JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **186**, 414–441 (1994)

Relation and Topology—Neighbor Element Structure and Convergence in Relations

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Received December 21, 1992

INTRODUCTION

In order to investigate the mutual relationships in general topology, fuzzy topology [1], topological molecular lattice [2], and latticized topology [3], the concepts of neighborhood structure and convergence are explored. Let R be a binary relation between the sets X and L. We studied several topological properties on X and L from the triad (X, L, R). The concepts and axiomatizations of *R-neighborhood structure and R-distance structure*, and their *risings on L* are investigated. The concept of *convergence in relation* is presented. Special emphasis is placed on studying the case where R is either a neighborhood relation or a distant relation. The neighborhood relation can be used to describe the traditional neighborhood and the fuzzy neighborhood, while the distance relation can be used to describe the distance field [2]. The neighborhood and distance relations can be mutually determined from each other. Furthermore, the risings of the neighborhood element structure and the distance element structure are useful tools for bridging the concepts of hypertopology and fuzzy hypertopology.

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0022-247X/94 $6.00

1. Adjoint Relation

Let X and L be sets and R be a binary relation between the set X and the set L. The elements in X are denoted by x, y, z etc., and the elements in L are denoted by α, β, γ etc. The subsets of X are denoted by A, B, C etc., and the subsets of L are denoted by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$. The negation and the inverse of R are denoted by R' and R^{-1}, respectively, where

$$(x, \alpha) \in R' \Leftrightarrow (x, \alpha) \notin R \tag{1-1}$$

for each pair $(x, \alpha) \in X \times L$, and

$$(\alpha, x) \in R^{-1} \Leftrightarrow (x, \alpha) \in R \tag{1-2}$$

for each pair $(\alpha, x) \in L \times X$. Suppose that $x \in X$ and $A \in \mathcal{P}(X)$. Let

$$R[x] = \{\alpha \in L | xR\alpha\}, \tag{1-3}$$

$$R[A] = \cup\{R[x] | x \in A\}. \tag{1-4}$$

Suppose that $\alpha \in L$ and $\mathcal{A} \in \mathcal{P}(L)$. Let

$$R^{-1}[\alpha] = \{x \in X | xR\alpha\}, \tag{1-5}$$

$$R^{-1}[\mathcal{A}] = \cup\{R^{-1}[\alpha] | \alpha \in \mathcal{A}\}. \tag{1-6}$$

Definition 1.1. Suppose that $\alpha, \beta \in L$. Let

$$\alpha R^* \beta \Leftrightarrow R^{-1}[\alpha] \subseteq R^{-1}[\beta]. \tag{1-7}$$

Suppose that $x, y \in X$. Let

$$xR_* y \Leftrightarrow R[y] \subseteq R[x]. \tag{1-8}$$

R^* and R_* are called the right adjoint relation and the left adjoint relation of R, respectively.

Proposition 1.1. *R^* and R_* are reflexive and transitive relations on L and on X, respectively.*

Proposition 1.2. *Suppose that $x, y, \in X$, and that $\alpha, \beta \in L$. Then*

$$xR\alpha R^* \beta \Rightarrow xR\beta, \tag{1-9}$$

$$xR_* yR\alpha \Rightarrow xR\alpha. \tag{1-10}$$

PROPOSITION 1.3. *The following equalities hold.*

$$(R')^* = (R^*)^{-1} = (R^{-1})_* \tag{1-11}$$

$$(R')_* = (R_*)^{-1} = (R^{-1})^*. \tag{1-12}$$

DEFINITION 1.2. A subset $\mathscr{A}$ of L is said to be o-adequate about R iff

$$\forall x \in X(\exists \alpha \in \mathscr{A}(xR\alpha)) \tag{1-13}$$

holds. A subset of L is said to be c-adequate about R iff it is o-adequate about R'. When L is o- (or c-)adequate about R, we say that R is o- (or c-)adequate.

DEFINITION 1.3. A subset $\mathscr{A}$ of L is said to be arbitrary (or finite) i-regular about R iff, for arbitrary (or finite) index set T,

$$\forall t \in T(xR\alpha_t) \Rightarrow \exists \alpha \in \mathscr{A}((xR\alpha) \quad \text{and} \quad (\forall t \in T(\alpha R^* \alpha_t))) \tag{1-14}$$

holds for each $x \in X$ and each $\alpha_t \in \mathscr{A}(t \in T)$. A subset $\mathscr{A}$ of L is said to be arbitrary (or finite) u-regular about R iff it is arbitrary (or finite) i-regular about R'. When L is arbitrary (or finite) i- (or u-)regular about R, we say that R is arbitrary (or finite) i- (or u-)regular.

DEFINITION 1.4. A subset $\mathscr{A}$ of L is said to be o-maximal about R, iff for each $\alpha \in L$, if

$$\forall x \in X(xR\alpha \Rightarrow \exists \gamma \in \mathscr{A}(xR\gamma R^*\alpha) \tag{1-15}$$

holds, then $\alpha \in L$. A subset $\mathscr{A}$ of L is said to be c-maximal about R if it is o-maximal about R'.

DEFINITION 1.5. Two subsets $\mathscr{A}$ and $\mathscr{B}$ of L are said to be o-equivalent about R, denoted by $\mathscr{A}\tilde{R}\mathscr{B}$, iff

$$\forall \alpha \in \mathscr{A} \forall x \in X(xR\alpha \Rightarrow \exists \beta \in \mathscr{B}(xR\beta R^*\alpha)), \tag{1-16}$$

$$\forall \beta \in \mathscr{B} \forall x \in X(xR\beta \Rightarrow \exists \alpha \in \mathscr{A}(xR\alpha R^*\beta)) \tag{1-17}$$

hold. Two subsets $\mathscr{A}$ and $\mathscr{B}$ of L are said to be c-equivalent about R, denoted by $\mathscr{A}\bar{R}\mathscr{B}$, iff they are o-equivalent about R'.

Suppose that $\mathscr{A} \in \mathscr{P}(L)$. Let

$$\mathscr{A}^{\tilde{R}} = \cup\{\mathscr{B} \in \mathscr{P}(L) | \mathscr{B}\tilde{R}\mathscr{A}\}, \tag{1-18}$$

$$\mathscr{A}^{\bar{R}} = \cup\{\mathscr{B} \in \mathscr{P}(L) | \mathscr{B}\bar{R}\mathscr{A}\}. \tag{1-19}$$

PROPOSITION 1.4. *Suppose that $\mathcal{A} \in \mathcal{P}(L)$. Then*

$$\mathcal{A}^{\mathring{R}}\, \mathring{R}\, \mathcal{A}, \tag{1-20}$$

$$\mathcal{A}^{\bar{R}}\, \bar{R}\, \mathcal{A}. \tag{1-21}$$

THEOREM 1.1. *Suppose that $\mathcal{A} \in \mathcal{P}(L)$. Then $\mathcal{A}^{\mathring{R}} = \mathcal{A}$ if and only if $\mathcal{A}$ is o-maximal about R, and $\mathcal{A}^{\bar{R}} = \mathcal{A}$ if and only if $\mathcal{A}$ is c-maximal about R.*

Proof. We prove the first part only. If $\alpha \in L$ satisfies the condition (1.15), then $(\mathcal{A} \cup \{\alpha\})\mathring{R}\mathcal{A}$, so that $\mathcal{A} \subseteq \mathcal{A} \cup \{\alpha\} \subseteq \mathcal{A}^{\mathring{R}}$. Now, if $\mathcal{A}^{\mathring{R}} = \mathcal{A}$, then $\alpha \in \mathcal{A}$. Hence $\mathcal{A}$ is o-maximal about R. Conversely, if $\mathcal{A}$ is o-maximal about R, then $\mathcal{B}\mathring{R}\mathcal{A}$ implies $\mathcal{B} \subseteq \mathcal{A}$. Hence $\mathcal{A}^{\mathring{R}}$.

PROPOSITION 1.5. *Suppose that $\mathcal{A}\mathring{R}\mathcal{B}$. Then $\mathcal{A}$ is o-adequate about R if and only if $\mathcal{B}$ is o-adequate about R, and $\mathcal{A}$ is arbitrary (or finite) i-regular about R if and only if $\mathcal{B}$ is arbitrary (or finite) i-regular about R.*

Suppose that $\mathcal{A}\bar{R}\mathcal{B}$. Then $\mathcal{A}$ is c-adequate about R if and only if $\mathcal{B}$ is c-adequate about R, and $\mathcal{A}$ is arbitrary (or finite) u-regular about R if and only if $\mathcal{B}$ is arbitrary (or finite) u-regular about R.

PROPOSITION 1.6. *We have*

$$\mathcal{A}\mathring{R}\mathcal{B} \Leftrightarrow \mathcal{A}^{\mathring{R}} = \mathcal{B}^{\mathring{R}} \tag{1-22}$$

$$\mathcal{A}\bar{R}\mathcal{B} \Leftrightarrow \mathcal{A}^{\bar{R}} = \mathcal{B}^{\bar{R}}. \tag{1-23}$$

2. NEIGHBOR ELEMENT SYSTEM AND NEIGHBOR ELEMENT STRUCTURE

DEFINITION 2.1. Suppose that R is o-adequate and finite i-regular. Let

$${}^{j}(X, L, R) = \{\mathcal{J} \in \mathcal{P}(L) | \mathcal{J} \text{ satisfies j.1–j.3}\}, \tag{2-1}$$

where

j.1. $\mathcal{J}$ is o-adequate about R,

j.2. $\mathcal{J}$ is finite i-regular about R,

j.3. $\mathcal{J}$ is o-maximal about R.

$\mathcal{J} \in {}^{j}(X, L, R)$ is called R-neighbor element system on X. We will denote ${}^{j}(X, L, R)$ by ${}^{j}X$ if there is no confusion.

DEFINITION 2.2. Suppose that R is o-adequate and finite i-regular. Let

$${}^{n}(X, L, R) = \{\mathcal{N} | \mathcal{N}: X \to \mathcal{P}(L) \backslash \{\emptyset\} \text{ satisfies n.1–n.4}\}, \tag{2-2}$$

where

n.1. $\alpha \in \mathcal{N}(x) \Rightarrow xR\alpha$

n.2. $\alpha, \beta \in \mathcal{N}(x) \Rightarrow \exists\gamma \in \mathcal{N}(x)(\gamma R^*\alpha$ and $\gamma R^*\beta)$

n.3. $\alpha \in \mathcal{N}(x)$ and $\alpha R^*\beta \Rightarrow \beta \in \mathcal{N}(x)$

n.4. $\alpha \in \mathcal{N}(x) \Rightarrow \exists\delta \in \mathcal{N}(x)(\delta R^*\alpha$ and $(yR\tau \Rightarrow \delta \in \mathcal{N}(y))$

$\mathcal{N} \in {}^n(X, L, R)$ is called R-neighbor element structure on X. We will denote ${}^n(X, L, R)$ by ${}^nX$ if there is no confusion.

