 – 1/R0 (i.e., 2 of the 10 nodes), we have made the infection spread to eventually die down.

The two vaccinated nodes 3 and 5 provide "Herd Immunity" to the non-vaccinated Nodes 9, 10 , 7 and 8 (that remain susceptible, but cannot get infected)

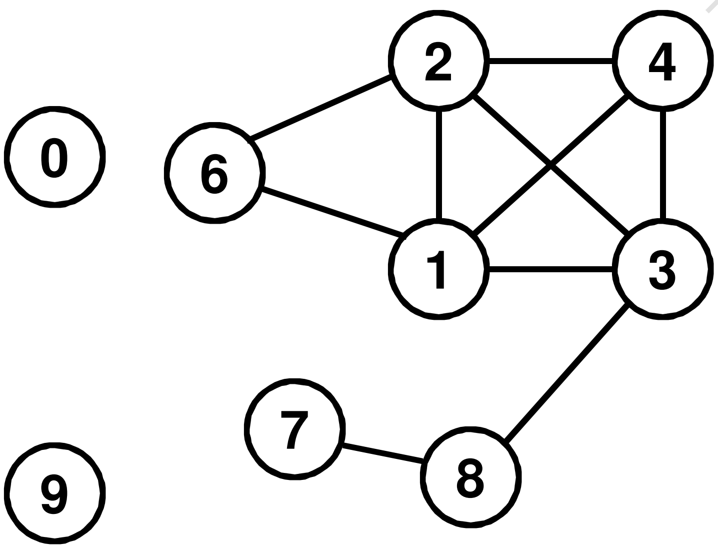
Nodes 3 and 5 are among the top-ranked bridge nodes of this graph and can be effectively identified using the NBNC tuple

# Bridge Node

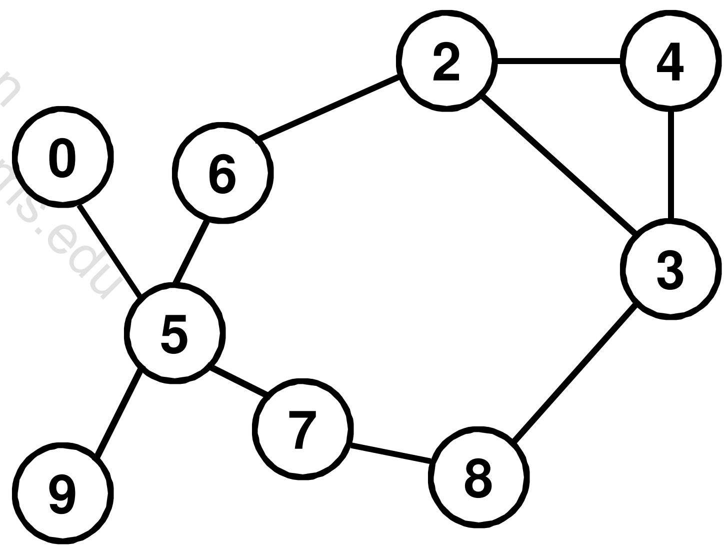
- We define a bridge node to be a node whose neighbor nodes are sparsely connected to each other and are likely to be part of different components/disjoint clusters if the node is removed from the network.