THEOREM 2.1. *Suppose that R is o-adequate and finite i-regular. Then there are a pair of mutually inverse maps between* ${}^iX$ *and* ${}^nX$

$$f_n^j : {}^jX \to {}^nX$$
$$\mathcal{J} \mapsto f_n^j(\mathcal{J})$$

where, for each $x \in X$,

$$f_n^j(\mathcal{J})(x) = \{\alpha \in L | \exists\delta \in \mathcal{J}(xR\delta R^*\alpha)\}$$

$$f_j^n : {}^nX \to {}^jX \tag{2-3}$$
$$\mathcal{N} \mapsto f_j^n(\mathcal{N}),$$

where

$$f_j^n(\mathcal{N}) = \{\delta \in L | \forall x \in X(xR\delta \Rightarrow \delta \in \mathcal{N}(x))\}. \tag{2-4}$$

Proof. Because R is o-adequate and finite i-regular, ${}^jX$ and ${}^nX$ are all nonempty. First we will extend the field of definition of the map f_n^j such that $A = \{\mathcal{J} \in \mathcal{P}(L) | \mathcal{J}$ satisfying j.1 and j.2$\}$, according to the expression (2-3), and the extended map is still denoted by f_n^j. Then the theorem can be proved according to the following six steps.

(1) f_n^j is a map from A to ${}^nX$.

(2) If $\mathcal{A}, \mathcal{B} \in A$, then $\mathcal{A}\tilde{R}\mathcal{B} \Leftrightarrow f_n^j(\mathcal{A}) = f_n^j(\mathcal{B})$.

(3) f_j^n is a map from ${}^nX$ to A.

(4) If $\mathcal{N} \in {}^nX$, then $f_n^j(f_j^n(\mathcal{N})) = \mathcal{N}$.

(5) If $\mathcal{N} \in {}^nX$, then $f_j^n(\mathcal{N})$ is o-maximal about R.

(6) If $\mathcal{J} \in {}^jX$, then $f_j^n(f_j^n(\mathcal{J})) = \mathcal{J}$.

DEFINITION 2.3. Suppose that R is c-adequate and finite u-regular. Let

$${}^e(X, L, R) = \{\mathcal{E} \in \mathcal{P}(L) | \mathcal{E} \text{ satisfies e.1–e.3}\}, \tag{2-5}$$

where

e.1. $\mathscr{E}$ is c-adequate about R,

e.2. $\mathscr{E}$ is finite u-regular about R,

e.3. $\mathscr{E}$ is c-maximal about R.

$\mathscr{E} \in {}^e(X, L, R)$ is called R-distant element system on X. We will denote ${}^e(X, L, R)$ by ${}^eX$ if there is no confusion.

DEFINITION 2.4. Suppose that R is c-adequate and finite u-regular. Let

$$^q(X, L, R) = \{\mathscr{Q} | \mathscr{Q} : X \to \mathscr{P}(L)\backslash\{\varnothing\} \text{ satisfies q.1–q.4}\}, \tag{2-6}$$

where

q.1. $\alpha \in \mathscr{Q}(x) \Rightarrow xR'\alpha$

q.2. $\alpha, \beta \in \mathscr{Q}(x) \Rightarrow \exists\gamma \in \mathscr{Q}(x)(\alpha R^*\gamma \text{ and } \beta R^*\gamma)$

q.3. $\alpha \in \mathscr{Q}(x)$ and $\beta R^*\alpha \Rightarrow \beta \in \mathscr{Q}(x)$

q.4. $\alpha \in \mathscr{Q}(x) \Rightarrow \exists\delta \in \mathscr{Q}(x)(\alpha R^*\delta \text{ and } (yR'\delta \to \delta \in \mathscr{Q}(y)).$

$\mathscr{Q} \in {}^q(X, L, R)$ is called R-distant element structure of X. We will denote ${}^q(X, L, R)$ by ${}^qX$ if there is no confusion.

THEOREM 2.2. *Suppose that R is c-adequate and finite u-regular. Then there a pair of mutually inverse maps between ${}^eX$ and ${}^qX$*

$$f_q^e : {}^eX \to {}^qX$$
$$\mathscr{E} \mapsto f_q^e(\mathscr{E}),$$

where, for each $x \in X$,

$$f_q^e(\mathscr{E})(x) = \{\alpha \in L | \exists\delta \in \mathscr{E}(xR'\delta \text{ and } \alpha R^*\delta)\}$$

$$f_e^q : {}^qX \to {}^eX \tag{2-7}$$
$$\mathscr{Q} \mapsto f_e^q(\mathscr{Q}),$$

where

$$f_e^q(\mathscr{Q}) = \{\delta \in L | \forall x \in X(xR'\delta \Rightarrow \delta \in \mathscr{Q}(x))\}. \tag{2-8}$$

3. OPEN ELEMENT SYSTEM AND CLOSED ELEMENT SYSTEM

In this section, we suppose that (L, R^*) becomes a complete lattice. We will discuss the relations between the neighbor element system and the open element system, and between the distant element system and the closed element system.

When $\alpha R^*\beta$, we say that α is less than or equal to β. The concepts of upper bound, supremum, lower bound, and infimum, conform, with usual rules. The operative symbols of the supremum and the infimum about R^* are denoted by $\bigvee_{R^*}$ and $\bigwedge_{R^*}$, respectively. The largest element and the least element in (L, R^*) are denoted by 1_{R^*} and 0_{R^*}, respectively.

DEFINITION 3.1. Let

$$^{g}(X, L, R) = \{\mathcal{G} \in \mathcal{P}(L) | \mathcal{G} \text{ satisfies o.1–o.3}\}, \tag{3-1}$$

where

o.1. $0_{R^*}, 1_{R^*} \in \mathcal{G}$,

o.2. $\delta_1, \delta_2 \in \mathcal{G} \Rightarrow \delta_1 \bigwedge_{R^*} \delta_2 \in \mathcal{G}$,

o.3. $\delta_t \in \mathcal{G}, t \in T \rightarrow \bigvee_{R^*}(\delta | t \in T) \in \mathcal{G}$,

where T is an arbitrary index set. $\mathcal{G} \in {}^{g}(X, L, R)$ is called R-open element system on X. We will denote $^{g}(X, L. R)$ by ^{g}X if there is no confusion.

$\mathcal{S} \in \mathcal{P}(L)$ satisfying condition o.2 is said to be closed under the formation of finite $\bigwedge_{R^*}$. $\mathcal{S} \in \mathcal{P}(L)$ satisfying condition o.3 is said to be closed under the formation of arbitrary $\bigvee_{R^*}$.

DEFINITION 3.2. Let

$$^{f}(X, L, R) = \{\mathcal{F} \in \mathcal{P}(L) | \mathcal{F} \text{ satisfies c.1–c.3}\}, \tag{3-2}$$

where

c.1. $O_{R^*}, 1_{R^*} \in \mathcal{F}$,

c.2. $\delta_1, \delta_2 \in \mathcal{F} \Rightarrow \delta_1 \bigvee_{R^*} \delta_2 \in \mathcal{F}$,

c.3. $\delta_t \in \mathcal{F}, t \in T \Rightarrow \bigwedge_{R^*}(\delta | t \in T) \in \mathcal{F}$,

where T is an arbitrary index set. $\mathcal{F} \in {}^{f}(X, L, R)$ is called R-closed element system on X. We will denote $^{f}(X, L, R)$ by ^{f}X if there is no confusion.

$\mathcal{S} \in \mathcal{P}(L)$ satisfying condition c.2 is said to be closed under the formation of finite $\bigvee_{R^*}$. $\mathcal{S} \in \mathcal{P}(L)$ satisfying condition c.3 is said to be closed and under the formation of arbitrary $\bigwedge_{R^*}$.

PROPOSITION 3.1. *R is o-adequate if and only if*

$$R^{-1}[1_{R^*}] = X. \tag{3-3}$$

R is c-adequate if and only if

$$R^{-1}[0_{R^*}] = \varnothing, \tag{3-4}$$

PROPOSITION 3.2. *If $\mathcal{S}$ is o-adequate and o-maximal about R, then $1_{R^*} \in \mathcal{S}$. Conversely, if $1_{R^*} \in \mathcal{S}$, then $\mathcal{S}$ is o-adequate about R under the supposition that R is o-adequate.*

PROPOSITION 3.3. *If $\mathcal{S}$ is c-adequate and c-maximal about R, then $0_{R^*} \in \mathcal{S}$. Conversely, if $0_{R^*} \in \mathcal{S}$, then $\mathcal{S}$ is c-adequate about R under the supposition that R is c-adequate.*

PROPOSITION 3.4. *Suppose that T is an arbitrary (or finite) index set. Then $\mathcal{S}$ is arbitrary (or finite) i-regular if and only if*

$$\forall t \in T(xR\delta_t) \Rightarrow \exists \delta \in \mathcal{S}(xR\delta R^*(\wedge_{R^*}(\delta_t | t \in T))), \tag{3-5}$$

where $x \in X$, $\delta_t \in \mathcal{S}$, $(t \in T)$.

PROPOSITION 3.5 *Suppose that T is an arbitrary (or finite) index set. Then $\mathcal{S}$ is arbitrary (or finite) u-regular if and only if*

$$\forall t \in T(xR'\delta_t) \Rightarrow \exists \delta \in \mathcal{S}(xR'\delta \text{ and } (\vee_{R^*}(\delta_t | t \in T))R^*\delta), \tag{3-6}$$

where $x \in X$, $\delta_t \in \mathcal{S}$, $(t \in T)$.

PROPOSITION 3.6. *Suppose that T is an arbitrary (or finite) index set. Then R is arbitrary (or finite) i-regular if and only if*

$$\forall t \in T(xR\delta_t) \rightarrow xR(\wedge_{R^*}(\delta_t | t \in T)) \tag{3-7}$$

holds. Also, the formula (3-7) *is equivalent to*

$$R^{-1}[\wedge_{R^*}(\delta_t | t \in T)] = \cap\{R^{-1}[\delta_t] | t \in T\}, \tag{3-8}$$

where $x \in X$, $\delta_t \in \mathcal{S}$, $(t \in T)$.

PROPOSITION 3.7. *Suppose that T is an arbitrary (or finite) index set. Then R is arbitrary (or finite) u-regular if and only if*

$$xR(\vee_{R^*}(\delta_t | t \in T)) \Rightarrow \exists t \in T(xR\delta_t) \tag{3-9}$$

holds. Also, the formula (3-9) *is equivalent to*

$$R^{-1}[\vee_{R^*}(\delta_t | t \in T)] = \cup\{R^{-1}[\delta_t] | t \in T\}, \tag{3-10}$$

where $x \in X$, $\delta_t \in \mathcal{S}$, $(t \in T)$.