**Network**




**Network after the removal of Node '5'**




**Network after the removal of Node '1'**

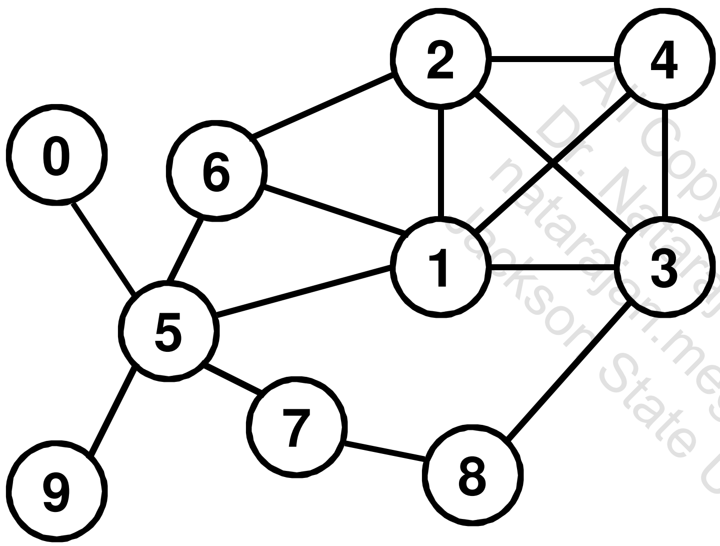
# Applications of Bridge Nodes

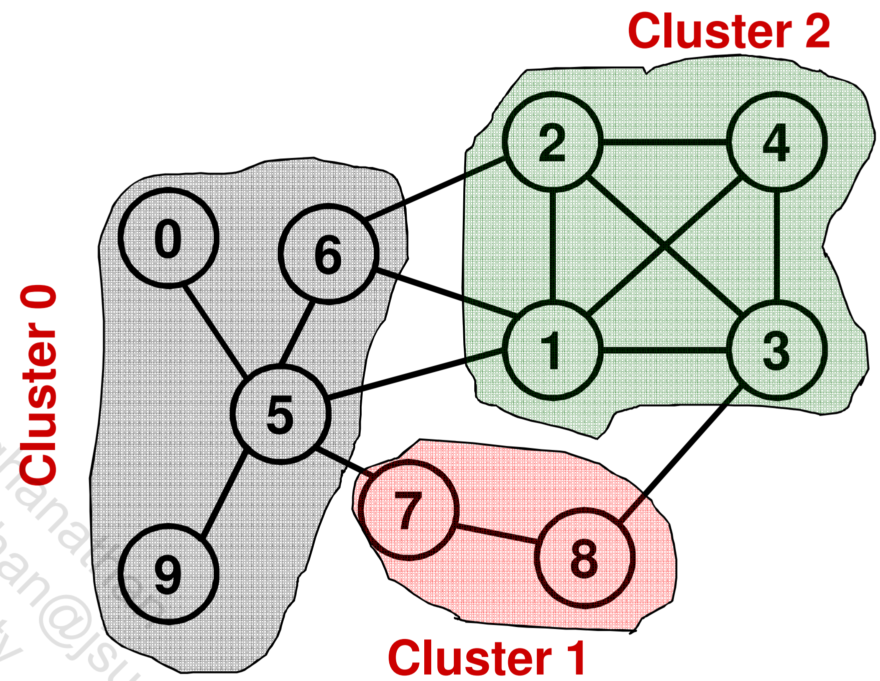
- Due to their likely proximity to several clusters, bridge nodes are the perfect choice for cluster heads in communication networks.
- For information cascade to be successful/complete in social networks, the bridge nodes of a cluster need to first adopt a decision before the internal nodes of the cluster can adopt the decision.
- To prevent an infection spread within a community/cluster, the bridge nodes of the cluster need to be meticulously identified and vaccinated so that the internal nodes could be protected.
- In collaboration networks, bridge nodes could indicate authors who have collaborated with people who have themselves not collaborated with each other. The identification of such bridge nodes in collaboration networks could lead to new research collaborations among individuals of diverse expertise.
- Bridge nodes in social networks could lead to acquaintance (could eventually lead to close friendship) between people who had not known each other until then.

# Topological Positions of Bridge Nodes

- We envision the following three topological positions (one or more of which) could be taken up by a bridge node.
- <u>Bridge "Hub" Node</u>: The links incident on the node are critical for the nodes within its *home* cluster to belong to that cluster.
- <u>Bridge "Border" Node</u>: The links incident on the node are critical to connect the node's home cluster to other *alien* clusters.
- <u>Bridge "between-clusters" Node</u>: The node connects two or more clusters or nodes that would otherwise be not connected or sparsely connected.