PROPOSITION 3.8. *Suppose that R is arbitrary (or finite) i-regular. If $\mathcal{S}$ is closed under the formation of arbitrary (or finite) $\wedge_{R^*}$, then $\mathcal{S}$ is arbitrary (or finite) i-regular about R.*

PROPOSITION 3.9. *Suppose that R is arbitrary (or finite) u-regular. If $\mathcal{S}$ is closed under the formation of arbitrary (or finite) $\bigvee_{R^*}$, then $\mathcal{S}$ is arbitrary (or finite) u-regular about R.*

PROPOSITION 3.10. *If $\mathcal{S}$ is closed under the formation of arbitrary $\bigvee_{R^*}$, and it satisfies*

$$\forall \alpha \in L(R^{-1}[\alpha] = \varnothing \Rightarrow \alpha \in \mathcal{S}), \tag{3-11}$$

then $\mathcal{S}$ is o-maximal about R. Conversely, suppose that R is arbitrary (or finite) u-regular. If $\mathcal{S}$ is o-maximal about R, then $\mathcal{S}$ is closed under the formation of arbitrary (or finite) $\bigvee_{R^*}$, and it satisfies the conditions (3-11).

PROPOSITION 3.1.1. *If $\mathcal{S}$ is closed under the formation of arbitrary $\bigwedge_{R^*}$, and it satisfies*

$$\forall \alpha \in L(R^{-1}[\alpha] = X \Rightarrow \alpha \in \mathcal{S}), \tag{3-12}$$

then $\mathcal{S}$ is c-maximal about R. Conversely, suppose that R is arbitrary (or finite) i-regular. If $\mathcal{S}$ is c-maximal about R, then it is closed under the formation of arbitrary (or finite) $\bigwedge_{R^*}$ and satisfies the condition (3-12).

PROPOSITION 3.12. *If $\mathcal{S}$ is o-maximal about R and arbitrary (or finite) i-regular about R, then $\mathcal{S}$ is closed under the formation of arbitrary (or finite) $\bigwedge_{R^*}$.*

PROPOSITION 3.13. *If $\mathcal{S}$ is c-maximal about R and arbitrary (or finite) u-regular about R, then $\mathcal{S}$ is closed under the formation of arbitrary (or finite) $\bigvee_{R^*}$.*

PROPOSITION 3.14. *If there exists $\alpha \in L$ such that $R^{-1}[\alpha] = \varnothing$, then $\alpha = 0_{R^*}$, if there exists $\alpha \in L$ such that $R^{-1}[\alpha] = X$, then $\alpha = 1_{R^*}$.*

DEFINITION 3.3. Suppose that R is o-adequate and c-adequate. Then R is called neighbor relation iff R is both finite i-regular and arbitrary u-regular. R is called distant relation iff R is both finite u-regular and arbitrary i-regular.

By the above propositions, we can obtain the following theorems.

THEOREM 3.1. *Suppose that R is a neighbor relation. Then*

$$^{j}(X, L, R) = {}^{g}(X, L, R). \tag{3-13}$$

THEOREM 3.2. *Suppose that R is a distant relation. Then*

$$^{c}(X, L, R) = {}^{j}(X, L, R). \tag{3-14}$$

4. The Rising of Neighbor Element Structure and Distant Element Structure

In this section, we suppose that (L, R^*) represents a complete lattice, and that R^* is anti-symmetric. We will discuss the rising of the neighbor element structure and the distant element structure onto the complete lattice L.

PROPOSITION 4.1. *Suppose that R is o-adequate and arbitrary i-regular. For every $x \in X$, let*

$$\langle x \rangle = \wedge_{R^*}(\alpha \in L | xR\alpha). \tag{4-1}$$

We have

$$xR\langle x \rangle \tag{4-2}$$

$$R[x] = R^*[\langle x \rangle] \tag{4-3}$$

$$xR\alpha \Leftrightarrow \langle x \rangle R^* \alpha \tag{4-4}$$

$$xR\langle y \rangle \Leftrightarrow xR{*}y \tag{4-5}$$

hold, where $x, y \in X$ and $\alpha \in L$.

Proof. Because R is o-adequate, $\{\alpha \in L | xR\alpha\}$ is nonempty. By (3-7) we know that (4-2) holds.

PROPOSITION 4.2. *Suppose that R is o-adequate and arbitrary i-regular. Then $R{*}$ is anti-symmetric if and only if one of the conditions*

$$R[x] = R[y] \Rightarrow x = y \tag{4-6}$$

$$xR\langle y \rangle \quad \text{and} \quad yR\langle x \rangle \Rightarrow x = y \tag{4-7}$$

$$\langle x \rangle = \langle y \rangle \Rightarrow x = y, \tag{4-8}$$

where $x, y \in X$, holds.

This proposition shows that, under the suppositions that $R{*}$ is anti-symmetric, and that R is o-adequate and arbitrary i-regular, the map

$$i : x \mapsto \langle x \rangle \tag{4-9}$$

is an injection from X to L. Thus we regard x and $\langle x \rangle$ as the same. In this case, X can be regarded as a subset of L, and we have $R^*|_{X \times L} = R$ and $R|_{X \times X} = R{*}$.

PROPOSITION 4.3. *Suppose that R is o-adequate and arbitrary i-regular. Suppose $\alpha \in L$. When $R^{-1}[\alpha] \neq \emptyset$, we have*

$$\alpha = \bigvee_{R^*}(\langle x\rangle | x \in X,\ xR\alpha). \tag{4-10}$$

When R is arbitrary u-regular, we have similar results.

PROPOSITION 4.4. *Suppose that R is c-adequate and arbitrary u-regular. For every $x \in X$, let*

$$<x> = \bigvee_{R^*}(\alpha \in L | xR'\alpha). \tag{4-11}$$

We have

$$xR'<x> \tag{4-12}$$

$$R'[x] = (R^*)^{-1}[<x>] \tag{4-13}$$

$$xR'\alpha \Leftrightarrow \alpha R^*<x> \tag{4-14}$$

$$xR'<y> \Leftrightarrow yR*x \tag{4-15}$$

hold, where $x, y \in X$ and $\alpha \in L$.

PROPOSITION 4.5. *Suppose that R is c-adequate and arbitrary u-regular. Then $R*$ is anti-symmetric if and only if one of the conditions*

$$R[x] = R[y] \rightarrow x = y \tag{4-16}$$

$$xR'<y> \& \ yR<x> \rightarrow x = y \tag{4-17}$$

$$<x> = <y> \rightarrow x = y, \tag{4-18}$$

Where $x, y \in X$, holds.

This proposition shows that, under the suppositions that $R*$ is anti-symmetric, and that R is c-adequate and arbitrary u-regular, the map

$$j : x \mapsto <x> \tag{4-19}$$

is an injection from X to L. Thus we regard x and $<x>$ as the same. In this case, X can be regarded as a subset of L, and we have $(R^*)^{-1}|_{X\times L} = R'$ and $R'|_{X\times X} = (R*)^{-1}$.

PROPOSITION 4.6. *Suppose that R is c-adequate and arbitrary u-regular. Suppose $\alpha \in L$. When $R^{-1}[\alpha] \neq X$, we have*

$$\alpha = \bigwedge_{R^*}(<x> | x \in X,\ xR'\alpha). \tag{4-20}$$

PROPOSITION 4.7. *Suppose that* $\mathcal{N} \in {}^{n}(X, L, R)$. *Then*

$$xR^*y \Rightarrow \mathcal{N}(x) \supseteq \mathcal{N}(y) \tag{4-21}$$

holds for $x, y \in X$.

DEFINITION 4.1. Suppose that R is o-adequate and c-adequate, and that R is arbitrary i-regular. Let $\mathcal{N} \in {}^{n}(X, L, R)$. For every $\alpha \in L$, let

$$\hat{\mathcal{N}}(\alpha) = \cap\{\mathcal{N}(x)|x \in X, xR\alpha\}, \qquad \alpha \neq 0_{R^*} \tag{4-22}$$

$$\hat{\mathcal{N}}(0_{R^*}) = L. \tag{4-23}$$

$\hat{\mathcal{N}}$ is called the rising of R-neighbor element structure $\mathcal{N}$.

We will suppose that R satisfies the conditions in Definition 4.1 until we specify it again. We note that ${}^{g}(X, L, R) \subseteq {}^{j}(X, L, R)$ in this case.

PROPOSITIONS 4.8. *For every* $x \in X$, *we have*

$$\hat{\mathcal{N}}(\langle x\rangle) = \mathcal{N}(x). \tag{4-24}$$

PROPOSITION 4.9. *For every* $\alpha, \beta \in L$, *we have*

$$\alpha R^*\beta \Rightarrow \hat{\mathcal{N}}(\alpha) \supseteq \hat{\mathcal{N}}(\beta). \tag{4-25}$$

LEMMA 4.1. *Suppose that* $\mathcal{G} \in {}^{g}X$, *and that* $\mathcal{N} \in {}^{n}X$ *is corresponding with* $\mathcal{G}$. *Then the rising* $\hat{\mathcal{N}}$ of $\mathcal{N}$ *satisfies*

$$\hat{\mathcal{N}}(\alpha) = \{\beta \in L|\exists\delta \in \mathcal{G}(\alpha R^*\delta R^*\beta)\}, \qquad \alpha \in L. \tag{4-26}$$

Proof. When $\alpha = 0_{R^*}$, both sides of the equality (4-26) are L. We can suppose $\alpha \neq 0_{R^*}$, so that $R^{-1}[\alpha] \neq \varnothing$. Let $\beta \in \hat{\mathcal{N}}(\alpha)$. Thus for every $x \in X$, if $xR\alpha$ then $\beta \in \mathcal{N}(x)$. Because $\mathcal{N}(x) = \{\alpha \in L|\exists\delta \in \mathcal{G}(xR\delta R^*\alpha)\}$, there exists $\delta_x \in \mathcal{G}$ such that $xR\delta_x R^*\beta$. Let $\delta = \bigvee_{R^*}(\delta_x|xR\beta)$. Hence $\delta \in \mathcal{G}$, because $\mathcal{G}$ is closed under the formation of arbitrary $\bigvee_{R^*}$. Hence, $xR\delta R^*\delta$ for $xR\alpha$, namely, $\langle x\rangle R^*\delta R^*\beta$. By $\alpha = \bigvee_{R^*}(\langle x\rangle|xR\alpha)$, we have $\alpha R^*\delta R^*\beta$. Conversely, suppose $\beta \in L$ such that $\alpha R^*\delta R^*\beta$; we have $xR\delta R^*\beta$ for $xR\alpha$, namely, $\beta \in \mathcal{N}(x)$, so that $\beta \in \cap\{\mathcal{N}(x)|xR\alpha\} = \hat{\mathcal{N}}(\alpha)$.

The following definition is from [3].