# Example: Roles for a Bridge Node



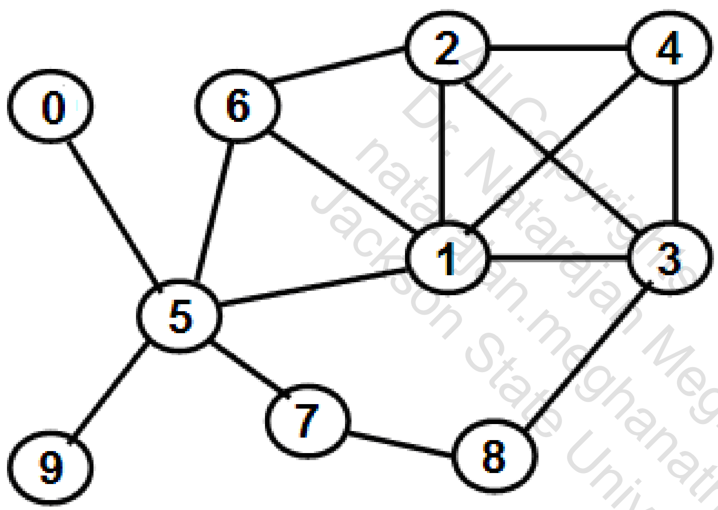



| | Bridge: Hub | Bridge: Border | Bridge: between-clusters |
|---|---|---|---|
| Node 5 | Yes | Yes | Yes, but weak |
| Node 3 | No | Yes | Yes, but weak |
| Nodes 7, 8 | No | Yes | Yes, but weak |
| Node 1 | No | Yes, but weak | No |

# Algebraic Connectivity (AC)

- AC of a network: a measure of the connectivity of the nodes in the network.
- The AC of a graph is computed by conducting spectral analysis of the Laplacian matrix of the graph.
- The Laplacian matrix of a graph is a square symmetric matrix whose diagonal entries correspond to the degree of the vertex.
  - A non-diagonal entry (*i*, *j*) would be -1 if there is an edge between vertices *i* and *j* in the graph or 0 if there is no edge between *i* and
- Spectral analysis of an *n* x *n* Laplacian matrix would return *n* eigenvalues (in the sorted order, starting from 0) and their corresponding eigenvectors.
  - The eigenvalues of the Laplacian matrix would be either 0 or positive.
- The # 0s among the eigenvalues of the Laplacian matrix of a graph would correspond to the number of components in the graph.
- Hence, if the entire graph is connected, there will be only one zero among the eigenvalues of its Laplacian matrix.
- For a connected graph, the smallest non-zero positive eigenvalue of the Laplacian matrix of a graph is a measure of the connectivity of the graph and is referred to as the Algebraic connectivity (AC).

# Laplacian Matrix for an entire Graph






| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 5 | -1 | -1 | -1 | -1 | -1 | 0 | 0 | 0 |
| 2 | 0 | -1 | 4 | -1 | -1 | 0 | -1 | 0 | 0 | 0 |
| 3 | 0 | -1 | -1 | 4 | -1 | 0 | 0 | 0 | -1 | 0 |
| 4 | 0 | -1 | -1 | -1 | 3 | 0 | 0 | 0 | 0 | 0 |
| 5 | -1 | -1 | 0 | 0 | 0 | 5 | -1 | -1 | 0 | -1 |
| 6 | 0 | -1 | -1 | 0 | 0 | -1 | 3 | 0 | 0 | 0 |
| 7 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 2 | -1 | 0 |
| 8 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | -1 | 2 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 1 |

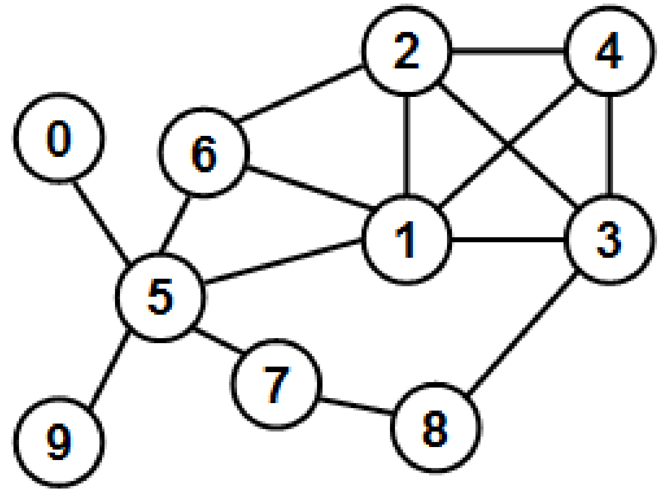
0.0000
0.6536
1.0000
1.0455
2.3065
3.0000
4.6651
5.0000
5.7224
6.6069

**Algebraic Connectivity of the Graph**

**The 10 Eigenvalues of the Laplacian Matrix**

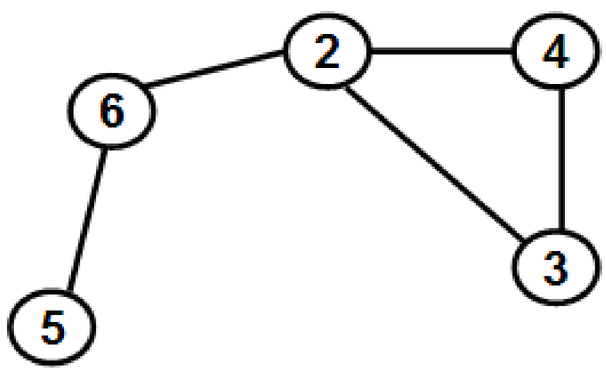
# Neighborhood Graph (NG) of a Node

- The neighborhood graph of a vertex *i* (denoted *NGi*) comprises the neighbors of the vertex as the vertices and the edges connecting the neighbors.
  - Note that the neighborhood graph of a vertex does not include the vertex and the edges to its neighbors.





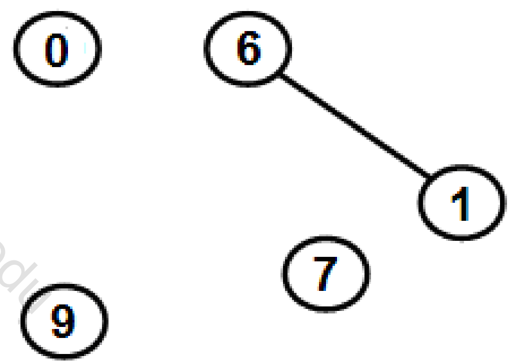

**Spectral Analysis of NG(1)**




| | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| 2 | 3 | -1 | -1 | 0 | -1 |
| 3 | -1 | 2 | -1 | 0 | 0 |
| 4 | -1 | -1 | 2 | 0 | 0 |
| 5 | 0 | 0 | 0 | 1 | -1 |
| 6 | -1 | 0 | 0 | -1 | 2 |

0.0000
0.5188
2.3111
3.0
4.1701

**Spectral Analysis of NG(5)**




| | 0 | 1 | 6 | 7 | 9 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | -1 | 0 | 0 |
| 6 | 0 | -1 | 1 | 0 | 0 |
| 7 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 |

0.0000
0.0000
0.0000
0.0000
2.0000

# Neighborhood-based Bridge Node Centrality (NBNC) Tuple

**NBNC(i) =** $\left( NG_i^{\#comp}, NG_i^{ACR}, | NG_i | \right)$

$NG_i^{\#comp}$ **# components in the neighborhood graph *NGi* of node *i***

$NG_i^{ACR}$ **is the ratio of the algebraic connectivity of *NGi* and # nodes in *NGi***

**| NGi |** **# nodes in *NGi***

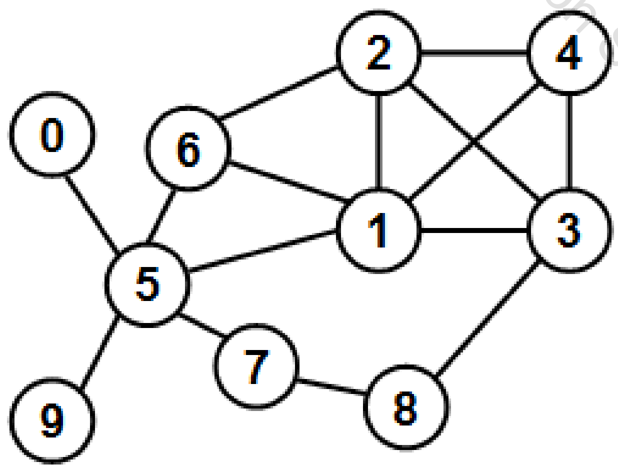
**Note that** $NG_i^{ACR} = 0$ **if** $NG_i^{\#comp} > 1$

**Note that** $NG_i^{ACR} = 1$ **if** $NG_i^{\#comp} = 1$ **and | NGi | = 1**

**Publication:** **Meghanathan, N. Neighborhood-based bridge node centrality tuple for complex network analysis.**
***Appl Netw Sci* 6, 47 (2021).**
**https://doi.org/10.1007/s41109-021-00388-1**