DEFINITION 4.2. Suppose that R is o-adequate and c-adequate, and that R is arbitrary i-regular. Let

$${}^{\hat{n}}(X, L, R) = \{\hat{\mathcal{N}}|\hat{\mathcal{N}} : L \to \mathcal{P}(L)\backslash\{\varnothing\} \text{ satisfies } \hat{\text{n}}.1\text{–}\hat{\text{n}}.6\}, \tag{4.27}$$

where

n̂.1. $\beta \in \hat{\mathcal{N}}(\alpha) \Rightarrow \alpha R^*\beta$.

n̂.2. $\beta, \gamma \in \hat{\mathcal{N}}(\alpha) \Rightarrow \beta \wedge_{R^*}\gamma \in \hat{\mathcal{N}}(\alpha)$.

n̂.3. $\beta \in \hat{\mathcal{N}}(\alpha)$ and $\beta R^*\gamma \Rightarrow \gamma \in \hat{\mathcal{N}}(\alpha)$.

n̂.4. $\beta \in \hat{\mathcal{N}}(\alpha) \Rightarrow \exists \delta \in \hat{\mathcal{N}}(\alpha)(\delta R^*\beta$ and $\delta \in \hat{\mathcal{N}}(\delta))$.

n̂.5. $\hat{\mathcal{N}}(\vee_{R^*}(\alpha_t|t \in T)) = \cap\{\hat{\mathcal{N}}(\alpha_t)|t \in T\}$.

n̂.6. $\hat{\mathcal{N}}(0_{R^*}) = L$,

where T is an arbitrary index set. $\hat{\mathcal{N}} \in {}^{\hat{n}}(X, L, R)$ is called R-upper neighbor element structure on L. ${}^{\hat{n}}(X, L, R)$ will be denoted by ${}^{\hat{n}}L$ is there is no confusion.

Suppose that $\hat{\mathcal{N}} \in {}^{\hat{n}}L$. Let

$$\mathcal{N}(x) = \hat{\mathcal{N}}(\langle x \rangle), \qquad x \in X. \tag{4-28}$$

$\mathcal{N}$ is called the restriction of R-upper neighbor element structure $\hat{\mathcal{N}}$.

THEOREM 4.1. *Suppose that $\mathcal{G} \in {}^{g}X$, and that $\mathcal{N} \in {}^{n}X$ correspond with $\mathcal{G}$. Then its rising $\hat{\mathcal{N}} \in {}^{\hat{n}}L$, and the restriction of $\hat{\mathcal{N}}$ is just $\mathcal{N}$. Conversely, if $\hat{\mathcal{N}} \in {}^{\hat{n}}L$, then its restriction $\hat{\mathcal{N}} \in {}^{n}X$, and the rising of $\mathcal{N}$ is just $\hat{\mathcal{N}}$. Moreover, if $\mathcal{S} \in {}^{j}X$ corresponds with the restriction $\mathcal{N}$ of $\hat{\mathcal{N}}$, then $\mathcal{S} \in {}^{g}X$.*

Proof. Let $\mathcal{G} \in {}^{g}X$. Suppose that $\mathcal{N} \in {}^{n}X$ is corresponding with $\mathcal{S}$, and that $\hat{\mathcal{N}}$ is the rising of $\mathcal{N}$. By Lemma 4.1, to prove $\hat{\mathcal{N}} \in {}^{\hat{n}}L$, we need to prove that $\hat{\mathcal{N}}$ satisfies $\hat{n}.5$ only. By Proposition 4.9, $\cap\{\hat{\mathcal{N}}(\alpha_t)|t \in T\} \supseteq \hat{\mathcal{N}}(\vee_{R^*}(\alpha_t|t \in T)$. Now, suppose $\alpha \in \cap\{\hat{\mathcal{N}}(\alpha_t)|t \in T\}$. By Lemma 4.1, there exist $\delta_t \in \mathcal{S}$ $(t \in T)$ such that $\alpha_t R^*\delta_t R^*\alpha (t \in T)$. Let $\delta = \vee_{R^*}(\delta_t|t \in T)$. Hence $\vee_{R^*}(\alpha_t|t \in T)R^*\delta R^*\alpha$. Again, by Lemma 4.1, $\alpha \in \hat{\mathcal{N}}(\vee_{R^*}(\alpha_t|_t \in T)$.

Conversely, let $\hat{\mathcal{N}} \in {}^{\hat{n}}L$, and suppose that $\mathcal{N}$ is the restriction of $\hat{\mathcal{N}}$. Obviously we have that $\mathcal{N} \in {}^{n}X$, and that the rising of $\mathcal{N}$ is just $\hat{\mathcal{N}}$, by n̂.5. Now we suppose that $\mathcal{S} \in {}^{j}X$ is corresponding with the restriction $\mathcal{N}$ of $\hat{\mathcal{N}}$. We have $\delta \in \mathcal{S}$ if and only if, for every $x \in X$, $xR\delta$ implies $\delta \in \mathcal{N}(x)$. So we have

$$\delta \in \mathcal{S} \Leftrightarrow \delta \in \hat{\mathcal{N}}(\delta). \tag{4-29}$$

Hence $0_{R^*} \in \mathcal{S}$ by $0_{R^*} \in \hat{\mathcal{N}}(0_{R^*})$. Let $\delta_t \in \mathcal{S}$ $(t \in T)$, so that $\delta_t \in \hat{\mathcal{N}}(\delta_t)$ $(t \in T)$. Hence $\vee_{R^*}(\delta_t|t \in T) \in \cap\{\hat{\mathcal{N}}(\delta_t)|t \in T\}$. By n̂.5, $\vee_{R^*}(\delta_t|t \in T) \in \hat{\mathcal{N}}(\vee_{R^*}(\delta_t|t \in T))$, so that $\vee_{R^*}(\delta_t|t \in T) \in \mathcal{S}$, so that $\mathcal{G} \in {}^{g}X$.

DEFINITION 4.3. Suppose that R is neighbor relation. Let $\mathcal{N} \in {}^{n}(X, L, R)$. For every $\alpha \in$ L, let

$$\check{\mathcal{N}}(\alpha) = \cup\{\mathcal{N}(x)|x \in X, xR\alpha\}, \qquad \alpha \neq 1_{R^*} \tag{4-30}$$

$$\check{\mathcal{N}}(1_{R^*}) = \varnothing. \tag{4-31}$$

$\check{\mathcal{N}}$ is called the quasi-rising of R-neighbor element structure $\mathcal{N}$.

We will suppose that R is a neighbor relation until we specify it again. We note that ${}^{i}(X, L, R) = {}^{g}(X, L, R)$ in this case.

PROPOSITION 4.10. *For every $x \in X$, we have*

$$\check{\mathcal{N}}(< x >) = \mathcal{N}(x). \tag{4-32}$$

PROPOSITION 4.11. *For every α, $\beta \in L$, we have*

$$\alpha R^*\beta \Rightarrow \check{\mathcal{N}}(\alpha) \supseteq \check{\mathcal{N}}(\beta). \tag{4-33}$$

LEMMA 4.2. *Suppose that $\mathcal{N} \in {}^{n}X$ is corresponding with $\mathcal{G} \in {}^{g}X$, and that $\check{\mathcal{N}}$ is the quasi-rising of $\mathcal{N}$. Then we have*

$$\check{\mathcal{N}}(\alpha) = \{\beta \in L | \exists \delta \in \mathcal{G}(\delta R^*\alpha \text{ and } \delta(R^*)'\beta)\}, \qquad \alpha \in L. \tag{4-34}$$

Proof. When $\alpha = 1_{R^*}$, both sides of the equality (4-34) are $\varnothing$. We can suppose $\alpha \neq 1_{R^*}$. Let $\beta \in \check{\mathcal{N}}(\alpha)$, then there exists $x \in X$ such that $xR'\alpha$ and $\beta \in \mathcal{N}(x)$, so that there exists $\delta \in \mathcal{G}$ such that $xR\delta R^*\beta$. From $xR'\alpha$ and $xR\delta$, we know $\delta(R^*)'\alpha$. Conversely, suppose $\beta \in L$ such that $\delta(R^*)'\alpha$ and $\delta R^*\beta$. By $\alpha = \bigwedge_{R^*}(> x < | x \in X, xR'\alpha)$, there exists $x \in X$ such that $xR'\alpha$ and $\delta(R^*)' <x>$, namely, $xR\delta$. Hence $\beta \in \mathcal{N}(x) \subset \check{\mathcal{N}}(\alpha)$.

DEFINITION 4.4. Suppose that R is a neighbor relation. Let

$${}^{\check{n}}(X, L, R) = \{\check{\mathcal{N}} | \check{\mathcal{N}} : L \to \mathcal{P}(L) \text{ satisfies } \check{n}.1\text{–}\check{n}.6\}, \tag{4-35}$$

where

ň.1. $\beta \in \check{\mathcal{N}}(\alpha) \Rightarrow \beta(R^*)'\alpha$.

ň.2. $\alpha, \beta \in \check{\mathcal{N}}(< x >) \Rightarrow \alpha \wedge_{R^*}\beta \in \check{\mathcal{N}}(< x >)$.

ň.3. $\beta \in \mathcal{N}(\alpha)$ and $\beta R^*\gamma \Rightarrow \gamma \in \check{\mathcal{N}}(\alpha)$.

ň.4. $\beta \in \check{\mathcal{N}}(\alpha) \Rightarrow \exists \delta \in \check{\mathcal{N}}(\alpha)(\delta R^*\beta$ and $(\delta(R^*)'\gamma \Rightarrow \delta \in \check{\mathcal{N}}(\gamma)))$.

ň.5. $\check{\mathcal{N}}(\bigwedge_{R^*}(\alpha_t | t \in T)) = \cup\{\check{\mathcal{N}}(\alpha_t) | t \in T\}$.

ň.6. $\hat{\mathcal{N}}(1_{R^*}) = \varnothing$,

where T is an arbitrary index set. $\check{\mathcal{N}} \in {}^{\check{n}}(X, L, R)$ is called R-lower neighbor element structure on L. ${}^{\check{n}}(X, L, R)$ will be denoted by ${}^{\check{n}}L$ if there is no confusion.

Suppose $\breve{\mathcal{N}} \in {}^{\breve{n}}L$. Let

$$\mathcal{N}(x) = \breve{\mathcal{N}}(< x >),\ x \in X. \tag{4-36}$$

$\mathcal{N}$ is called the restriction of R-lower neighbor element structure $\breve{\mathcal{N}}$.

THEOREM 4.2. *Suppose that $\mathcal{N} \in {}^{n}X$ corresponds with $\mathcal{G} \in {}^{g}X$, and that $\breve{\mathcal{N}}$ is the quasi-rising of $\mathcal{N}$. Then $\breve{\mathcal{N}} \in {}^{\breve{n}}L$, and the restriction of $\breve{\mathcal{N}}$ is just $\mathcal{N}$. Conversely, if $\breve{\mathcal{N}} \in {}^{\breve{n}}L$, then its restriction $\mathcal{N} \in {}^{n}X$, and the quasi-rising of $\mathcal{N}$ is just $\breve{\mathcal{N}}$.*

Proof. Suppose that $\breve{\mathcal{N}}$ is the quasi-rising of $\mathcal{N} \in {}^{n}X$. By Lemma 4.2, to prove $\breve{\mathcal{N}} \in {}^{\breve{n}}L$, we need to prove that $\breve{\mathcal{N}}$ satisfies $\breve{n}$.5 only. By Proposition 4.11, $\cup\{\breve{\mathcal{N}}(\alpha_t)|t \in T\} \subseteq \breve{\mathcal{N}}(\wedge_{R^*}(\alpha_t|t \in T))$. Now, suppose $\alpha \in \breve{\mathcal{N}}(\wedge_{R^*}(\alpha_t|t \in T))$. By Lemma 4.2, there exists $\delta \in \mathcal{G}$ such that $\delta(R^*)'(\wedge_{R^*}(\alpha_t|t \in T))$ and $\delta R^*\alpha$, so that there exists $t \in T$ such that $\delta(R^*)'\alpha_t$. Hence $\alpha \in \breve{\mathcal{N}}(\alpha_t) \subseteq \cup\{\breve{\mathcal{N}}(\alpha_t|t \in T)$.

Conversely, let $\breve{\mathcal{N}} \in {}^{\breve{n}}L$, and suppose that $\mathcal{N}$ is the restriction of $\breve{\mathcal{N}}$. Obviously we have that $\mathcal{N} \in {}^{n}X$, and that the quasi-rising of $\mathcal{N}$ is just $\breve{\mathcal{N}}$, by $\breve{n}$.5.

In the following we will discuss the risings of R-distant element structure. We will give their results without proofs because their proofs are similar to the above proofs.

PROPOSITION 4.12. *Suppose that $\mathcal{Q} \in {}^{q}(X, L, R)$. Then*

$$xR^*y \Rightarrow \mathcal{Q}(x) \subseteq \mathcal{Q}(y) \tag{4-37}$$

holds for $x, y \in X$.

DEFINITION 4.5. Suppose that R is o-adequate and c-adequate, and that R is arbitrary u-regular. Let $\mathcal{Q} \in {}^{q}(X, L, R)$. For every $\alpha \in L$, let

$$\mathcal{Q}(\alpha) = \cap\{\mathcal{Q}(x)|x \in X, xR'\alpha\}, \qquad \alpha \neq 1_{R^*} \tag{4-38}$$

$$\breve{\mathcal{Q}}(1_{R^*}) = L. \tag{4-39}$$

$\breve{\mathcal{Q}}$ is called the rising of R-distant element structure $\mathcal{Q}$.

We will suppose that R satisfies the conditions in Definition 4.5 until we specify it again. We note that ${}^{f}(X, L, R) \subseteq {}^{e}(X, L, R)$ in this case.

PROPOSITION 4.13. *For every $x \in X$, we have*

$$\breve{\mathcal{Q}}(< x >) = \mathcal{Q}(x). \tag{4-40}$$

PROPOSITION 4.14. *For every α, $\beta \in L$, we have*

$$\alpha R^*\beta \Rightarrow \check{\mathfrak{Q}}(\alpha) \subseteq \check{\mathfrak{Q}}(\beta). \tag{4-41}$$

LEMMA 4.3. *Suppose that $\mathfrak{F} \in {}^fX$, and that $\mathfrak{Q} \in {}^qX$ is corresponding with $\mathfrak{F}$. Then the rising $\check{\mathfrak{Q}}$ of $\mathfrak{Q}$ satisfies*

$$\mathfrak{Q}(\alpha) = \{\beta \in L | \exists \delta \in \mathfrak{F}(\beta R^*\delta R^*\alpha)\}, \qquad \alpha \in L \tag{4-42}$$

The following definition is from [3].

DEFINITION 4.6. Suppose that R is o-adequate and c-adequate, and that R is arbitrary u-regular. Let

$$^{\check{q}}(X, L, R) = \{\check{\mathfrak{Q}} | \check{\mathfrak{Q}} : L \to \mathcal{P}(L) \backslash \{\emptyset\} \text{ satisfies } \check{q}.1\text{–}\check{q}.6\}, \tag{4-43}$$

where

$\check{q}$.1. $\beta \in \check{\mathfrak{Q}}(\alpha) \Rightarrow \beta R^*\alpha$.
$\check{q}$.2. $\beta, \gamma \in \check{\mathfrak{Q}}(\alpha) \Rightarrow \beta \vee_{R^*} \gamma \in \check{\mathfrak{Q}}(\alpha)$.
$\check{q}$.3. $\beta \in \check{\mathfrak{Q}}(\alpha)$ and $\gamma R^*\beta \Rightarrow \gamma \in \check{\mathfrak{Q}}(\alpha)$.
$\check{q}$.4. $\beta \in \check{\mathfrak{Q}}(\alpha) \Rightarrow \exists \delta \in \check{\mathfrak{Q}}(\alpha)(\beta R^*\delta$ and $\delta \in \check{\mathcal{N}}(\delta))$.
$\check{q}$.5. $\check{\mathfrak{Q}}(\wedge_{R^*}(\alpha_t | T \in T)) = \cap\{\check{\mathfrak{Q}}(\alpha_t) | t \in T\}$.
$\check{q}$.6. $\check{\mathfrak{Q}}(1_{R^*}) = L$,

where T is an arbitrary index set. $\check{\mathfrak{Q}} \in {}^{\check{q}}(X, L, R)$ is called R-lower distant element structure on L. $^{\check{q}}(X, L, R)$ will be denoted by ^{q}L is there is no confusion.

Suppose $\check{\mathfrak{Q}} \in {}^{q}L$. Let

$$\mathfrak{Q}(x) = \check{\mathfrak{Q}}(< x >), \qquad x \in X. \tag{4-44}$$

$\mathfrak{Q}$ is called the restriction of R-lower distant element structure $\check{\mathfrak{Q}}$.

THEOREM 4.3. *Suppose that $\mathfrak{F} \in {}^fX$, and that $\mathfrak{Q} \in {}^qX$ corresponds with $\mathfrak{F}$. Then its rising $\check{\mathfrak{Q}} \in {}^gL$, and the restriction of $\check{\mathfrak{Q}}$ is just $\mathfrak{Q}$. Conversely, if $\check{\mathfrak{Q}} \in {}^gL$, then its restriction $\mathfrak{Q} \in {}^qX$, and the rising of $\mathfrak{Q}$ is just $\check{\mathfrak{Q}}$. Moreover, if $\mathcal{G} \in {}^fX$ corresponds with the restriction $\mathfrak{Q}$ of $\check{\mathfrak{Q}}$, then $\mathcal{G} \in {}^fX$.*

DEFINITION 4.7. Suppose that R is distant relation. Let $\mathfrak{Q} \in {}^q(X, L, R)$. For every $\alpha \in L$, let

$$\hat{\mathfrak{Q}}(\alpha) = \cup\{\mathfrak{Q}(x) | x \in X, xR\alpha\}, \qquad \alpha \neq 0_{R^*} \tag{4-45}$$

$$\hat{\mathfrak{Q}}(0_{R^*}) = \varnothing. \tag{4-46}$$

$\hat{\mathfrak{Q}}$ is called quasi-rising of R-distant element structure $\mathfrak{Q}$.

We will suppose that R is a distant relation until we specify it again. We note that ${}^e(X, L, R) = {}^f(X, L, R)$ in this case.

PROPOSITION 4.15. *For every $x \in X$, we have*

$$\hat{\mathfrak{Q}}(\langle x \rangle) = \mathfrak{Q}(x). \tag{4-47}$$

PROPOSITION 4.16. *For every $\alpha, \beta \in L$, we have*

$$\alpha R^* \beta \Rightarrow \hat{\mathfrak{Q}}(\alpha) \subseteq \hat{\mathfrak{Q}}(\beta). \tag{4-48}$$

LEMMA 4.4. *Suppose that $\mathfrak{Q} \in {}^qX$ is corresponding with $\mathcal{F} \in {}^fX$, and that $\hat{\mathfrak{Q}}$ is the rising of $\mathfrak{Q}$. Then*

$$\hat{\mathfrak{Q}}(\alpha) = \{\beta \in L | \exists \delta \in \mathcal{F}(\alpha(R^*)'\delta \text{ and } \beta R^* \delta)\}, \qquad \alpha \in L. \tag{4-49}$$

DEFINITION 4.8. Suppose that R is a distant relation. Let

$${}^{\hat{q}}(X, L, R) = \{\hat{\mathfrak{Q}} | \hat{\mathfrak{Q}} : L \to \mathcal{P}(L) \text{ satisfies } \hat{\mathrm{q}}.1\text{–}\hat{\mathrm{q}}.6\}, \tag{4-50}$$

where

- $\hat{\mathrm{q}}.1.$ $\beta \in \hat{\mathfrak{Q}}(\alpha) \Rightarrow \alpha(R^*)'\beta$.
- $\hat{\mathrm{q}}.2.$ $\alpha, \beta \in \hat{\mathfrak{Q}}(\langle x \rangle) \Rightarrow \alpha \vee_{R^*} \beta \in \hat{\mathfrak{Q}}(\langle x \rangle)$.
- $\hat{\mathrm{q}}.3.$ $\beta \in \hat{\mathfrak{Q}}(\alpha)$ and $\gamma R^* \beta \Rightarrow \gamma \in \hat{\mathfrak{Q}}(\alpha)$.
- $\hat{\mathrm{q}}.4.$ $\beta \in \hat{\mathfrak{Q}}(\alpha) \Rightarrow \exists \delta \in \hat{\mathfrak{Q}}(\alpha)(\beta R^* \delta$ and $(\gamma(R^*)'\delta \Rightarrow \delta \in \hat{\mathfrak{Q}}(\gamma)))$.
- $\hat{\mathrm{q}}.5.$ $\hat{\mathfrak{Q}}(\vee_{R^*}(\alpha_t | t \in T)) = \cup\{\hat{\mathfrak{Q}}(\alpha_t) | t \in T\}$.
- $\hat{\mathrm{q}}.6.$ $\hat{\mathfrak{Q}}(0_{R^*}) = \varnothing$,

where T is an arbitrary index set. $\hat{\mathfrak{Q}} \in {}^{\hat{q}}(X, L, R)$ is called R-upper distant element structure on L. ${}^{\hat{q}}(X, L, R)$ will be denoted by ${}^{\hat{q}}L$ if there is no confusion.

Suppose $\hat{\mathfrak{Q}} \in {}^{\hat{q}}L$. Let

$$\mathfrak{Q}(x) = \hat{\mathfrak{Q}}(\langle x \rangle), \qquad x \in X. \tag{4-51}$$

$\mathfrak{Q}$ is called the restriction of R-upper distant element structure $\hat{\mathfrak{Q}}$.

THEOREM 4.4. *Suppose that $\mathfrak{Q} \in {}^qX$ is corresponding with $\mathcal{F} \in {}^fX$, and that $\hat{\mathfrak{Q}}$ is the quasi-rising of $\mathfrak{Q}$. then $\hat{\mathfrak{Q}} \in {}^{\hat{q}}L$, and the restriction of $\hat{\mathfrak{Q}}$*

is just $\mathfrak{Q}$. Conversely, if $\hat{\mathfrak{Q}} \in {}^{\hat{n}}L$, then its restriction $\mathfrak{Q} \in {}^{q}X$, and the quasi-rising of $\mathfrak{Q}$ is just $\hat{\mathfrak{Q}}$.