# NBNC Tuple: Example Graph

**NBNC(i) =** $\left(NG_i^{\#comp}, NG_i^{ACR}, |NG_i|\right)$




| Node ID | *NBNC* Tuple |
|---|---|
| 0 | (1, 1.0, 1) |
| 1 | (1, 0.1038, 5) |
| 2 | (1, 0.25, 4) |
| 3 | (2, 0.0, 4) |
| 4 | (1, 1.0, 3) |
| 5 | (4, 0.0, 5) |
| 6 | (1, 0.3333, 3) |
| 7 | (2, 0.0, 2) |
| 8 | (2, 0.0, 2) |
| 9 | (1, 1.0, 1) |

# NBNC Tuple: Ranking Rules

- Vertex v is ranked above vertex u

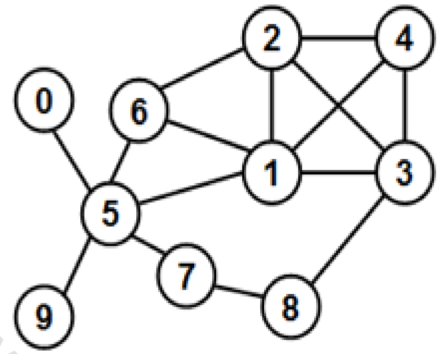
NBNC(v) > NBNC(u) per the following rules

$$(1)\ \text{If}\ (NG_v^{\#comp} > NG_u^{\#comp})\ \text{then,}$$
$$\text{NBNC}(v) > \text{NBNC}(u)$$

$$(2)\ \text{If}\ (NG_v^{\#comp} = NG_u^{\#comp})$$
$$\text{and}\ (NG_v^{ACR} < NG_u^{ACR})\ \text{then,}$$
$$\text{NBNC}(v) > \text{NBNC}(u)$$

$$(3)\ \text{If}\ (NG_v^{\#comp} = NG_u^{\#comp})$$
$$\text{and}\ (NG_v^{ACR} = NG_u^{ACR})$$
$$\text{and}\ (|NG_v| > |NG_u|)\ \text{then,}$$
$$\text{NBNC}(v) > \text{NBNC}(u)$$

# NBNC Tuple-based Ranking for the Example Graph




| Node ID | *NBNC* Tuple |
|---|---|
| 0 | (1, 1.0, 1) |
| 1 | (1, 0.1038, 5) |
| 2 | (1, 0.25, 4) |
| 3 | (2, 0.0, 4) |
| 4 | (1, 1.0, 3) |
| 5 | (4, 0.0, 5) |
| 6 | (1, 0.3333, 3) |
| 7 | (2, 0.0, 2) |
| 8 | (2, 0.0, 2) |
| 9 | (1, 1.0, 1) |

## Our initial analysis

| | Bridge: Hub | Bridge: Border | Bridge: between-clusters |
|---|---|---|---|
| Node 5 | Yes | Yes | Yes, but weak |
| Node 3 | No | Yes | Yes, but weak |
| Nodes 7, 8 | No | Yes | Yes, but weak |
| Node 1 | No | Yes, but weak | No |



| Tentative Rank | Node ID | NBNC Tuple | | Final Rank | Node ID | NBNC Tuple |
|---|---|---|---|---|---|---|
| 0 | 5 | (4, 0.0, 5) | | 0 | 5 | (4, 0.0, 5) |
| 1 | 3 | (2, 0.0, 4) | | 1 | 3 | (2, 0.0, 4) |
| 2 | 8 | (2, 0.0, 2) | → | 2.5 | 8 | (2, 0.0, 2) |
| 3 | 7 | (2, 0.0, 2) | → | 2.5 | 7 | (2, 0.0, 2) |
| 4 | 1 | (1, 0.1038, 5) | | 4 | 1 | (1, 0.1038, 5) |
| 5 | 2 | (1, 0.25, 4) | | 5 | 2 | (1, 0.25, 4) |
| 6 | 6 | (1, 0.3333, 3) | | 6 | 6 | (1, 0.3333, 3) |
| 7 | 4 | (1, 1.0, 3) | | 7 | 4 | (1, 1.0, 3) |
| 8 | 9 | (1, 1.0, 1) | → | 8.5 | 9 | (1, 1.0, 1) |
| 9 | 0 | (1, 1.0, 1) | → | 8.5 | 0 | (1, 1.0, 1) |