5. Dually Negative Relation

In this section, we suppose that there is a one–one complement map c on L relative of R^*, namely, c satisfies that, for α, $\beta \in L$

(1) $\alpha^c R^* \beta^c$ if and only if $\beta R^* \alpha$,

(2) $(\alpha^c)^c = \alpha$.

Definition 5.1. For any $x \in X$ and $\alpha \in L$, let

$$xR\theta\alpha \Leftrightarrow xR'\alpha^c. \tag{5-1}$$

R^θ is called the dually negative relation of R.

Proposition 5.1. *We have*

$$xR^\theta\alpha^c \Leftrightarrow xR'\alpha \tag{5-2}$$

$$x(R^\theta)'\alpha \Leftrightarrow xR\alpha^c \tag{5-3}$$

$$x(R^\theta)'\alpha^c \Leftrightarrow xR\alpha. \tag{5-4}$$

Proposition 5.2. *We have*

$$(R^\theta)^* = R^* \tag{5-5}$$

$$(R^{-1}[\alpha^c])' = (R^\theta)^{-1}[\alpha] \tag{5-6}$$

$$(R^{-1}[\alpha])' = (R^\theta)^{-1}[\alpha^c] \tag{5-7}$$

$$((R^\theta)^{-1}[\alpha])' = R^{-1}[\alpha^c] \tag{5-8}$$

$$((R^\theta)^{-1}[\alpha^c])' = R^{-1}[\alpha], \tag{5-9}$$

where ′ is the operative symbol of the complement of set.

Suppose that $\mathcal{S} \in \mathcal{P}(L)$, and that $f: X \to \mathcal{P}(L)$. Let

$$\mathcal{S}^{\backslash c} = \{\delta^c | \delta \in \mathcal{S}\} \tag{5-10}$$

$$f^{\backslash c}(x) = f(x)^{\backslash c}, \qquad \forall x \in X. \tag{5-11}$$

Proposition 5.3. *Suppose that $\mathcal{S} \in \mathcal{P}(L)$. Then*

(1) *$\mathcal{S}$ is o- (or c-)adequate about R^θ if and only if $\mathcal{S}^{\backslash c}$ is c- (or o-)adequate about R;*

(2) *$\mathcal{S}$ is arbitrary (or finite) i- (or u-)regular about R_θ if and only if $\mathcal{S}^{\backslash c}$ is arbitrary (or finite) u- (or i-)regular about R.*

(3) *$\mathcal{S}$ is o-(or c-)maximal about R_θ is and only if $\mathcal{S}^{\backslash c}$ is c- (or o-)maximal about R.*

By this proposition, we know that R^θ is o- (or c-)adequate if and only if R is c- (or o-)adequate, and that R^θ is arbitrary (or finite) i- (or u-)regular if and only if R is arbitrary (or finite) u- (or i-)regular.

PROPOSITION 5.4. *R is a neighbor (or distant) relation if and only if R^θ is a distant (or neighbor) relation.*

THEOREM 5.1. *Suppose that R is o-adequate and finite i-regular. Then*

$$\mathcal{J} \in {}^{j}(X, L, R) \Leftrightarrow \mathcal{J}^{\backslash c} \in {}^{e}(X, L, R^\theta), \tag{5-12}$$

$$\mathcal{N} \in {}^{n}(X, L, R) \Leftrightarrow \mathcal{N}^{\backslash c} \in {}^{q}(X, L, R^\theta). \tag{5-13}$$

Thus, under the supposition in this theorem, by Theorems 2.1, 2.2, and 5.2, we can obtain the following commutative diagram:

$$\begin{array}{cccccc}
\mathcal{N} & \in {}^{n}(X, L, R) & \overset{\backslash c}{\leftrightarrow} & \mathcal{Q} & \in {}^{q}(X, L, R^\theta) \\
\updownarrow & \text{(Th. 2.1)} & & \updownarrow & \text{(Th. 2.2)} \\
\mathcal{J} & \in {}^{j}(X, L, R) & \overset{\backslash c}{\leftrightarrow} & \mathcal{E} & \in {}^{e}(X, L, R^\theta).
\end{array}$$

THEOREM 5.2. *Suppose that R is c-adequate and finite u-regular. Then*

$$\mathcal{E} \in {}^{e}(X, L, R) \Leftrightarrow \mathcal{E}^{\blacktriangleleft c} \in {}^{j}(X, L, R^\theta), \tag{5-14}$$

$$\mathcal{Q} \in {}^{q}(X, L, R) \Leftrightarrow \mathcal{Q}^{\blacktriangleleft c} \in {}^{n}(X, L, R^\theta). \tag{5-15}$$

Thus, under the supposition in this theorem, by Theorems 2.1, 2.2, and 5.2, we can obtain the following commutative diagrams:

$$\begin{array}{cccccc}
\mathcal{Q} & \in {}^{q}(X, L, R) & \overset{\backslash c}{\leftrightarrow} & \mathcal{N} & \in {}^{n}(X, L, R^\theta) \\
\updownarrow & \text{(Th. 2.2)} & & \updownarrow & \text{(Th. 2.1)} \\
\mathcal{E} & \in {}^{e}(X, L, R) & \overset{\backslash c}{\leftrightarrow} & \mathcal{J} & \in {}^{j}(X, L, R^\theta).
\end{array}$$

This theorem shows that, making use of dually negative relation of R, we can construct a distant element structure from a neighbor element structure, and we can also construct a neighbor element structure from a distant element structure.

Now we suppose that (L, R^*) becomes a complete lattice. By $(R^\theta)^* = R^*$, ${}^{g}(X, L, R) = {}^{o}(X, L, R^\theta)$ and ${}^{f}(X, L, R) = {}^{f}(X, L, R^\theta)$.

THEOREM 5.3. *Suppose that R is a neighbor relation. Then*

$$\mathscr{G} \in {}^{g}(X, L, R) \Leftrightarrow \mathscr{G}^{\backslash c} \in {}^{f}(X, L, R^{\theta}). \tag{5-16}$$

Suppose that R is a distant relation. Then

$$\mathscr{F} \in {}^{f}(X, L, R) \Leftrightarrow \mathscr{F}^{\backslash c} \in {}^{g}(X, L, R^{\theta}). \tag{5-17}$$

Further, we suppose that $R*$ is anti-symmetric. When R is a neighbor relation, we have the following commutative diagram:

$$\begin{array}{llcll}
\hat{\mathscr{N}} & \in {}^{\hat{n}}(X, L, R^{\theta}) & \overset{\backslash c}{\leftrightarrow} & \breve{\mathfrak{Q}} & \in {}^{\breve{q}}(X, L, R) \\
\updownarrow & \quad \text{(Th. 4.1)} & & \updownarrow & \quad \text{(Th. 4.3)} \\
\mathscr{G} & \in {}^{g}(X, L, R) & \overset{\backslash c}{\leftrightarrow} & \mathscr{F} & \in {}^{f}(X, L, R). \\
\updownarrow & \quad \text{(Th. 4.1)} & & \updownarrow & \quad \text{(Th. 4.3)} \\
\mathscr{N} & \in {}^{n}(X, L, R) & \overset{\backslash c}{\leftrightarrow} & \mathfrak{Q} & \in {}^{q}(X, L, R^{\theta}) \\
\updownarrow & \quad \text{(Th. 4.2)} & & \updownarrow & \quad \text{(Th. 4.4)} \\
\breve{\mathscr{N}} & \in {}^{\breve{n}}(X, L, R) & \overset{\backslash c}{\leftrightarrow} & \hat{\mathfrak{Q}} & \in {}^{\hat{q}}(X, L, R^{\theta}).
\end{array}$$

When R is a distant relation, we will have a similar commutative diagram.

6. CONVERGENCE IN RELATION

In this section, we will discuss the problem of convergence of nets in the set X and in the set L.

Let D be a nonempty set. Let $\geq$ be a semi-order on D. The pair $(D, \geq)$ is called a directed set, directed by $\geq$, iff, for every pair $m, n \in D$, there exists a $p \in D$ such that $p \geq m$ and $p \geq n$.

Let $(D, \geq)$ be a directed set. The function $w: D \to X$ is called a net in X. For $n \in D$, $w(n)$ is often denoted by w_n and hence a net w is often denoted by $\{w_n : n \in D\}$.

Let A be a subset of X, and let $\{w_n : n \in D\}$ be a net in X. By $w \in A$ means, for all $n \in D$, $w_n \in A$. By $w \in A$ eventually means, there exists a $m \in D$, when $n \geq m$, $w_n \in A$. By $w \in A$ often means, for every $m \in D$, there exists a $n \in D$ such that $w_n \in A$.

DEFINITION 6.1. Suppose that w is a net in X, $x \in X$, and that $\mathscr{S}$ is a subset of L. If

$$\forall \delta \in \mathscr{S}(xR\delta \Rightarrow wR\delta \text{ eventually}) \tag{6.1}$$

holds, then the net w is said to converge to x in relation R relative to $\mathscr{S}$ (or, for short, is said to converge to x in $R - \mathscr{S}$), and denoted by $w \to x$ $(R - \mathscr{S})$.

First we give the following important theorem.

THEOREM 6.1. *Suppose that w is a net in X, $x \in X$, and that $\mathscr{S}$ is a subset of L. Let*

$$\mathscr{C}(R - \mathscr{S}) = \{(w, x) | w \to (R - \mathscr{S})\}; \tag{6-4}$$

then $\mathscr{C}(R - \mathscr{S})$ is a convergence class on X.

Proof. We need to verify the following four conditions.

(1) If w is a net such that $w_n = x$ for all n, then $w \to x$ $(R - \mathscr{S})$.

(2) If $w \to x$ $(R - \mathscr{S})$, then for all subnets w' of w, $w' \to x$ $(R - \mathscr{S})$,

the above two conditions are obviously true.

(3) If $w \not\to x$ $(R - \mathscr{S})$, then there is a subnet w' of w such that there is no subnet of w' which converges to x in $R - \mathscr{S}$. In fact, suppose $w \not\to x$ $(R - \mathscr{S})$; then there is a $\delta \in \mathscr{S}$ such that $xR\delta$, but $w \notin R^{-1}[\mathscr{C}]$ often. Hence there is a subnet w' such that $w' \notin R^{-1}[\delta]$. The subnet w' cannot be of any subnet which converges to x in $R - \mathscr{S}$.

(4) Suppose that D is a directed set, and that for every m, E_m is also a directed set. Let $F = D \times \times\{E_m : m \in D\}$. If nets $(w_n^m : n \in E_m)$ converge to w_m in $R - \mathscr{S}$, for all $m \in D$, and the net $(w_m : m \in D)$ converges to x in $R - \mathscr{S}$, then the net $(w_{(m,f)} : (m, f) \in F)$ converges to x in $R - \mathscr{S}$, where $w_{(m,f)} = w_{f(m)}^m$. In fact, because $(w_m : m \in D)$ converges to x in $R - \mathscr{S}$, we have, $\forall \delta \in \mathscr{S}(xR\delta)$,

$$\exists m_0 \in D(m \geq m_0 \Rightarrow w_m R\delta). \tag{6-5}$$

Because $(w_n^m : n \in E_m)$ converge to w_m in $R - \mathscr{S}$, we have, by (6-5), when $m \geq m_0$, $w_n R\delta$,

$$\exists (n_m)_0 \in E_m(n_m \geq (n_m)_0 \to w_{n_m}^m R\delta. \tag{6-6}$$

Take $f_0 \in \times\{E_m | \, m \in D\}$ when $m \geq m_0, f_0(m) \geq (n_m)_0$, so that when $(m, f) \geq (m, f_0)$, $f(m) \geq f_0(m) \geq (n_m)_0$, we have $w_{(m,f)} R\delta$ by (6-6). Hence $w_{(m,f)} \to x$ $(R - \mathscr{S})$.

This theorem shows that $\mathscr{C}(R - \mathscr{S})$ can be used to induce a topology on the set X. We will denote its open set system by $T(R - \mathscr{S})$ and its closed set system by $F(R - \mathscr{S})$.

Note that the relation R and the subset $\mathscr{S}$ in Definition 6.1 and Theorem 6.1 are arbitrary. For a given triad (X, L, R), we can consider the relations R and R', when we need to consider the topology on X; we can also consider the relations R^*, $(R')^*$, $(R^*)'$ and $((R')^*)'$ when we need to consider the topology on L. Moreover, we can obtain many useful results, especially when the subset $\mathscr{S}$ belongs to ${}^{i}(X, L, R)$ or to ${}^{c}(X, L, R)$.

Now we will give several special notations.

Suppose that w is a net in X, $x \in X$. When $\mathscr{I} \in {}^{i}(X, L, R)$, the notations

$$w \to x\ (R - \mathscr{I}), \qquad T(R - \mathscr{I}), \qquad \text{and } F(R - \mathscr{I})$$

will be replaced with

$$w \to x\ (R \rightharpoonup \mathscr{I}), \qquad T(R \rightharpoonup \mathscr{I}), \qquad \text{and } F(R \rightharpoonup \mathscr{I}),$$

respectively. When $\mathscr{C} \in {}^{c}(X, L, R)$, the notations

$$w \to x\ (R' - \mathscr{C}), \qquad T(R' - \mathscr{C}), \qquad \text{and } F(R' - \mathscr{C})$$

will be replaced by

$$w \to x\ (R \rightharpoondown \mathscr{C}), \qquad T(R \rightharpoondown \mathscr{C}), \qquad \text{and } F(R \rightharpoondown \mathscr{C}),$$

respectively.

Suppose that w is a net in L, $\alpha \in L$. We can replace X in Definition 6.1 and Theorem 6.1 with L, and we can replace R in Definition 6.1 and Theorem 6.1 with R^*, $(R')*$, $(R^*)'$, and $((R')^*)'$, respectively. When $\mathscr{I} \in {}^{i}(X, L, R)$, the notations

$$w \to x\ (R^* - \mathscr{I}), \qquad T(R^* - \mathscr{I}), \qquad \text{and } F(R^* - \mathscr{I})$$

will be replaced with

$$w \to x\ (R - \mathscr{I}(1, 0)), \qquad T_{(1,0)}(R - \mathscr{I}), \qquad \text{and } F_{(1,0)}(R - \mathscr{I}),$$

respectively; the notations

$$w \to x\ (((R')^*)' - \mathscr{I}), \qquad T(((R')^*)' - \mathscr{I}), \qquad \text{and } F(((R')^*)' - \mathscr{I})$$

will be replaced with

$$w \to x\ (R - \mathscr{I}(2, 0)), \qquad T_{(2,0)}(R - \mathscr{I}), \qquad \text{and } F_{(2,0)}(R - \mathscr{I}),$$

respectively. When $\mathscr{E} \in {}^e(X, L, R)$, the notations

$$w \to x\ ((R')^* - \mathscr{E}), \qquad T((R')^* - \mathscr{E}), \qquad \text{and } F((R')^* - \mathscr{E})$$

will be replaced with

$$w \to x\ (R - \mathscr{E}(0, 1)), \qquad T_{(0,1)}(R - \mathscr{E}), \qquad \text{and } F_{(0,1)}(R - \mathscr{E}),$$

respectively; the notations

$$w \to x\ ((R^*)' - \mathscr{E}), \qquad T((R^*)' - \mathscr{E}), \qquad \text{and } F((R^*)' - \mathscr{E})$$

will be replaced with

$$w \to x\ (R - \mathscr{E}(0, 2)), \qquad T_{(0,2)}(R - \mathscr{E}), \qquad \text{and } F_{(0,2)}(R - \mathscr{E}).$$

PROPOSITION 6.1. *Suppose that* $\mathscr{J} \in {}^j(X, L, R)$, *and that* $\mathscr{N} \in {}^n(X, L, R)$ *corresponds with* $\mathscr{J}$. *Then* $w \to x\ (R \rightharpoonup \mathscr{J})$ *if and only if*

$$\forall \alpha \in L(\alpha \in \mathscr{N}(x) \Rightarrow wR\alpha \text{ eventually}). \tag{6-2}$$

PROPOSITION 6.2. *Suppose that* $\mathscr{E} \in {}^e(X, L, R)$, *and that* $\mathscr{Q} \in {}^q(X, L, R)$ *corresponds, with* $\mathscr{E}$. *Then* $w \to x\ (R \rightharpoonup \mathscr{E})$ *if and only if*

$$\forall \alpha \in L(\alpha \in \mathscr{Q}(x) \to wR'\alpha \text{ eventually}). \tag{6.3}$$

PROPOSITION 6.3. *Suppose that* $xR{*}y$. *If* $w \to x\ (R - \mathscr{S})$, *then* $w \to y\ (R - \mathscr{S})$. *If* $w \to y(R' - \mathscr{S})$, *then* $w \to x\ (R' - \mathscr{S})$.

PROPOSITION 6.4. *Suppose that* $\mathscr{S} \in \mathscr{P}(L)$. *Then* $\mathscr{S}$ *is o-adequate about* R *if and only if* $R^{-1}\{\mathscr{S}\}$ *is a topological subbase of* $T(R - \mathscr{S})$; $\mathscr{S}$ *is o-adequate and finite i-regular about* R *if and only if* $R^{-1}\{\mathscr{S}\}$ *is a topological base of* $T(R - \mathscr{S})$, *where*

$$R^{-1}\{\mathscr{S}\} = \{R^{-1}[\delta] \| \delta \in \mathscr{S}\}. \tag{6-7}$$

PROPOSITION 6.5. *Suppose that* $\mathscr{A}, \mathscr{B} \in \mathscr{P}(L)$ *are o-adequate about* R. *Then*

$$\mathscr{A}\tilde{R}\mathscr{B} \Rightarrow T(R - \mathscr{A}) = T(R - \mathscr{B}). \tag{6-8}$$

If $\mathscr{A}, \mathscr{B}$ *are also i-regular about* R, *then*

$$\mathscr{A}\tilde{R}\mathscr{B} \leftrightarrow T(R - \mathscr{A}) = T(R - \mathscr{B}). \tag{6-9}$$

PROPOSITION 6.6. *Suppose that* $\mathcal{S} \in \mathcal{P}(L)$ *is o-adequate and finite i-regular about* R. *Then* $\mathcal{J}$ *is o-maximal about* R *if and only if*

$$R^{-1}\{\mathcal{S}\} = T(R - \mathcal{S}). \tag{6-10}$$

PROPOSITION 6.7. *Suppose that* $\mathcal{S} \in \mathcal{P}(L)$. *Then* $\mathcal{S}$ *is c-adequate about* R *if and only if* $(R')^{-1}\{\mathcal{S}\}$ *is a topological subbase of* $T(R' - \mathcal{S})$; $\mathcal{S}$ *is c-adequate and finite u-regular about* R *if and only if* $(R')^{-1}\{\mathcal{S}\}$ *is a topological base of* $T(R' - \mathcal{S})$, *where u-regular about* R *if and only if* $(R')^{-1}\{\mathcal{S}\}$ *is a topological base of* $T(R' - \mathcal{S})$, *where*

$$(R')^{-1}\{\mathcal{S}\} = \{(R')^{-1}[\delta] \| \delta \in \mathcal{S}\}. \tag{6-11}$$

PROPOSITION 6.8. *Suppose that* $\mathcal{A}$, $\mathcal{B} \in \mathcal{P}(L)$ *are c-adequate about* R. *Then*

$$\mathcal{A}\bar{R}\mathcal{B} \Rightarrow T(R' - \mathcal{A}) = T(R' - \mathcal{B}). \tag{6-12}$$

If $\mathcal{A}$, $\mathcal{B}$ *are also u-regular about* R, *then*

$$\mathcal{A}\bar{R}\mathcal{B} \Leftrightarrow T(R' - \mathcal{A}) = T(R' - \mathcal{B}). \tag{6-13}$$

PROPOSITION 6.9. *Suppose that* $\mathcal{S} \in \mathcal{P}(L)$ *is c-adequate and finite u-regular about* R. *Then* $\mathcal{S}$ *is c-maximal about* R *if and only if*

$$R^{-1}\{\mathcal{S}\} = T(R' - \mathcal{S}). \tag{6-14}$$

PROPOSITION 6.10. *Suppose that* R *is o-adequate and finite i-regular, and that* $\mathcal{J} \in {}^{j}(X, L, R)$. *Then*

$$T(R \rightarrow \mathcal{J}) = T(R^{\theta} \rightarrow \mathcal{J}^{\backslash c}). \tag{6-15}$$

Suppose that R *is c-adequate and finite u-regular, and that* $\mathcal{E} \in {}^{e}(X, L, R)$. *Then*

$$T(R \rightarrow \mathcal{E}) = T(R^{\theta} \rightarrow \mathcal{E}^{\backslash c}). \tag{6-16}$$

This proposition shows that R-neighbor element structure and R^{θ}-distant structure can play the same roles in describing the convergence of nets in X.

Now we suppose that (L, R^*) becomes a complete lattice, and that $R*$ is antisymmetric. We next discuss the problem of convergence of nets in L.

PROPOSITION 6.11. *Suppose that R is both o-adequate and c-adequate, and arbitrary i-regular. Let* $\mathcal{G} \in {}^g(X, L, R)$ *be corresponding with* $\hat{\mathcal{N}} \in {}^{\hat{n}}L$. *Let* $\mathcal{N} \in {}^nX$ *be the restriction of* $\hat{\mathcal{N}}$. *For a net w in L, and* $\alpha \in L$, *we have* $w \to \alpha\ (R - \mathcal{G}(1, 0))$ *if and only if*

$$\forall \beta \in L(\beta \in \hat{\mathcal{N}}(\alpha) \Rightarrow wR^*\beta \text{ eventually}) \tag{6-17}$$

holds. For a net w in X, and x in X, we have

$$w \to x\ (R \to \mathcal{G}) \leftrightarrow \langle w \rangle \Leftrightarrow \langle x \rangle \qquad (R - \mathcal{G}(1, 0)). \tag{6-18}$$

PROPOSITION 6.12. *Suppose that R is a neighbor relation. Let* $\mathcal{G} \in {}^g(X, L, R)$ *be corresponding with* $\check{\mathcal{N}} \in {}^{\check{n}}L$. *Let* $\mathcal{N} \in {}^nX$ *be the restriction of* $\check{\mathcal{N}}$. *For a net w in L, and* $\alpha \in L$, *we have* $w \to \alpha\ (R - \mathcal{G}(2, 0))$ *if and only if*

$$\forall \beta \in L(\beta \in \check{\mathcal{N}}(\alpha) \Rightarrow \beta(R^*)'w \text{ eventually}) \tag{6-19}$$

holds. For a net w in X, and x in X, we have

$$w \to x\ (R \to \mathcal{G}) \leftrightarrow \langle w \rangle \to \langle x \rangle \qquad (R - \mathcal{G}(2, 0)). \tag{6-20}$$

PROPOSITION 6.13. *Suppose that R is both o-adequate and c-adequate, and arbitrary u-regular. Let* $\mathcal{F} \in {}^f(X, L, R)$ *be corresponding with* $\check{\mathfrak{Q}} \in {}^qL$. *Let* $\mathfrak{Q} \in {}^qX$ *be the restriction of* $\check{\mathfrak{Q}}$. *For a net w in L, and* $\alpha \in L$, *we have* $w \to \alpha\ (R - \mathcal{F}(0, 1))$ *if and only if*

$$\forall \beta \in L(\beta \in \check{\mathfrak{Q}}(\alpha) \Rightarrow \beta R^* w \text{ eventually}) \tag{6-21}$$

holds. For a net w in X, and x in X, we have

$$w \to x\ (R \to \mathcal{F}) \Leftrightarrow \langle w \rangle \to \langle x \rangle \qquad (R - \mathcal{F}(0, 1)). \tag{6-22}$$

PROPOSITION 6.14. *Suppose that R is a distant relation. Let* $\mathcal{F} \in {}^f(X, L, R)$ *be corresponding with* $\hat{\mathfrak{Q}} \in {}^qL$. *Let* $\mathfrak{Q} \in {}^qX$ *be the restriction of* $\hat{\mathfrak{Q}}$. *For a net w in L, and* $\alpha \in L$, *we have* $w \to \alpha\ (R - \mathcal{F}(0, 2))$ *if and only if*

$$\forall \beta \in L(\beta \in \hat{\mathfrak{Q}}(\alpha) \Rightarrow w(R^*)\beta \text{ eventually}) \tag{6-23}$$

holds. For a net w in X, and x in X, we have

$$w \to x\ (R \to \mathcal{F}) \Leftrightarrow \langle w \rangle \to \langle x \rangle \qquad (R - \mathcal{F}(0, 2)). \tag{6-24}$$

PROPOSITION 6.15. *We have*

$$w \to \alpha (R - \mathcal{G}(1,0)) \quad \text{and} \quad \alpha R^* \beta \Rightarrow w \to \beta (R - \mathcal{G}(1,0)) \tag{6-13}$$

$$w \to \alpha (R - \mathcal{G}(2,0)) \quad \text{and} \quad \alpha R^* \beta \Rightarrow w \to \beta (R - \mathcal{G}(2,0)) \tag{6-14}$$

$$w \to \alpha (R - \mathcal{F}(0,1)) \quad \text{and} \quad \beta R^* \alpha \Rightarrow w \to \beta (R - \mathcal{F}(0,1)) \tag{6-15}$$

$$w \to \alpha (R - \mathcal{F}(0,2)) \quad \text{and} \quad \beta R^* \alpha \Rightarrow w \to \beta (R - \mathcal{F}(0,2)). \tag{6-16}$$

This proposition and Proposition 6.3 show us that the convergence in relation of nets in X and in L is of the one-sided property. In order to exclude the one-sided property, when R is a neighbor relation, we suppose that $\mathcal{G} \in {}^g(X, L, R)$, and that $\mathcal{F} = \mathcal{G}^{\prime c} \in {}^f(X, L, R^\theta)$, and we set

$$\begin{aligned} & w \to \alpha \quad \text{in } R(1,1)(\mathcal{G}) \\ \Leftrightarrow\ & w \to \alpha \quad (R - \mathcal{G}(1,0)) \quad \text{and} \quad w \to \alpha (R^\theta - \mathcal{F}(0,1)) \end{aligned} \tag{6-17}$$

$$\begin{aligned} & w \to \alpha \quad \text{in } R(1,2)(\mathcal{G}) \\ \Leftrightarrow\ & w \to \alpha \quad (R - \mathcal{G}(1,0)) \quad \text{and} \quad w \to \alpha (R^\theta - \mathcal{F}(0,2)) \end{aligned} \tag{6-18}$$

$$\begin{aligned} & w \to \alpha \quad \text{in } R(2,1)(\mathcal{G}) \\ \Leftrightarrow\ & w \to \alpha \quad (R - \mathcal{G}(2,0)) \quad \text{and} \quad w \to \alpha (R^\theta - \mathcal{F}(0,1)) \end{aligned} \tag{6-19}$$

$$\begin{aligned} & w \to \alpha \quad \text{in } R(2,2)(\mathcal{G}) \\ \Leftrightarrow\ & w \to \alpha \quad (R - \mathcal{G}(2,0)) \quad \text{and} \quad w \to \alpha (R^\theta - \mathcal{F}(0,2)). \end{aligned} \tag{6-20}$$

By now, if $\mathcal{F} = \mathcal{G}^{\prime c}$, then we have defined eight kinds of convergence relations on L relative to $\mathcal{G}$. So we can define eight kinds of general topologies on L.

7. THE CONNECTION WITH OTHER KINDS OF TOPOLOGIES

1. *General Topology*

Suppose that X is a set, and that $\mathcal{P}(X)$ is the power set of X. Consider two kinds of relations between X and $\mathcal{P}(X)$, the belonging relation $\in$ and non-belonging relation $\notin$.

$\in^*$ is $\subseteq$; $\in_*$ is $=$; $\in$ is both a neighbor relation and a distant relation; the dually negative relation of $\in$ is just $\in$. $\in$-Neighbor element structure on X is just neighborhood structure; $\in$-neighbor element system on X and $\in$-open element system on X are equal, and they are all open set system of X; $\in$-distant element system on X and $\in$-closed element system on X are equal, and they are all closed set system on X.

$\notin$ is $\supseteq$; $\notin*$ is $=$; $\notin$ is both a neighbor relation and a distant relation; the dually negative relation of $\notin$ is just $\notin$. $\notin$-Distant element structure on X is just neighborhood structure; $\notin$-distant element system on X and $\notin$-closed element system on X are equal, and they are all open set system on X; $\notin$-neighbor element system on X and $\notin$-open element system on X are equal, and they are all closed set system on X.

The risings of $\in$-neighbor element structure and the risings of $\notin$-distant element structure are equal, and they are neighborhood structure of subsets in X, while their quasi-risings are new concepts. Meanwhile, $\in$-distant element structure and $\notin$-neighbor element structure are equal, and they are also new concepts, as are their risings and quasi-risings.

Moreover, suppose that $\mathcal{G}$ is an open set system on X; we can define eight kinds of topologies on $\mathcal{P}(X)$ relative to $\mathcal{G}$ according to the results in Section 6. They are hypertopologies on $(X, \mathcal{G})$.

2. *Fuzzy Topology*

Suppose that X is a set, $\Phi(X)$ is the set of all fuzzy points in X, and $\mathcal{F}(X)$ is the set of all fuzzy subsets on X. Consider two kinds of relations between $\Phi(X)$ and $\mathcal{F}(X)$: the belonging relation $\in$ and the quasi-coincident relation[1] q. We recall that, for $x_\lambda \in \Phi$, and $A \in \mathcal{F}(X)$,

$$x_\lambda \in A \Leftrightarrow 0 < \lambda \leq \mu_A(x) \tag{7-1}$$

$$x_\lambda q A \Leftrightarrow \lambda + \mu_A(x) > 1. \tag{7-2}$$

$\in^*$ is $\subseteq$; the restriction of $\in^*$ on Φ is equal to $\in*$; $\in$ is a distant relation; the dually negative relation of $\in$ is just q.

q^* is $\subseteq$; the restriction of q^* on Φ is equal to $q*$; q is a neighbor relation; the dually negative relation of q is just $\in$.

$\in$-Distant element structure is just the distant field structure on Φ; $\in$-distant element system which is equal to $\in$-closed element system is just the fuzzy closed set system. q-Neighbor element structure is just the q-neighborhood structure on X; q-neighbor element system which is equal to q-open element system is just the fuzzy open set system.

Suppose that $\mathcal{G}$ is a fuzzy open set system on X, and that $\mathcal{F}$ is the corresponding fuzzy closed set system. Then, on Φ,

$$T(q \rightarrow \mathcal{G}) = T(\in \rightarrow \mathcal{F}), \tag{7-3}$$

and we can define eight kinds of topologies on $\mathcal{P}(X)$ relative to $\mathcal{G}$ according to the results in Section 6. They are fuzzy hypertopologies.

3. *Topological Molecular Lattice*

Suppose that $L(\Pi)$ is a topological molecular lattice, Π is the set of molecules in $L(\Pi)$[2]. Consider the less than or equal to relation $\leq$ between Π and $L(\Pi)$.

$\leq^*$ is just $\leq$; $\leq$ is a distant relation; $\leq$-distant element system which is equal to $\leq$-closed element system is just the closed element system on $L(\Pi)$; $\leq$-distant element structure is just the distant field structure on Π; the quasi-rising of $\leq$-distant element structure is just the distant field structure on $L(\Pi)$.

According to our results in this paper, we can introduce the dually negative relation $\leq^\theta$ of $\leq$. The $\leq^\theta$ is a neighbor relation, so we can study topological properties on $L(\Pi)$ by a neighborhood structure.

4. *Latticized Topology*

Suppose that $(L, \leq)$ is a complete lattice, and that its largest element and its least element are denoted by 1 and 0, respectively. Take $X = X\backslash\{0\}$. Consider the relation $\leq$. Then $\leq^*$ is still $\leq$; $\leq$ is both o-adequate and c-adequate, and arbitrary i-regular. Theorem 4.1 gives a 1–1 map between the upper neighbor element structure and the upper open element system. Take $X = X\backslash\{1\}$. Consider the relation $\geq$. Then $\geq^*$ is still $\geq$; $\geq$ is both o-adequate and c-adequate, and arbitrary i-regular. Theorem 4.1 gives a 1–1 map between the lower neighbor element structure and the lower open element system.

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